Recall. · $C^{2,\alpha}$ regularity and estimate of Newtonian potential

· Estimate. $\alpha \in (0,1)$ $f \in C^{\alpha}(\overline{B_R(x_0)})$.
If $u \in L^\infty(B_R(x_0)) \cap C^2(B_R(x_0))$ satisfies $\Delta u = f$ in $B_R(x_0)$
then $u \in C^{2,\alpha}(\overline{B_{R/2}(x_0)})$ and

$$R\|\nabla u\|_{L^\infty(B_{R/2}(x_0))} + R^2\|\nabla^2 u\|_{L^\infty(B_{R/2}(x_0))} + R^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{R/2}(x_0))}$$
$$\le C(\alpha, n)\left(\|u\|_{L^\infty(B_R(x_0))} + R^2\|f\|_{L^\infty(B_R(x_0))} + R^{2+\alpha}[f]_{C^\alpha(B_R(x_0))}\right)$$

· Dirichlet Problem of Poisson's equation in balls.

Lecture 7. Schauder theory (I)

Definition. $k \in \mathbb{Z}_{\ge 0}$ $\alpha \in (0,1)$

$$\|u\|^*_{C^{k,\alpha}(B_R(x_0))} = \sum_{i=0}^{k} R^i\|\nabla^i u\|_{L^\infty(B_R(x_0))} + R^{k+\alpha}[\nabla^k u]_{C^\alpha(B_R(x_0))} \quad \|u\|^*_{C^k(B_R(x_0))} = \|u\|^*_{C^{k,0}(B_R(x_0))}$$

$$\|u\|^*_{C^\alpha(B_R(x_0))} = \|u\|^*_{C^{0,\alpha}(B_R(x_0))}$$

Rewrite the above estimate

Theorem. $\alpha \in (0,1)$ $f \in C^\alpha(\overline{B_R(x_0)})$ if $u \in L^\infty(B_R(x_0)) \cap C^2(B_R(x_0))$
satisfies $\Delta u = f$ in $B_R(x_0)$. then $u \in C^{2,\alpha}(\overline{B_{R/2}(x_0)})$

$$\|u\|^*_{C^{2,\alpha}(B_{R/2}(x_0))} \le C(\alpha, n)\left(\|u\|_{L^\infty(B_R(x_0))} + R^2\|f\|^*_{C^\alpha(B_R(x_0))}\right)$$

Goal: Generalize the above theorem to $Lu = f$

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$$

1. Interpolation inequalities

Lemma. $\alpha \in (0,1)$. $\tau \in (0,1)$. $\forall u \in C^{2,\alpha}(\bar{B}_R)$

(1) $\tau^\alpha R^\alpha [u]_{C^\alpha(B_R)} \le C(\alpha, n)\left(\tau R\|\nabla u\|_{L^\infty(B_R)} + \|u\|_{L^\infty(B_R)}\right)$

(2) $\tau R\|\nabla u\|_{L^\infty(B_R)} \le C(\alpha, n)\left((\tau R)^{1+\alpha}[\nabla u]_{C^\alpha(B_R)} + \|u\|_{L^\infty(B_R)}\right)$

(3) $\tau R\|\nabla u\|_{L^\infty(B_R)} \le C(\alpha, n)\left((\tau R)^2\|\nabla^2 u\|_{L^\infty(B_R)} + \|u\|_{L^\infty(B_R)}\right)$

Proof. By scaling method. assume $R=1$

(1). $\forall x', x'' \in B_1$

Case 1. if $|x'-x''| \le \tau$ mean value theorem

$$\frac{|u(x')-u(x'')|}{|x'-x''|^\alpha} \le \frac{\|\nabla u\|_{L^\infty(B_1)}|x'-x''|}{|x'-x''|^\alpha} \le \|\nabla u\|_{L^\infty(B_1)}\tau^{1-\alpha}$$

Case 2. if $|x'-x''| > \tau$.

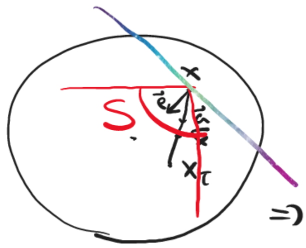
$$\frac{|u(x')-u(x'')|}{|x'-x''|^\alpha} \le \frac{2\|u\|_{L^\infty(B_1)}}{\tau^\alpha}$$

$$\Rightarrow [u]_{C^\alpha(B_1)} \le \tau^{1-\alpha}\|\nabla u\|_{L^\infty(B_1)} + 2\tau^{-\alpha}\|u\|_{L^\infty(B_1)} \Rightarrow (1)$$

(2) $\forall x\in B_1$. set $\vec{e} = -\frac{x}{|x|}$ (if $x=0$. then choose $\vec{e}=(1,0,\cdots,0)$)

$$S = \{\vec{a} \text{ is a unit vector} \mid \text{angle}(\vec{a},\vec{e}) \le \tfrac{\pi}{4}\}$$

$$\forall \vec{a}\in S \Rightarrow x_\tau := x+\tau\vec{a} \in B_1 \quad \& \quad |x_\tau - x| = \tau\in(0,1)$$



Mean value theorem $\Rightarrow \exists\, \bar{x} \in \overline{x x_\tau}$

$$|\partial_{\vec{a}}u(\bar{x})| = \frac{|u(x_\tau)-u(x)|}{|x_\tau-x|} \le \frac{2\|u\|_{L^\infty(B_1)}}{\tau}$$

$$\Rightarrow |\partial_{\vec{a}} u(x)| \le |\partial_{\vec{a}}u(\bar{x})| + |\partial_{\vec{a}}u(\bar{x})-\partial_{\vec{a}}u(x)|$$

$\bar{x}\in\overline{x x_\tau}$
$$\le 2\tau^{-1}\|u\|_{L^\infty(B_1)} + [\nabla u]_{C^\alpha(B_1)}|\bar{x} - x|^\alpha$$

$\Downarrow$

$|\bar{x} - x| \le |x-x_\tau| = \tau$
$$\le 2\tau^{-1}\|u\|_{L^\infty(B_1)} + \tau^\alpha[\nabla u]_{C^\alpha(B_1)}$$

Case 1. If $\nabla u(x)\neq\vec{0}$. $\exists\, \vec{a}\in S$ s.t.

$$\text{angle}(\vec{a},\nabla u(x)) \le \tfrac{\pi}{4} \quad \text{or} \quad \text{angle}(\vec{a},-\nabla u(x)) \le \tfrac{\pi}{4}$$

$$\Rightarrow |\vec{a}\cdot\nabla u(x)| \ge \frac{1}{\sqrt{2}}|\nabla u(x)|$$

$$\Rightarrow |\nabla u(x)| \le \sqrt{2}\,|\partial_{\vec{a}}u(x)| \le 2\sqrt{2}\,\tau^{-1}\|u\|_{L^\infty(B_1)} + \sqrt{2}\,\tau^\alpha[\nabla u]_{C^\alpha(B_1)} \quad (*)$$

Case 2. If $\nabla u(x) = \vec{0}$. $(*)$ is trivial.

$\Rightarrow$ (2).

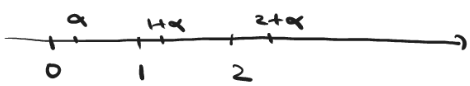
(3). Similar argument of (2).

(3) "=" the case $\alpha=1$ in (2). #

Corollary $\alpha\in(0,1)$. $\tau\in(0,1)$.

$$\sum_{i=1}^{2}(\tau R)^i\|\nabla^i u\|_{L^\infty(B_R)} + \sum_{i=0}^{1}(\tau R)^{i+\alpha}[\nabla^i u]_{C^\alpha(B_R)}$$

$$\le C(\alpha,n)\left((\tau R)^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_R)} + \|u\|_{L^\infty(B_R)}\right)$$



Proof. By scaling method. assume $R=1$. For example, we prove

$$\tau^2 \|\nabla^2 u\|_{L^\infty(B_1)} \le C(\alpha,n)\left(\tau^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}\right) \quad (*)$$

Apply Lemma (2) to ∇u. $\Rightarrow$

$$\|\nabla^2 u\|_{L^\infty(B_1)} \le C(\alpha,n)\left(\tau^\alpha [\nabla^2 u]_{C^\alpha(B_1)} + \tau^{-1}\|\nabla u\|_{L^\infty(B_1)}\right)$$

Apply Lemma (3) to u $\Rightarrow$

$$\|\nabla u\|_{L^\infty(B_1)} \le C(\alpha,n)\left(\hat{\tau}\|\nabla^2 u\|_{L^\infty(B_1)} + \hat{\tau}^{-1}\|u\|_{L^\infty(B_1)}\right)$$

$\hat{\tau}$ is a constant to be determined.

$$\Rightarrow \|\nabla^2 u\|_{L^\infty(B_1)} \le C(\alpha,n)\left(\tau^\alpha [\nabla^2 u]_{C^\alpha(B_1)} + \tau^{-1}\hat{\tau}\|\nabla^2 u\|_{L^\infty(B_1)} + (\tau\hat{\tau})^{-1}\|u\|_{L^\infty(B_1)}\right)$$

choose $\hat{\tau} = \dfrac{\tau}{2C(\alpha,n)} \in (0,1) \Rightarrow \tau^{-1}\hat{\tau} \le \dfrac{1}{2C(\alpha,n)}$.

$\Rightarrow (*)$

Corollary. $\alpha \in (0,1)$. $\varepsilon \in (0,1)$.

$$\|u\|^*_{C^2(B_R)} \le \varepsilon R^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_R)} + C(\varepsilon,\alpha,n)\|u\|_{L^\infty(B_R)} \quad (**)$$

Proof. Previous corollary $\Rightarrow$

$$\sum_{i=0}^{2}(\tau R)^i\|\nabla^i u\|_{L^\infty(B_1)} \le C(\alpha,n)\left((\tau R)^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_R)} + \|u\|_{L^\infty(B_R)}\right)$$

$\tau \in (0, C^{-1}(\alpha,n))$ $\Rightarrow$

$$\|u\|^*_{C^2(B_R)} = \sum_{i=0}^{2} R^i\|\nabla^i u\|_{L^\infty(B_R)} \le \tau^{-2}\sum_{i=0}^{2}(\tau R)^i\|\nabla^i u\|_{L^\infty(B_R)}$$

$$\le \tau^{-2}C(\alpha,n)\left((\tau R)^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_R)} + \|u\|_{L^\infty(B_R)}\right)$$

Choose τ s.t. $\tau^\alpha = \dfrac{\varepsilon}{C(\alpha,n)}$ $\Rightarrow (**)$. #

2. Extremely special case.

Proposition. $A=(a_{ij})$ constant symmetry matrix.

$\lambda I_n \le A \le \Lambda I_n$ $(\lambda,\Lambda>0)$. If $u \in L^\infty(B_R)\cap C^2(B_R)$.

Satisfies $\sum_{i,j} a_{ij}\partial_{ij}u = f$ in B_R

$\Updownarrow$

$tr(A \cdot \nabla^2 u) = f$ in B_R

for $f \in C^{\alpha}(\overline{B}_R)$. $\alpha \in (0,1)$. then $u \in C^{2,\alpha}(\overline{B}_{R/2})$

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \leq C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_R)} + R^2 \|f\|^*_{C^{\alpha}(B_R)}\right).$$

Proof. By scaling method, assume $R=1$.

$\exists$ Orthogonal matrix P s.t.

$PAP^T = diag(\lambda_1, \cdots, \lambda_n)$ $\quad (\lambda \leq \lambda_i \leq \Lambda)$

Define $D = diag(\lambda_1^{-\frac{1}{2}}, \cdots, \lambda_n^{-\frac{1}{2}})$ and $Q = DP$.

$\Rightarrow QAQ^T = I_n \Rightarrow Q^{-1}Q^{-T} = A$. $\quad (Q^{-T} = (Q^{-1})^T)$

Consider the map $x \mapsto Qx$. denote it by $Q: B_1 \to Q(B_1)$

Define $u_Q(x) = u(Q^{-1}x)$, $f_Q(x) = f(Q^{-1}x)$ $\quad x \in Q(B_1) \to$ ellipsoid

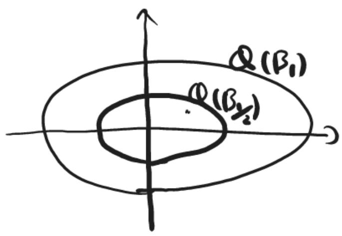
$$\Delta u_Q(x) = tr(\nabla^2 u_Q(x)) = tr(Q^{-T} \cdot \nabla^2 u(Q^{-1}x) \cdot Q^{-1})$$

$$= tr(Q^{-1}Q^{-T}\nabla^2 u(Q^{-1}x)) = tr(A\nabla^2 u(Q^{-1}x))$$

$$= \sum_{i,j} a_{ij}\partial_{ij}u(Q^{-1}x) = f(Q^{-1}x) = f_Q(x)$$

$$|Qx|^2 = x^T Q^T Q x = x^T A^{-1} x \Rightarrow \Lambda^{-\frac{1}{2}}|x| \leq |Qx| \leq \lambda^{-\frac{1}{2}}|x|.$$

For $Q(B_{1/2}) \subset Q(B_1)$



$d(Q(B_{1/2}), \partial Q(B_1)) \geq \Lambda^{-\frac{1}{2}} dist(B_{1/2}, \partial B_1) = \frac{\Lambda^{-\frac{1}{2}}}{2}$

Set $\gamma = \frac{\Lambda^{-\frac{1}{2}}}{4}$.

$\exists N(\lambda, \Lambda, n)$ & $y_1, \cdots, y_N \in Q(B_{1/2})$

s.t. $Q(B_{1/2}) \subset \bigcup_{i=1}^{N} B_{\gamma/4}(y_i)$

Note that $B_\gamma(y_i) \subset Q(B_1)$

Apply previous result (Poisson's equation) to $B_\gamma(y_i)$.

$\Rightarrow u_Q \in C^{2,\alpha}(\overline{B_{\gamma/2}(y_i)})$ &

$$\|u_Q\|^*_{C^{2,\alpha}(B_{\frac{\gamma}{2}}(y_i))} \le C(\alpha,n)\left(\|u_Q\|_{L^\infty(B_\gamma(y_i))} + \gamma^2 \|f\|^*_{C^\alpha(B_\gamma(y_i))}\right) \quad (*)$$

Claim: $[\nabla^2 u_Q]_{C^\alpha(Q(B_{1/2}))} \le C(\alpha)\left(\sum_{i=1}^N [\nabla^2 u_Q]_{C^\alpha(B_{\frac{\gamma}{2}}(y_i))} + \gamma^{-2-\alpha}\sum_{i=1}^N \|u_Q\|_{L^\infty(B_{\frac{\gamma}{2}}(y_i))}\right)$

$(*)$, claim, interpolation inequality $\Rightarrow u_Q \in C^{2,\alpha}(\overline{Q(B_{\frac{1}{2}})})$

& $\|u_Q\|_{C^{2,\alpha}(Q(B_{1/2}))} \le C(\alpha,n,N,\gamma)\left(\|u_Q\|_{L^\infty(Q(B_1))} + \|f_Q\|_{C^\alpha(Q(B_1))}\right)$

Definitions of $u_Q, f_Q \Rightarrow u \in C^{2,\alpha}(\overline{B}_{1/2})$

& $\|u\|_{C^{2,\alpha}(B_{1/2})} \le C(\lambda,\Lambda,\alpha,n)\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right)$

$\Rightarrow \quad \|u\|^*_{C^{2,\alpha}(B_{1/2})} \le C(\lambda,\Lambda,\alpha,n)\left(\|u\|_{L^\infty(B_1)} + 1^2\cdot\|f\|^*_{C^\alpha(B_1)}\right)$

Proof of claim.

$\forall x', x'' \in Q(B_{1/2}) \subset \bigcup_{i=1}^N B_{\frac{\gamma}{4}}(y_i)$

Assume $x' \in B_{\frac{\gamma}{4}}(y_k)$

Case 1. If $|x'-x''| \le \frac{\gamma}{100}$, then $x'' \in B_{\frac{\gamma}{2}}(y_k)$.

$$\Rightarrow \quad \frac{|\nabla^2 u_Q(x') - \nabla^2 u_Q(x'')|}{|x'-x''|^\alpha} \le [\nabla^2 u_Q]_{C^\alpha(B_{\frac{\gamma}{2}}(y_k))} \le \sum_{i=1}^N [\nabla^2 u_Q]_{C^\alpha(B_{\frac{\gamma}{2}}(y_i))}$$

Case 2. If $|x'-x''| > \frac{\gamma}{100}$, then $x'' \in B_{\frac{\gamma}{4}}(y_\ell)$

$$\Rightarrow \quad \frac{|\nabla^2 u_Q(x') - \nabla^2 u_Q(x'')|}{|x'-x''|^\alpha} \le \frac{\|\nabla^2 u_Q\|_{L^\infty(B_{\frac{\gamma}{4}}(y_k))} + \|\nabla^2 u_Q\|_{L^\infty(B_{\frac{\gamma}{4}}(y_\ell))}}{(\gamma/100)^\alpha}$$

$$\le C(\alpha)\gamma^{-\alpha}\sum_{i=1}^N \|\nabla^2 u_Q\|_{L^\infty(B_{\frac{\gamma}{2}}(y_i))}$$

Interpolation inequality $\le C(\alpha)\gamma^{-\alpha}\sum_{i=1}^N\left(\left(\frac{\gamma}{2}\right)^\alpha [\nabla^2 u_Q]_{C^\alpha(B_{\frac{\gamma}{2}}(y_i))} + \left(\frac{\gamma}{2}\right)^{-2}\|u_Q\|_{L^\infty(B_{\frac{\gamma}{2}}(y_i))}\right)$

$\Rightarrow$ Claim. #

3. Special Case (Small oscillation)

Proposition. $L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$. $a_{ij}, b_i, c \in C^\alpha(\bar{B}_R)$ $\alpha \in (0,1)$

$\lambda I_n \le (a_{ij}) \le \Lambda I_n$ $(\lambda, \Lambda > 0)$. $\|a_{ij}\|^*_{C^\alpha(B_R)}, R\|b_i\|^*_{C^\alpha(B_R)}, R^2\|c\|^*_{C^\alpha(B_R)} \le \Lambda$.

Suppose that $u \in L^\infty(B_R) \cap C^2(B_R)$ satisfies $Lu = f$ in B_R

for some $f \in C^\alpha(\bar{B}_R)$. Then $\exists\, \varepsilon(\lambda, \Lambda, \alpha, n)$. s.t.

if $\sup_{x', x'' \in B_R} |a_{ij}(x') - a_{ij}(x'')| \le \varepsilon$.

then $u \in C^{2,\alpha}(\bar{B}_{R/2})$ and

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_R)} + R^2\|f\|^*_{C^\alpha(B_R)}\right)$$