Recall: · Laplace's equation $\Delta u = 0$

· Fundamental solution $\Gamma(x) = \begin{cases} \frac{1}{2\pi}\log|x| & n=2 \\ \frac{1}{n(2-n)\omega_n}|x|^{2-n} & n\geq 3 \end{cases}$

· Green's identity

$$u(x) = \int_\Omega \Gamma(x-y)\Delta_y u(y)\,dy - \int_{\partial\Omega}\left[\Gamma(x-y)\frac{\partial u}{\partial \vec{n}}(y) - u(y)\frac{\partial \Gamma}{\partial \vec{n}_y}(x-y)\right]dS_y$$

Lecture 2. Laplace's equation (II)

Goal: Solve Dirichlet Problem in $B_R = B_R(0)$: $\begin{cases} \Delta u = 0 & \text{in } B_R \\ u = \varphi & \text{on } \partial B_R \end{cases}$

1. Green's function

Definition. $\Omega \subset \mathbb{R}^n$: bounded domain with C^1 boundary.

Suppose that $\Phi(x,y)$ defined in $\Omega\times\Omega$ satisfies

for any $x\in\Omega$. $\begin{cases} \Delta_y \Phi(x,y) = 0 & \text{in } \Omega \\ \Phi(x,y) = \Gamma(x-y) & \text{on } \partial\Omega. \end{cases}$

Then $G(x,y) = \Gamma(x-y) - \Phi(x,y)$ is called Green's function of Ω.

Lemma. If $u\in C^2(\bar\Omega)$ satisfies $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$

then $$u(x) = \int_{\partial\Omega}\varphi(y)\frac{\partial G}{\partial \vec{n}_y}(x,y)\,dS_y.$$

Proof. By Green's identity,

$$u(x) = \int_\Omega \Gamma(x-y)\Delta_y u(y)\,dy - \int_{\partial\Omega}\left[\Gamma(x-y)\frac{\partial u}{\partial \vec{n}}(y) - u(y)\frac{\partial \Gamma}{\partial \vec{n}_y}(x-y)\right]dS_y$$

$$(\Gamma = G+\Phi) \quad = \quad 0 - \int_{\partial\Omega}\Gamma(x-y)\frac{\partial u}{\partial \vec{n}}(y)\,dS_y + \int_{\partial\Omega}u(y)\left[\frac{\partial G}{\partial \vec{n}_y}(x,y) + \frac{\partial \Phi}{\partial \vec{n}_y}(x,y)\right]dS_y$$

$$(\Phi = \Gamma \text{ on } \partial\Omega) \quad = \quad \int_{\partial\Omega}u(y)\frac{\partial G}{\partial \vec{n}_y}(x,y)\,dS_y + \int_{\partial\Omega}\left[u(y)\frac{\partial \Phi}{\partial \vec{n}_y}(x,y) - \Phi(x,y)\frac{\partial u}{\partial \vec{n}}(y)\right]dS_y$$

$$(u=\varphi \text{ on } \partial\Omega) \quad = \quad \int_{\partial\Omega}\varphi(y)\frac{\partial G}{\partial \vec{n}_y}(x,y)\,dS_y + \underbrace{\int_\Omega\left[u(y)\Delta_y\Phi(x,y) + \Phi(x,y)\Delta u(y)\right]dS_y}_{=0}$$

#

Question 1. Existence of Green's function?

Question 2. Is this formal solution a classical solution?

2. Green's function in $B_R = B_R(0)$

Proposition. Define $\Phi(x,y)$ by

when $n=2$. $\Phi(x,y) = \begin{cases} \frac{1}{2\pi}\log R & x=0.\ y \in B_R \\ \frac{1}{2\pi}\log\left(\frac{|x|}{R}y - \frac{R}{|x|}x\right) & x \neq 0.\ y \in B_R \end{cases}$

when $n \geq 3$. $\Phi(x,y) = \begin{cases} \frac{1}{n(2-n)\omega_n}R^{2-n} & x=0.\ y \in B_R \\ \frac{1}{n(2-n)\omega_n}\left|\frac{|x|}{R}y - \frac{R}{|x|}x\right|^{2-n} & x \neq 0.\ y \in B_R. \end{cases}$

Then $G(x,y) = \Gamma(x-y) - \Phi(x,y)$ is a Green's fuction of B_R.

Proof. It suffices to verify $\begin{cases} \Delta_y \Phi(x,y) = 0 & y \in B_R \\ \Phi(x,y) = \Gamma(x-y) & y \in \partial B_R. \end{cases}$

Suppose $n \geq 3$ ($n=2$ similarly)

Case 1. $x=0$.

$$\Phi(0,y) = \frac{1}{n(2-n)\omega_n}R^{2-n} \Rightarrow \Delta_y \Phi(0,y) = 0 \quad y \in B_R$$

$$\Gamma(0-y) = \frac{1}{n(2-n)\omega_n}|y|^{2-n} \Rightarrow \Phi(0,y) = \Gamma(0-y) \quad y \in \partial B_R.$$

Case 2. $x \neq 0$

$$\Phi(x,y) = \frac{1}{n(2-n)\omega_n}\left|\frac{|x|}{R}y - \frac{R}{|x|}x\right|^{2-n}$$

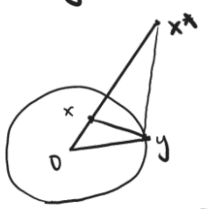
$$= \left(\frac{|x|}{R}\right)^{2-n}\frac{1}{n(2-n)\omega_n}\left|y - \frac{R^2}{|x|^2}x\right|^{2-n}$$

$$= \left(\frac{|x|}{R}\right)^{2-n}\Gamma(y-x^*)$$

where $x^* = \frac{R^2}{|x|^2}\cdot x$.

$$|x^*| = \frac{R^2}{|x|^2}|x| = \frac{R}{|x|}\cdot R > R \Rightarrow x^* \notin \overline{B_R}$$

$\Rightarrow \Gamma(y-x^*)$ is harmonic in $B_R \Rightarrow \Delta_y \Phi(x,y) = 0 \quad y \in B_R$



$\forall\, y \in \partial B_R$.

$$\frac{Oy}{Ox^*} = \frac{R}{|x^*|} = \frac{|x|}{R} = \frac{Ox}{Oy} \Rightarrow \triangle Oxy \sim \triangle Oyx^*$$

$$\Rightarrow \frac{|y-x|}{|y-x^*|} = \frac{Ox}{Oy} = \frac{|x|}{R}$$

$$\Rightarrow |y-x| = \frac{|x|}{R}\cdot|y-x^*|. \quad (1)$$

$$\Rightarrow \forall y \in \partial B_R. \quad \Phi(x,y) = \left(\frac{|x|}{R}\right)^{2-n} \Gamma(y-x^*)$$

$$= \left(\frac{|x|}{R}\right)^{2-n} \frac{1}{n(2-n)\omega_n} |y-x^*|^{2-n}$$

$$(1) \quad = \left(\frac{|x|}{R}\right)^{2-n} \frac{1}{n(2-n)\omega_n} \left(\frac{R}{|x|}\right)^{2-n} |y-x|^{2-n}$$

$$= \Gamma(y-x) \qquad \#$$

3. Poisson kernel in B_R

Recall: Lemma: $\begin{cases} \Delta u = 0 & \text{in } B_R \\ u = \varphi & \text{on } \partial B_R \end{cases} \Rightarrow u(x) = \int_{\partial B_R} \varphi(y) \frac{\partial G}{\partial \vec{n}_y}(x,y) dS_y.$

Definition. Poisson kernel: $K(x,y) = \frac{\partial G}{\partial \vec{n}_y}(x,y)$ for $x \in B_R$, $y \in \partial B_R$.

Lemma. $K(x,y) = \frac{R^2 - |x|^2}{n\omega_n R |x-y|^n}$.

Proof. Suppose $n \geq 3$. ($n=2$ similarly)

Recall: $G(x,y) = \Gamma(x-y) - \Phi(x,y)$

Case 1. $x = 0$.

$$G(0,y) = \frac{1}{n(2-n)\omega_n} \left(|y|^{2-n} - R^{2-n}\right)$$

$$\frac{\partial G}{\partial y_i}(0,y) = \frac{1}{n\omega_n} \cdot \frac{y_i}{|y|^n}$$

$$y \in \partial B_R \Rightarrow \vec{n} = \frac{y}{R} \Rightarrow n_i = \frac{y_i}{R}.$$

$$\Rightarrow \frac{\partial G}{\partial \vec{n}_y}(0,y) = \frac{R^2}{n\omega_n R |y|^n}.$$

Case 2. $x \neq 0$.

Recall $\Phi(x,y) = \left(\frac{|x|}{R}\right)^{2-n} \Gamma(y-x^*) \qquad x^* = \frac{R^2}{|x|^2} x$

$$G(x,y) = \frac{1}{n(2-n)\omega_n} \left[|y-x|^{2-n} - \left(\frac{|x|}{R}\right)^{2-n} |y-x^*|^{2-n} \right]$$

Recall $|y-x^*| = \frac{R}{|x|} \times |y-x|. \qquad x^* = \frac{R^2}{|x|^2} x$

$$\frac{\partial G}{\partial y_i}(x,y) = \frac{1}{n\omega_n}\left[\frac{y_i-x_i}{|y-x|^n} - \left(\frac{|x|}{R}\right)^{2-n}\frac{y_i-\frac{R^2}{|x|^2}x_i}{\left(\frac{R}{|x|}\right)^n \times |y-x|^n}\right]$$

$$= \frac{1}{n\omega_n}\left[\frac{y_i-x_i}{|y-x|^n} - \frac{\frac{|x|^2}{R^2}y_i - x_i}{|y-x|^n}\right]$$

$$= \frac{R^2-|x|^2}{n\omega_n R^2|y-x|^n}\cdot y_i$$

$$\vec{n} = \frac{y}{R} \Rightarrow \frac{\partial G}{\partial \vec{n}_y}(x,y) = \frac{R^2-|x|^2}{n\omega_n R|y-x|^n}$$ #

Lemma $K(x,y) = \dfrac{R^2-|x|^2}{n\omega_n R|x-y|^n}$ $x\in B_R$ $y\in\partial B_R$

(1). $K(x,y)$ is smooth and positive

(2) $\int_{\partial B_R} K(x,y)\,dS_y = 1$

(3) $\Delta_x K(x,y) = 0$

Proof. (1) is trivial

(2) Recall Lemma $\begin{cases}\Delta u = 0 & \text{in } B_R\\ u=\varphi & \text{on } \partial B_R\end{cases}$

$\Rightarrow u(x) = \int_{\partial B_R}\varphi(y)K(x,y)\,dS_y$

Choose $u\equiv 1$ $\Rightarrow \int_{\partial B_R} K(x,y)\,dS_y = 1$

(3). Method 1. $G(x,y)$ can be extended to $\overline{B}_R\times\overline{B}_R\setminus\{x=y\}$

Fact: $G(x,y) = G(y,x)$

$G(x,y) = \Gamma(x-y) - \Phi(x,y) \Rightarrow \Delta_y G(x,y) = 0$

Fact $\Rightarrow \Delta_x G(x,y) = 0$

$\Rightarrow \Delta_x \frac{\partial G}{\partial \vec{n}_y}(x,y) = 0 \Rightarrow \Delta_x K(x,y) = 0$.

Method 2. Direct calculation

$$\frac{\partial K}{\partial x_i}(x,y) = -\frac{1}{n\omega_n R}\left[2x_i|x-y|^{-n} + n(R^2-|x|^2)|x-y|^{-n-2}(x_i-y_i)\right]$$

$$\frac{\partial^2 K}{\partial x_i \partial x_i}(x,y) = -\frac{1}{n\omega_n R}\Big[2|x-y|^{-n} - 4n|x-y|^{-n-2}(x_i^2 - x_i y_i)$$
$$- n(n+2)(R^2-|x|^2)|x-y|^{-n-4}(x_i-y_i)^2$$
$$+ n(R^2-|x|^2)|x-y|^{-n-2}\Big]$$

$$\Delta_x K(x,y) = -\frac{1}{n\omega_n R}\Big[2n|x-y|^{-n} - 4n|x-y|^{-n-2}(|x|^2 - x\cdot y) - 2n(R^2-|x|^2)|x-y|^{-n-2}\Big]$$
$$= -\frac{1}{n\omega_n R} \times 2n \times |x-y|^{-n-2}\Big[|x-y|^2 - 2(|x|^2 - x\cdot y) - (R-|x|^2)\Big]$$
$$= -\frac{2}{\omega_n R|x-y|^{n+2}}\Big[|x|^2 - 2x\cdot y + |y|^2 - 2|x|^2 + 2x\cdot y - R^2 + |x|^2\Big]$$
$$(y \in \partial B_R) \quad = 0$$

4. Dirichlet Problem in B_R

Theorem. For $\varphi \in C(\partial B_R)$. Write $u(x) = \int_{\partial B_R} \varphi(y) K(x,y)\, dS_y$. Then

(1) u is smooth and $\Delta u = 0$ in B_R

(2) For any $y_0 \in \partial B_R$. $\lim_{x\to y_0} u(x) = \varphi(y_0)$.

Proof. (1). $K(x,y)$ is smooth $\Rightarrow$ u is smooth.

$$\Delta_x K(x,y) = 0 \Rightarrow \Delta u = \int_{\partial B_R} \varphi(y)\, \Delta_x K(x,y)\, dS_y = 0$$

(2). Goal: $\forall \varepsilon > 0$. $\exists \delta$. s.t. (such that)

$$|u(x) - \varphi(y_0)| \le \varepsilon \quad \forall x \in B_\delta(y_0) \cap B_R.$$

Recall: $\int_{\partial B_R} K(x,y)\, dS_y = 1 \Rightarrow \varphi(y_0) = \int_{\partial B_R} \varphi(y_0) K(x,y)\, dS_y$.

$$\Rightarrow u(x) - \varphi(y_0) = \int_{\partial B_R} (\varphi(y) - \varphi(y_0))\, K(x,y)\, dS_y$$
$$= \int_{\partial B_R \cap B_{\delta_1}(y_0)} \sim \; + \int_{\partial B_R \setminus B_{\delta_1}(y_0)} \sim \; = I_1 + I_2$$

where δ_1 is to be determined later.

Term I_1: $\varphi \in C(\partial B_R)$. $\exists\, \delta_1$. s.t.

$$|\varphi(y)-\varphi(y_0)| < \varepsilon/2 \quad \forall y \in B_{\delta_1}(y_0) \cap \partial B_R$$

$$\Rightarrow |I_1| \le \varepsilon/2 \int_{\partial B_R} K(x,y)\, dS_y = \varepsilon/2.$$

Term I_2. $\quad K(x,y) = \dfrac{R^2-|x|^2}{n\omega_n R|x-y|^n}$.

For $\delta<\delta_1$ $\quad x \in B_R \cap B_\delta(y_0)$. $\quad y \in \partial B_R \setminus B_{\delta_1}(y_0)$

$$0 < K(x,y) \le \frac{(R+|x|)(R-|x|)}{n\omega_n R(\delta_1-\delta)^2} \le \frac{2R}{n\omega_n R(\delta_1-\delta)^n}|y_0-x|$$

$$\uparrow$$
$$R-|x| \le |y_0-x|$$

$$\le \frac{2\delta}{n\omega_n(\delta_1-\delta)^n}$$

Set $M = \max\limits_{\partial B_R} |\varphi|$ $\quad$ choose $\delta << 1$.

$$0 < K(x,y) \le \frac{\varepsilon}{2M\omega_n R^{n-1}} \quad \forall\, x \in B_R \cap B_\delta(y_0).\ y \in \partial B_R \setminus B_{\delta_1}(y_0).$$

$$\Rightarrow |I_2| \le \int_{\partial B_R \setminus B_{\delta_1}(y_0)} \frac{\varepsilon}{2M\omega_n R^{n-1}} \left(|\varphi(y)| + |\varphi(y_0)| \right) dS_y$$

$$\le \frac{\varepsilon}{2M\omega_n R^{n-1}} \times 2M \times |\partial B_R| = \frac{\varepsilon}{2}.$$

Hence for $x \in B_\delta(y_0) \cap B_R$.

$$|u(x)-\varphi(y_0)| \le |I_1| + |I_2| \le \varepsilon. \Rightarrow \lim_{x\to y_0} u(x) = \varphi(y_0) \qquad \#$$

Remark. $\quad u(x) = \displaystyle\int_{\partial B_R} \varphi(y) K(x,y)\, dS_y$.

is a harmonic function in B_R

The above theorem $\Rightarrow$

(1) $u \in C^\infty(B_R) \cap C(\bar{B}_R)$.

(2) u is the solution of Dirichlet Problem

$$\begin{cases} \Delta u = 0 & \text{in } B_R \\ u = \varphi & \text{on } \partial B_R \end{cases}$$

5. Harmonic functions.

Theorem. Ω: domain in $\mathbb{R}^n$. If $u \in C^2(\Omega)$ is harmonic in Ω.

then u satisfies the mean value property in Ω.

i.e., (that is) $\forall\ B_r(x_0) \subset\subset \Omega$ (i.e., $\bar{B}_r(x_0) \subset \Omega$)

$$u(x_0) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x_0)} u(y)\,dy = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u(y)\,dy$$

Proof. After translation assume $x_0 = 0$. Otherwise consider

$$V(x) = u(x + x_0).$$

$\forall\ s \in (0, r]$. we compute

$$0 = \int_{B_s} \Delta u\,dy = \int_{\partial B_s} \frac{\partial u}{\partial \vec{n}}\,dS_y = \int_{\partial B_s} \sum_{i=1}^n \frac{y_i}{s} * \partial_{y_i} u(y)\,dS_y$$

$$\left(W = \frac{y}{s}\right) \qquad = s^{n-1} \int_{\partial B_1} \sum_{i=1}^n W_i * \partial_{y_i} u(sw)\,dS_w$$

$$= s^{n-1} \int_{\partial B_1} \frac{\partial u}{\partial s}(sw)\,dS_w$$

$$= s^{n-1} \frac{\partial}{\partial s} \int_{\partial B_1} u(sw)\,dS_w$$

$$\Rightarrow \int_{\partial B_1} u(sw)\,dS_w = \text{Constant}.$$

$$\lim_{s \to 0} \int_{\partial B_1} u(sw)\,dS_w = \lim_{s \to 0} \int_{\partial B_1} \left(u(0) + O(s)\right) dS_w = n\omega_n u(0) + 0.$$

$$\Rightarrow \int_{\partial B_1} u(sw)\,dS_w = n\omega_n u(0)$$

$$\Rightarrow u(0) = \frac{1}{n\omega_n}\int_{\partial B_1} u(sw)\,dS_w \overset{y=sw}{=} \frac{1}{n\omega_n s^{n-1}}\int_{\partial B_s} u(y)\,dS_y \quad (1)$$

Choose $s=r$ in (1) $\Rightarrow$ $u(0) = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r} u(y)\,dS_y$

$$(1) \Rightarrow \quad n\omega_n s^{n-1} u(0) = \int_{\partial B_s} u(y)\,dy$$

$$\Rightarrow \int_0^r n\omega_n s^{n-1} u(0)\,ds = \int_0^r \int_{\partial B_s} u(y)\,dy = \int_{B_r} u(y)\,dy$$

$$\Rightarrow \quad \omega_n r^n u(0) = \int_{B_r} u(y)\,dy$$

$$\Rightarrow \quad u(0) = \frac{1}{\omega_n r^n}\int_{B_r} u(y)\,dy. \qquad \#$$

Theorem. If $u \in C^\infty(\Omega)$ is harmonic in Ω. For $B_R(x_0) \subset\subset \Omega$. $r \in (0,R)$ and $k \in \mathbb{Z}_+$

$$\sup_{B_r(x_0)} |\nabla^k u| \le \frac{C(n,k)}{(R-r)^k} \max_{B_R(x_0)} |u| \qquad (*_k)$$

Where $C(n,k)$ denotes a constant depending only on n and k.

Proof. After translation, assume $x_0 = 0$.

when $k=1$. $\forall\ x \in B_r$.

$$u(x) = \frac{1}{\omega_n (R-r)^n}\int_{B_{R-r}(x)} u(y)\,dy = \frac{1}{\omega_n (R-r)^n}\int_{B_{R-r}} u(x+y)\,dy$$

$$\Rightarrow \partial_i u(x) = \frac{1}{\omega_n (R-r)^n}\int_{B_{R-r}} \partial_{x_i} u(x+y)\,dy$$

$$= \frac{1}{\omega_n (R-r)^n}\int_{B_{R-r}} \partial_{y_i} u(x+y)\,dy$$

$$= \frac{1}{\omega_n (R-r)^n}\int_{\partial B_{R-r}} u(x+y)\cdot n_i\,dy \ , \quad \text{where } \vec{n} = (n_i)$$

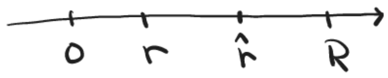
$$\Rightarrow |\partial_i u(x)| \le \frac{n}{R-r} \sup_{B_R} |u|$$

$$\Rightarrow \sup_{B_r} |\nabla u| \le \frac{C(n)}{R-r} \sup_{B_R} |u| \qquad (*_1) \ \checkmark$$

Argue by induction. Assume $(*_k)$ holds.

$\Delta u = 0 \Rightarrow \Delta(\nabla^k u) = 0 \Rightarrow \nabla^k u$ is harmonic.

Set $\hat{r} = \frac{R-r}{2}$



Applying $(*_1)$ to $\nabla^k u$ (replacing r, R by $r, \hat{r}$)

$$\sup_{B_r}|\nabla^{k+1}u| \le \frac{C(n,1)}{\hat{r}-r}\sup_{B_{\hat{r}}}|\nabla^k u| = \frac{2C(n,1)}{R-r}\sup_{B_{\hat{r}}}|\nabla^k u| \quad (1)$$

By $(*_k)$ (replace r, R by $\hat{r}, R$)

$$\sup_{B_{\hat{r}}}|\nabla^k u| \le \frac{C(n,k)}{(R-\hat{r})^k}\sup_{B_R}|u| = \frac{2^k C(n,k)}{(R-r)^k}\sup_{B_R}|u| \quad (2)$$

(1), (2) $\Rightarrow (*_{k+1})$ ✓ $\qquad C(n,k+1) = 2^{k+1}C(n,1)C(n,k)$ #