Lecture 5

De Rham spaces

1. Backgroup: Riemann-Hilbert problem (Hilbert's 21st problem. 1900)

original problem: show the existence of linear systems of differential equations with prescribed monodromy group.

$\Updownarrow$

the map Fuchsian equations $\longrightarrow$ equiv. classes of representations is surjective.

regular singularities i.e. 1st order singularity.

1908. J. Plemelj gave a proof

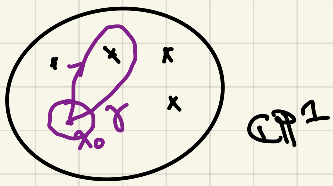
1990. A.A. Bolibrukh found a counterexample to Plemelj's proof

Consider the following linear system of ODEs.

$$\frac{d\Psi(z)}{dz} = A(z)\,\Psi(z) \qquad (*)$$

defined on $\Sigma := \mathbb{CP}^1 - \{z_1, \dots, z_s\}$

$A(z) \in \mathfrak{gl}_n(\mathbb{C}(z))$ is analytic on Σ.



Fix $x_0 \in \Sigma$

a fundamental solution to $(*)$ is $Y(z) \in \mathfrak{gl}_n(\mathbb{C}(z))$ with $\det Y(z)\big|_{x_0} \neq 0$.

satisfies $(*)$. always exists on some disc $D_r(x_0)$.

Taking γ a loop based at x_0.

analytic continuity of $Y(z)$ along γ $\rightsquigarrow$ $Y'(z)$

$Y(z) = Y'(z) \cdot g_\gamma \qquad g_\gamma \in GL_n(\mathbb{C})$

Moreover. g_γ only depends on the homotopy class of γ

$\rightsquigarrow$ $(*) \Rightarrow \quad \rho : \pi_1(\Sigma, x_0) \longrightarrow GL_n(\mathbb{C})$

$$[\gamma] \longmapsto g_\gamma$$

called the monodromy of the ODEs $(*)$.

When $g \geq 2$ $\rightsquigarrow$ it's more "natural" to formulate the problem to the relationship between parallel sections of connections and the monodromies.

$\rightsquigarrow$

" Riemann-Hilbert correspondence ".

Today: an analytic version of <u>Deligne's R-H correspondence</u>.

X alg. variety $/\mathbb{C}$ $\qquad X^{an} := (X(\mathbb{C}), \text{analytic top})$ the associated analytic space

X^{top} underlying topological space.

<u>Thm</u> (Deligne) Fix $n \in \mathbb{Z}_{>0}$. equivalence of categories

$$\mathcal{C}^{reg}_{Flat}(X, n) \simeq \mathcal{C}_{Flat}(X^{an}, n) \simeq \mathcal{C}_{Loc}(X^{an}, n)$$

$$\simeq \mathcal{C}_{Rep}(X^{top}, n)$$

algebraic flat connections with regular singularity

In particular. if X smooth projective variety $/\mathbb{C}$

<u>Cor</u>: $\mathcal{C}_{Flat}(X, n) \simeq \mathcal{C}_{Flat}(X^{an}, n) \simeq \cdots$

§2. Holomorphic structures and Dolbeault operators

$\mathbb{k} = \mathbb{C}$. X complex mfld $\dim_{\mathbb{C}} X = m$

$\pi : E \to X$ smooth complex vector bundle. $\mathrm{rk} = n$

$0 \le k \le 2m$. space of C^∞ k-forms on E:

$$A^k(X, E) := \Gamma(X, \Lambda^k_{\mathbb{C}} X \otimes E)$$
$$= \Gamma(X, \Lambda^k(T^*_{\mathbb{C}} X) \otimes E)$$

$0 \le p, q \le m$. space of C^∞ (p,q)-forms on E

$$A^{p,q}(X, E) := \Gamma(X, \Lambda^{p,q} X \otimes E)$$
$$= \Gamma(X, \Lambda^p(T^*_{1,0} X) \otimes_{\mathbb{C}} \Lambda^q(T^*_{0,1} X) \otimes E)$$

$\Lambda^k_{\mathbb{C}} X = \bigoplus_{p+q=k} \Lambda^{p,q} X \quad \rightsquigarrow$

$$A^k(X, E) = \bigoplus_{p+q=k} A^{p,q}(X, E)$$

<u>Recall</u>: holo. bundle $\iff$ transition functions $\{g_{ij} : U_i \cap U_j \to GL_n \mathbb{C}\}$ are holomorphic

Denote $\mathcal{E}$ as a holomorphic bundle.

Given $\mathcal{E}$ holo. let E the underlying $\mathbb{C}$-bundle.

$\Omega^0(X, E)$: space of <u>holomorphic sections</u> on $\mathcal{E}$.

$s \in A^0(X, E)$ is holomorphic if $\forall x \in X$. $\exists$ $x \in U \subseteq X$ open s.t. under the local trivialization

$$\varphi_U : E|_U := \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^n$$

the map

$$\begin{aligned} \xi : U &\longrightarrow \mathbb{C}^n \\ z &\longmapsto \xi(z) \end{aligned}$$

determined by

$$s|_U : U \to E|_U \xrightarrow{\simeq} U \times \mathbb{C}^n$$
$$z \longmapsto (z, \xi(z))$$

is holomorphic.

$\Omega^p(X, E)$: space of holomorphic p-forms on E.

$\alpha \otimes s$, $\alpha \in A^{p,0}(X)$. holo.

$s \in A^0(X, E)$ holo.

Prop 2.2 $\pi : E \to X$ cplex bundle. then E is holomorphic iff it admits an operator $\bar\partial_E : A^0(X, E) \to A^{0,1}(X, E)$ that satisfies the following Leibniz rule:

$$\bar\partial_E(fs) = \bar\partial(f) \otimes s + f \bar\partial_E(s)$$

$\forall f \in C^\infty(X, \mathbb{C})$. $s \in A^0(X, E)$

and
$$\bar\partial_E^2 = 0$$

under the natural extension

$$\bar\partial_E : A^{p,q}(X, E) \to A^{p,q+1}(X, E)$$
$$\underset{A^{p,q}(X)}{\alpha} \otimes \underset{A^0(X,E)}{s} \longmapsto \bar\partial\alpha \otimes s + (-1)^{p+q} \alpha \wedge \bar\partial_E s$$

Rmk So defined $\bar\partial_E$ is called a Dolbeault operator.

Pf: Suppose E is holomorphic. of rk n. under a trivialization $(U, \varphi_U : E|_U \simeq U \times \mathbb{C}^n)$. choose local holomorphic frame $\{s_1, \dots s_n\}$ on this chart.

locally define
$$\bar\partial_E : A^0(X, E|_U) \to A^{0,1}(X, E|_U)$$
$$s = \textstyle\sum_i f_i s_i \mapsto \sum_i \bar\partial(f_i)\, s_i$$

This is well-defined to give a global defined operator.

Indeed. take $(V, \varphi_V : E|_V \simeq V \times \mathbb{C}^n)$ s.t. $U \cap V \neq \phi$

$\{s_1', \dots s_n'\}$, then $\exists$ holo. functions g_{ij} s.t. on $U \cap V$.

$$s_i' = \sum_j g_{ij} s_j$$

Suppose $s = \sum_i f_i' s_i' \Rightarrow f_i = \sum_j f_j' g_{ij}$

$$\Rightarrow \bar{\partial}_E(s) = \bar{\partial}_E\left(\sum_{i,j} f_i' g_{ij} s_j\right)$$

$$= \sum_{i,j} \bar{\partial}(f_i') g_{ij} s_j$$

$$= \sum_{i,j} \bar{\partial}(f_i') s_i'$$

Conversely. if we have $\bar{\partial}_E$, then $\forall x \in X$. $\exists x \in U \subseteq X$ open. s.t.

$$E|_U \simeq U \times \mathbb{C}^n$$

we may choose a local frame $\{s_1, \dots s_n\}$ s.t.

$$\bar{\partial}_E(s_i) = 0 \qquad 1 \le i \le n$$

need PDE technique! As long as this holds, we can find that the transition function are holo. $\rightsquigarrow$ E is holomorphic bundle.

$\boxtimes$

From now on. write $\mathcal{E} = (E, \bar{\partial}_E)$ for $\bar{\partial}_E$ the Dolbeault operator.

Rmk: $\Omega^p(X, E) = \ker(\bar{\partial}_E : A^{p,0}(X, E) \to A^{p,1}(X, E))$

§3. Smooth and holomorphic connections

$\pi : E \to X$ complex v.b. of rk n

Def 3.1 A C^∞ connection on E is an operator

$$\nabla : A^0(X, E) \longrightarrow A^1(X, E)$$

that satisfies the following Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

$\forall f \in C^\infty(X, \mathbb{C})$, $s \in A^0(X, E)$.

It is clearly that ∇ extends inductively to an operator

$$\nabla : A^k(X, E) \longrightarrow A^{k+1}(X, E)$$

$$\underset{\underset{A^k(X,\mathbb{C})}{\cap}}{\alpha} \otimes \underset{\underset{A^0(X,E)}{\cap}}{s} \longmapsto d(\alpha) \otimes s + (-1)^k \alpha \wedge \nabla(s)$$

If moreover, $\nabla^2 = 0$, then ∇ is called a flat C^∞ connection.

and the pair (E, ∇) is called a C^∞ flat bundle.

Rmk If ∇ is a flat C^∞ connection on E.

let $\nabla^{1,0} := \pi_{1,0} \circ \nabla$

$\nabla^{0,1} := \pi_{0,1} \circ \nabla$

for $\pi_{1,0} : A^1(X,E) \to A^{1,0}(X,E)$, $\pi_{0,1} : A^1(X,E) \to A^{0,1}(X,E)$ proj

$\Rightarrow$

$$\nabla^2 = 0 \iff \begin{cases} (\nabla^{1,0})^2 = 0 \\ (\nabla^{0,1})^2 = 0 \\ [\nabla^{1,0}, \nabla^{0,1}] = 0 \end{cases}$$

$$[\nabla^{1,0}, \nabla^{0,1}] = \nabla^{1,0} \circ \nabla^{0,1} + \nabla^{0,1} \circ \nabla^{1,0}$$

$\rightsquigarrow$ induces a holomorphic bundle $(E, \bar\partial_E := \nabla^{0,1})$.

Conversely, suppose $\mathcal{E} := (E, \bar\partial_E)$ is holo. bundle.

Def 3.3 A holomorphic connection on $\mathcal{E}$ is an operator

$$D : \Omega^0(X, E) \longrightarrow \Omega^1(X, E)$$

that satisfies the Leibniz rule

$$D(fs) = df \otimes s + f D(s)$$

$\forall f \in \mathcal{O}_X(X)$ holomorphic function. $s \in \Omega^0(X, E)$

Similarly, D extends

$$D : \Omega^p(X, E) \to \Omega^{p+1}(X, E) \qquad \forall p$$

if $D^2 = 0$ under extension, then D is called a flat holomorphic connection.
$(\mathcal{E}, D)$ is called a holomorphic flat bundle.

Prop 3.4 Given a flat holomorphic connection on a holo. bundle is equivalent to given a flat C^∞ connection on its underlying complex bundle.

Pf.

If we have a holo. flat bundle $(\mathcal{E}, D) = (E, \bar\partial_E, D)$

$\rightsquigarrow$ $\nabla := D + \bar\partial_E$ is flat C^∞ connection.

Conversely, if we have a C^∞ flat bundle (E, ∇), then it follows from the above Remark that

$$\bar\partial_E := \nabla^{0,1} \qquad \text{holo. str.}$$

$$D := \nabla^{1,0} \qquad \text{is a holo. connection, which is flat.}$$

$\rightsquigarrow$ we get a holomorphic flat bundle $(\mathcal{E} := (E, \bar\partial_E), D)$.

$\square$

From now on, we write (E, ∇)

$(\mathcal{E}, D)$

Q: When does a holo. bundle admit a flat holo. connection, or equivalently, when does a complex bundle admit a flat C^∞ connection?

When $\dim_{\mathbb{C}} X = 1$ $(n = 1)$, this is well-known:

Prop 3.6 (Weil–Atiyah criterion)

A holo. bundle on a cpt connected R.S. admits a holo. connection

$\Updownarrow$ $n=1$

flat holo. conn.

$\left(\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell \quad \mathcal{E}_i \text{ indecomposable}\right)$

iff each direct summand of it is of degree 0.

Prop 3.7 (Conjecture?)

A holo. bundle $\mathcal{E}$ over a cpt. connected cplex mfld admits a flat holo connection iff each direct summand of it has all chern classes vanishing.

Conj 3.8 $m > 1$.

holomorphic connection induces a flat holo. connection.

§4. Riemann–Hilbert correspondence

$\mathcal{E}$ holo. bundle

$$D : \Omega^0(X, E) \to \Omega^1(X, E) \quad \text{holo. conn.}$$

Recall:

$$\{\text{holo. bundles of rk } n\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{locally free sheaves of } \mathcal{O}_X\text{-modules} \\ \text{of rk } n\end{array}\right\}$$

Viewing $\mathcal{E}$ as a locally free sheaf of $\mathcal{O}_X$-modules, then D can be viewed as a sheaf morphism

$$D : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$$

$\uparrow$ sheaf of holo. 1-forms on X

that satisfies the Leibniz rule

$$D(fs) = df \otimes s + f\, Ds \qquad \begin{array}{l} f \in \mathcal{O}_X \\ s \in \mathcal{E} \end{array}$$

Thm 4.1 (Riemann-Hilbert correspondence. I)

$\forall n \in \mathbb{Z}_{>0}$. then a $\mathbb{C}$-local system of rk n over X is equivalent to a holomorphic flat bundle of rk n over X.

Namely. we have the following one-to-one correspondence of categories:

$$\mathcal{C}_{Loc}(X, n) \simeq \mathcal{C}_{Flat}(X, n)$$

for

$\mathcal{C}_{Flat}(X,n)$: the category of holo. flat bundles of rk n over X

morphism: $(\mathcal{E}, D_{\mathcal{E}})$ & $(\mathcal{F}, D_{\mathcal{F}})$ flat bundles

a morphism is a morphism $f: \mathcal{E} \to \mathcal{F}$ s.t.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{F} \\ D_{\mathcal{E}} \downarrow & \circlearrowright & \downarrow D_{\mathcal{F}} \\ \mathcal{E}\otimes_{\mathcal{O}_X}\Omega_X^1 & \xrightarrow[f\otimes id_{\Omega_X^1}]{} & \mathcal{F}\otimes_{\mathcal{O}_X}\Omega_X^1 \end{array}$$

Pf: "$\Rightarrow$"

L local system of rk n.

then $\mathcal{E} := L \otimes_{\mathbb{C}} \mathcal{O}_X$ locally free sheaf of $\mathcal{O}_X$-modules of rk n

$D := 1 \otimes d$ for $d: \mathcal{O}_X \to \Omega_X^1$ the usual exterior diff. operator

$\Rightarrow D: \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1$ flat.

And moreover. $L = \ker(D) := \ker(L \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{1\otimes d} L \otimes \Omega_X^1) = L \otimes \mathbb{C} \simeq L$

"sheaf of parallel sections".

$\Leftarrow$ Given a holo. flat bundle $(\mathcal{E}, D)$. define $L^D := \ker(D) \subset \mathcal{E}$ which is a locally free subsheaf. Claim L^D is a local system.

$\mathcal{E} \simeq L^D \otimes \mathcal{O}_X$. $(\mathcal{E}, D) \simeq (L^D \otimes \mathcal{O}_X, 1 \otimes d)$

To show L^D is a locally constant sheaf. it suffices to show the transition function are locally constant.

by definition., sections of L are localy of D-parallel. chose local frame $\{s_1, \dots, s_n\}$. $\{s_1', \dots, s_n'\}$ $\rightsquigarrow$ $L|_{U_i} \simeq U_i \times \mathbb{C}^n$

$L|_{U_j} \simeq U_j \times \mathbb{C}^n$ $\quad U_i \cap U_j \neq \emptyset$

$\Rightarrow$ on $U_i \cap U_j$.

$$s_i = \sum_j g_{ij} s_j' \qquad 1 \le i \le n$$

for $g_{ij} : U_i \cap U_j \to GL_n\mathbb{C}$ transition functions

$$\Rightarrow \quad 0 = Ds_i = \sum_j dg_{ij} \otimes s_j' + \sum_j g_{ij} Ds_j'$$

$$= \sum_j dg_{ij} \otimes s_j'$$

$\rightsquigarrow$ $dg_{ij} = 0 \quad 1 \le i, j \le n$

$\rightsquigarrow$ L^D is a localy constant sheaf.

$\blacksquare$

Already shown:

$$\mathcal{C}_{Loc}(X, n) \simeq \mathcal{C}_{Rep}(X, n)$$

Thm 4.2 (Riemann-Hilbert correspondence. II).

For $n \in \mathbb{Z}_{>0}$. then a holo. flat bundle of rk n over X is equivalent. to a fundamental group representation into $GL_n(\mathbb{C})$.

Namely, we have the following one-to-one correspondence between categories

$$\mathcal{C}_{Flat}(X, n) \simeq \mathcal{C}_{Rep}(X, n)$$

Pf... "$\Leftarrow$": Given $\rho : \pi_1(X, x) \to GL_n\mathbb{C}$,

$$E_\rho := \tilde{X} \times_\rho \mathbb{C}^n = \tilde{X} \times \mathbb{C}^n /_\rho$$

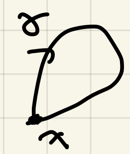
$$\gamma \cdot (\tilde{x}, v) := (\gamma \tilde{x}, \rho(\gamma) v)$$

$\tilde{X}/\pi_1(X, x) \simeq X \rightsquigarrow E_\rho$ admits a trivial flat connection.

"$\Leftarrow$" $(\mathcal{E}, D)$ holo. flat bundle. (E, ∇) associated C^∞ flat bundle.

Consider the parallel transport w.r.t. ∇:

$\forall\, \gamma : [0,1] \to X$ based at x, consider



$$\nabla(s)(\gamma'(t)) = 0 \qquad (**)$$

Taking local frame $\{s_1, \dots, s_n\}$. under the local frame, then we have

$$\nabla s_i = A_{ij} s_j \quad \text{for } A_{ij} \text{ 1-form}.$$

$s = \sum_i f_i s_i$, then $(**)$

$$\Leftrightarrow \quad \frac{d f_i(\gamma(t))}{dt} + \sum_{j=1}^{n} A_{ij}(\gamma'(t)) f_i(\gamma(t)) = 0 \qquad 1 \leq i \leq n \qquad (***)$$

By standard theory on the existence and uniqueness of solutions to linear systems of ODEs. $\forall$ initial values $f_i \in C^\infty(X, \mathbb{C})$ with

$$f_i(\gamma(0)) = f_i \qquad 1 \leq i \leq n$$

then $\exists$ a unique smooth solution to $(***)$.

$\Updownarrow$

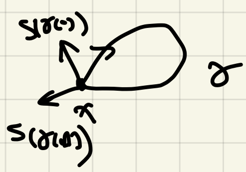
$\forall$ initial $v \in E_{\gamma(0)} = E_x$ with $s(\gamma(0)) = v$

then $\exists$ a unique section s along γ to $(**)$.

and the solution is independent on the homotopy class of γ

then

$$v = s(\gamma(0)) \longrightarrow s(\gamma(1))$$



$\Rightarrow$ uniquely determines an element $\underbrace{\ }_{g_\gamma}$ in $GL_n(E_x) \simeq GL_n\mathbb{C}$.

$$\rightsquigarrow \quad \rho : \pi_1(X, x) \to GL_n \mathbb{C}$$
$$[\gamma] \mapsto g_\gamma .$$

called the monodromy representation of (E, ∇)

$\square$

§5. Some preliminaries for later use

$E \xrightarrow{\pi} X$ C^∞ cplex bundle. a hermitian metric on E is

$$h : A^0(X, E) \times A^0(X, E) \to A^0(X, \mathbb{C})$$

which is an inner product on each fiber.

$$h_x : E_x \times E_x \to \mathbb{C}$$

(1) $h(\lambda v, w) = \lambda h(v, w)$

(2) $h(v, w) = \overline{h(w, v)}$

(3) $h(v, v) \geq 0$ & $h(v, v) = 0 \Leftrightarrow v = 0$

extends to

$$h : A^p(X, E) \times A^q(X, E) \to A^{p+q}(X, \mathbb{C})$$
$$(\alpha \otimes u, \quad \beta \otimes v) \mapsto \alpha \wedge \bar{\beta}\, h(u, v)$$

(E, ∇) C^∞ flat bundle. $\bar{\partial}_E := \nabla^{0,1}$. with a hermitian metric h

Prop 5.1 There is a unique $(1,0)$-type connection ∂_h s.t. $D_h := \partial_h + \bar{\partial}_E$ is a metric connection. i.e. D_h is C^∞ connection that satisfies

$$d(h(u, v)) = h(D_h(u), v) + h(u, D_h(v)) \qquad (5.1)$$
$$\forall\, u, v \in A^0(X, E)$$

∂_h is $(1,0)$-type :

$$\partial_h : A^0(X, E) \to A^{1,0}(X, E)$$

that satisfies the ∂-twisted Leibniz rule.

Rmk : Such a metric connection is called a Chern connection.

Pf.

(1) Uniqueness:

$E|_U \simeq U \times \mathbb{C}^n$.

$s_U := (s_1, \dots, s_n)$ local holo. frame.

$\omega_U = (\omega_\alpha^\beta)_{1 \le \alpha, \beta \le n}$ the local conn. form of D_h

$$\left(\text{i.e. } \underset{\partial_h + \bar{\partial}_E}{\overset{\|}{D_h}} s_\alpha = \sum_\beta \omega_\alpha^\beta s_\beta \right) \quad = \partial_h s_\alpha \qquad (5.2)$$

$\Rightarrow \omega_\alpha^\beta$ is type $(1,0)$.

(5.2) has a matrix expression

$$D_h s_U = s_U \omega_U$$

$$\Rightarrow \quad \partial h(s_\alpha, s_\beta) = h(\partial_h s_\alpha, s_\beta) + \cancel{h(s_\alpha, \bar{\partial}_E s_\beta)}$$
$$= h(D_h s_\alpha, s_\beta)$$

$$\Rightarrow \quad \partial H_U = (\omega_U)^T H_U$$

$$\Rightarrow \quad \omega_U = (\partial H_U \cdot H_U^{-1})^T$$

i.e. the connection form is uniquely determined by H_U.

(2) existence: localy defined is global defined.

$U \cap V \neq \emptyset$

$s_U = (s_1, \dots, s_n)$

$s_V = (s_1', \dots, s_n')$ local holo. frame.

$$s_V = s_U g_{UV} \qquad g_{UV} : U \cap V \to GL_n\mathbb{C} \text{ holo.}$$

$$\Rightarrow \quad s_U g_{UV} \omega_V = s_V \omega_V = D_h s_V$$

$$= D_h(s_U g_{UV})$$

$$= D_h(s_U) g_{UV} + s_U \, d g_{UV}$$

$$= s_U \omega_U g_{UV} + s_U \, d g_{UV}$$

$$\Rightarrow \quad \omega_V = g_{UV}^{-1} \omega_U g_{UV} + g_{UV}^{-1} \, d g_{UV}$$

on the other hand, H_U, H_V are related by

$$H_V = g_{UV}^{T} H_U \overline{g_{UV}}$$

$$\Rightarrow \quad \omega_V = (\partial H_V \cdot H_V^{-1})^T = \cdots$$

$$= g_{UV}^{-1} \, d g_{UV} + g_{UV}^{-1} \omega_U g_{UV}$$

$\rightsquigarrow$ globaly defined

$\rightsquigarrow$ well-defined.

$\boxtimes$