Recall: • Poisson's equation $\Delta u = f$

• Hölder space

• Newtonian potential

$$w = w_{\Omega, f} = \int_\Omega \Gamma(x-y) f(y)\, dy$$

when $\Omega = B_1$. $f \in C^\alpha(\bar{B}_1)$

(1) $w \in C^1(\bar{B}_1)$. $\partial_i w(x) = \int_{B_1} \partial_{x_i} \Gamma(x-y) f(y)\, dy$

$\& \ \|w\|_{C^1(B_1)} \le C(n) \|f\|_{L^\infty(B_1)}$

(2) $w \in C^2(\bar{B}_{1/2})$. $\partial_{ij} w(x) = \int_{B_1} \partial_{x_i} \partial_{x_j} \Gamma(x-y)\,(f(y) - f(x))\, dy$

$$- f(x) \int_{\partial B_1} \partial_{x_j} \Gamma(x-y)\, n_i \, dS_y$$

$\& \ \|w\|_{C^2(B_{1/2})} \le C(\alpha, n) \|f\|_{C^\alpha(B_1)}$

How to understand (2)?

Formally. $\partial_{ij} w(x) \text{ “}=\text{” } \int_{B_1} \partial_{x_i x_j} \Gamma(x-y) f(y)\, dy$

$\nabla^2 \Gamma \simeq |x-y|^{-n}$ $\quad x \in B_{1/2} \Rightarrow \int_{B_1} |\nabla^2 \Gamma(x-y)|\, dy \ge \underbrace{\frac{1}{C(n)} \int_{B_{\frac{1}{2}}(x)} |x-y|^{-n}\, dy}_{\text{“}\ \text{''}\ T}$

$$T = \frac{1}{C(n)} \int_0^{\frac{1}{2}} n \omega_n r^{n-1} \cdot r^{-n}\, dr = +\infty$$

$$\partial_{ij} W(x) \text{ “}=\text{” } \int_{B_1} \partial_{x_i x_j} \Gamma(x-y)\,(f(y) - f(x))\, dy$$

$$+ \underbrace{f(x) \int_{B_1} \partial_{x_i} \partial_{x_j} \Gamma(x-y)\, dy}$$

$$= - f(x) \int_{B_1} \partial_{y_i} \partial_{x_j} \Gamma(x-y)\, dy$$

$$= - f(x) \int_{\partial B_1} \partial_{x_j} \Gamma(x-y)\, n_i \, dy.$$

Lecture 6. Poisson's equation (II).

1. $C^{2,\alpha}$ regularity and estimate of w.

Lemma. If $\Omega = B_1$ and $f \in C^{\alpha}(\bar{B}_1)$ for some $\alpha \in (0,1)$. then

$$(*) \quad w \in C^{2,\alpha}(\bar{B}_{1/2}) \quad \& \quad \|w\|_{C^{2,\alpha}(B_{1/2})} \leq C(\alpha,n)\|f\|_{C^{\alpha}(B_1)}$$

Proof. $\forall\, x', x'' \in B_{1/2}$. set $r = |x'-x''|$.

Goal: $|\partial_{ij}w(x') - \partial_{ij}w(x'')| \leq C(\alpha,n)\|f\|_{C^{\alpha}(B_1)} \cdot r^{\alpha}$.

Goal + Lecture 5 $\Rightarrow$ $(*)$.

Case 1. $r \geq \frac{1}{4}$

$$|\partial_{ij}w(x') - \partial_{ij}w(x'')| \leq 2\|w\|_{C^2(B_{1/2})} \cdot 4^{\alpha} \cdot \frac{1}{4^{\alpha}} \leq C(\alpha)\|w\|_{C^2(B_{1/2})} r^{\alpha}$$

$$\leq C(\alpha,n)\|f\|_{C^{\alpha}(B_1)} \cdot r^{\alpha}$$

Case 2. $r < \frac{1}{4}$.

Set $x^* = \frac{1}{2}(x'+x'')$. (midpoint)

$$\partial_{ij}w(x') = \int_{B_1} \partial_{x_ix_j}\Gamma(x'-y)(f(y)-f(x'))dy - f(x')\int_{\partial B_1} \partial_{x_i}\Gamma(x'-y)n_j\,dS_y$$

$$\partial_{ij}w(x'') = \int_{B_1} \partial_{x_ix_j}\Gamma(x''-y)(f(y)-f(x''))dy - f(x'')\int_{\partial B_1} \partial_{x_i}\Gamma(x''-y)n_j\,dS_y$$

$$\partial_{ij}w(x') - \partial_{ij}w(x'') = \sum_{k=1}^{6} I_k.$$

New goal: $\sum_{k=1}^{6}|I_k| \leq C(\alpha,n)\|f\|_{C^{\alpha}(B_1)} \cdot r^{\alpha}$

New goal $\Rightarrow$ Goal. $\Rightarrow$ $(*)$.

$$I_1 = f(x'')\int_{\partial B_1}(\partial_{x_i}\Gamma(x''-y) - \partial_{x_i}\Gamma(x'-y))n_j\,dS_y$$

$$I_2 = (f(x'') - f(x'))\int_{\partial B_1}\partial_{x_i}\Gamma(x'-y)n_j\,dS_y$$

$$I_3 = \int_{B_r(x^*)}\partial_{x_ix_j}\Gamma(x'-y)(f(y)-f(x'))dy$$

$$I_4 = \int_{B_r(x^*)}\partial_{x_ix_j}\Gamma(x''-y)(f(x'')-f(y))dy$$

$$I_5 = (f(x'')-f(x'))\cdot\int_{B_1\setminus B_r(x^*)}\partial_{x_ix_j}\Gamma(x'-y)dy$$

$$I_6 = \int_{B_1\setminus B_r(x^*)}(\partial_{x_ix_j}\Gamma(x'-y) - \partial_{x_ix_j}\Gamma(x''-y))(f(y)-f(x''))dy$$

I_1 : Mean value theorem $\Rightarrow \exists\, \bar{x} \in \overline{x'x''}$ (line segment) s.t.

$$|\partial_{x_i}\Gamma(x''-y) - \partial_{x_i}\Gamma(x'-y)| \le |\nabla\partial_{x_i}\Gamma(\bar{x}-y)| \cdot |x'-x''|$$

$$\Rightarrow |I_1| \le \|f\|_{L^\infty(B_1)} \cdot |x'-x''| \int_{\partial B_1} |\nabla^2\Gamma(\bar{x}-y)|\, dS_y$$

$$\le \|f\|_{L^\infty(B_1)} \cdot r^{1-\alpha} \cdot r^\alpha \cdot \int_{\partial B_1} C(n)\, |\bar{x}-y|^{-n}\, dS_y$$

$$\left[\, x', x'' \in B_{1/2} \Rightarrow \bar{x} \in B_{1/2}, \quad r < \tfrac{1}{4} \,\right]$$

$$\le C(\alpha, n)\, \|f\|_{L^\infty(B_1)} \cdot r^\alpha$$

I_2 : $$|I_2| \le \|f\|_{C^\alpha(B_1)} \cdot |x'-x''|^\alpha \int_{\partial B_1} |\partial_i\Gamma(x'-y)|\, dS_y$$

$$\le \|f\|_{C^\alpha(B_1)} \cdot r^\alpha \cdot C(n)$$

I_3 : $B_r(x^*) \subset B_{\frac{3r}{2}}(x') \subset B_1 \quad (r < \tfrac{1}{4})$

$$|I_3| \le \int_{B_{\frac{3r}{2}}(x')} |\partial_{x_ix_j}\Gamma(x'-y)| \cdot |f(y) - f(x')|\, dy$$

$$\le \int_{B_{\frac{3r}{2}}(x')} C(n)\, |x'-y|^{-n} \cdot \|f\|_{C^\alpha(B_1)} \cdot |y-x'|^\alpha\, dy$$

$$= C(n)\, \|f\|_{C^\alpha(B_1)} \int_{B_{\frac{3r}{2}}(x')} |y-x'|^{-n+\alpha}\, dy$$

$$= C(\alpha, n)\, \|f\|_{C^\alpha(B_1)} \cdot r^\alpha$$

I_4 : Similar argument $\Rightarrow |I_4| \le C(\alpha,n)\, \|f\|_{C^\alpha(B_1)}\, r^\alpha$

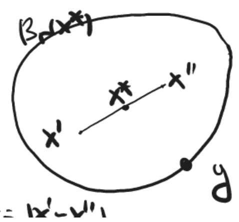
I_5 : $$\left| \int_{B_1 \setminus B_r(x^*)} \partial_{x_ix_j}\Gamma(x'-y)\, dy \right|$$

$$= \left| -\int_{B_1 \setminus B_r(x^*)} \partial_{y_i}\partial_{x_j}\Gamma(x'-y)\, dy \right|$$

$$= \left| -\int_{\partial(B_1 \setminus B_r(x^*))} \partial_{x_j}\Gamma(x'-y)\, n_i\, dS_y \right|$$

$$\le \int_{\partial B_r(x^*)} |\partial_{x_j}\Gamma(x'-y)|\, dS_y + \int_{\partial B_1} |\partial_{x_j}\Gamma(x'-y)|\, dS_y$$

$$\le \int_{\partial B_r(x^*)} C(n)\, |x'-y|^{1-n}\, dS_y + \int_{\partial B_1} C(n)\, |x'-y|^{1-n}\, dS_y$$



$r = |x'-x''|$

$\frac{r}{2} \le |x'-y| \le \frac{3}{2} r \qquad y \in \partial B_r(x^*)$

$x' \in B_{\frac{1}{2}} \Rightarrow \underline{|x'-y| \ge \tfrac{1}{2}} \qquad y \in \partial B_1$

$$\leq \int_{\partial B_r(x^*)} C(n)\, r^{1-n}\, dS_y + \int_{\partial B_1} C(n)\, dS_y$$

$$= C(n).$$

$$\Rightarrow |I_5| \leq |f(x') - f(x'')| \cdot C(n) \leq C(n) \|f\|_{C^\alpha(B_1)} r^\alpha.$$

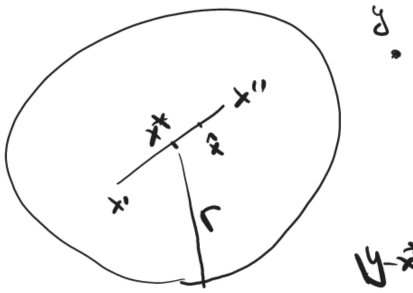
I_6: $\exists\ \hat{x} \in \overline{x'x''}$ (mean value theorem)

$$|\partial_{x_i x_j}\Gamma(x'-y) - \partial_{x_i x_j}\Gamma(x''-y)|$$

$$\leq |\nabla \partial_{x_i x_j}\Gamma(\hat{x}-y)| \cdot |x'-x''| \leq C(n) \cdot \frac{r}{|y-\hat{x}|^{n+1}}$$

$$\Rightarrow |I_6| \leq C(n) \cdot r \int_{B_1 \setminus B_r(x^*)} \frac{|f(y) - f(x'')|}{|y-\hat{x}|^{n+1}}\, dy$$

$$\leq C(n) \|f\|_{C^\alpha(B_1)} \cdot r \int_{B_1 \setminus B_r(x^*)} \frac{|y-x''|^\alpha}{|y-\hat{x}|^{n+1}}\, dy.$$



$$|y-x''| \leq |y-x^*| + |x^*-x''|$$

$$= |y-x^*| + \frac{r}{2}$$

$$|y-x^*| > r \longrightarrow \leq \frac{3}{2}|y-x^*|$$

$$|y-\hat{x}| \geq |y-x^*| - |x^*-\hat{x}|$$

$$\geq |y-x^*| - r$$

$$\geq \frac{1}{2}|y-x^*|.$$

$$\Rightarrow |I_6| \leq C(n) \|f\|_{C^\alpha(B_1)} \cdot r \int_{B_1 \setminus B_r(x^*)} C(\alpha,n) \cdot |y-x^*|^{\alpha-n-1}\, dy$$

$$\leq C(\alpha,n) \|f\|_{C^\alpha(B_1)} \cdot r \int_{B_2(x^*) \setminus B_r(x^*)} |y-x^*|^{\alpha-n-1}\, dy$$

$$= C(\alpha,n) \|f\|_{C^\alpha(B_1)} \cdot r \cdot C(\alpha,n)\, r^{\alpha-1}$$

$$= C(\alpha,n) \|f\|_{C^\alpha(B_1)} \cdot r^\alpha$$

#

Theorem. $\alpha\in(0,1)$. $f\in C^{\alpha}(\bar{B}_1)$. If $u\in L^\infty(B_1)\cap C(B_1)$ satisfies $\Delta u=f$ in B_1. then $u\in C^{2,\alpha}(\bar{B}_{1/2})$ and

$$\|u\|_{C^{2,\alpha}(B_{1/2})}\le C(\alpha,n)\left(\|u\|_{L^\infty(B_1)}+\|f\|_{C^\alpha(B_1)}\right)$$

Proof. W: Newtonian potential of f in B_1.

$$\begin{cases}\Delta W=f \quad \text{in } B_1 \qquad W\in C^1(\bar{B}_1)\cap C^{2,\alpha}(\bar{B}_{1/2})\cap C^2(B_1)\\ \|W\|_{C^1(B_1)}\le C(n)\|f\|_{L^\infty(B_1)}\\ \|W\|_{C^{2,\alpha}(B_{1/2})}\le C(\alpha,n)\|f\|_{C^\alpha(B_1)}\end{cases}$$

Define $V=u-W\Rightarrow \Delta V=\Delta u-\Delta W=f-f=0$ in B_1.

V is harmonic $\Rightarrow V\in C^\infty(B_1)$ (cf. Lecture 3)

& $\|V\|_{C^{2,\alpha}(B_{1/2})}\underset{\uparrow}{\le} C(n)\|V\|_{C^3(B_{1/2})}\le C(\alpha,n)\|V\|_{L^\infty(B_1)}$

mean-value theorem (cf. Lecture 2)

$u=W+V$ $\quad W\in C^{2,\alpha}(\bar{B}_{1/2})$. $V\in C^\infty(B_1)\Rightarrow u\in C^{2,\alpha}(\bar{B}_{1/2})$

$$\begin{aligned}\|u\|_{C^{2,\alpha}(B_{1/2})}&\le\|W\|_{C^{2,\alpha}(B_{1/2})}+\|V\|_{C^{2,\alpha}(B_{1/2})}\\&\le C(\alpha,n)\|f\|_{C^\alpha(B_1)}+C(\alpha,n)\|V\|_{L^\infty(B_1)}\\(V=u-W)\quad&\le C(\alpha,n)\|f\|_{C^\alpha(B_1)}+C(\alpha,n)\left(\|u\|_{L^\infty(B_1)}+\|W\|_{L^\infty(B_1)}\right)\\&\le C(\alpha,n)\|f\|_{C^\alpha(B_1)}+C(\alpha,n)\left(\|u\|_{L^\infty(B_1)}+C(n)\|f\|_{L^\infty(B_1)}\right)\\&\le C(\alpha,n)\left(\|f\|_{C^\alpha(B_1)}+\|u\|_{L^\infty(B_1)}\right)\end{aligned}$$

#.

Corollary. $\alpha\in(0,1)$. $f\in C^\alpha(\overline{B_R(x_0)})$

If $u\in L^\infty(B_R(x_0))\cap C^2(B_R(x_0))$ satisfies

$\Delta u=f$ in $B_R(x_0)$. then $u\in C^{2,\alpha}(\overline{B_{R/2}(x_0)})$ and

$$R\|\nabla u\|_{L^\infty(B_{R/2}(x_0))}+R^2\|\nabla^2u\|_{L^\infty(B_{R/2}(x_0))}+R^{2+\alpha}[\nabla^2u]_{C^\alpha(B_{R/2}(x_0))}$$

$$\le C(\alpha,n)\left(\|u\|_{L^\infty(B_R(x_0))}+R^2\|f\|_{L^\infty(B_R(x_0))}+R^{2+\alpha}[f]_{C^\alpha(B_R(x_0))}\right)$$

Proof. After translation assume $x_0=0$. then $B_R(x_0)=B_R$

Scaling method: Define $u_R(x)=u(Rx)$. $f_R(x)=f(Rx)$ $x\in B_1$

$\Rightarrow \quad \Delta u_R = R^2 f_R$ in B_1

Previous theorem $\Rightarrow$ $\|u_R\|_{C^{2,\alpha}(B_{1/2})} \le C(\alpha,n)\left(\|u_R\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right)$ (*)

For (*), We compute

$LHS = \|u\|_{L^\infty(B_{R/2})} + R\|\nabla u\|_{L^\infty(B_{R/2})} + R^2\|\nabla^2 u\|_{L^\infty(B_{R/2})} + R^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{R/2})}$

$RHS = C(\alpha,n)\left(\|u\|_{L^\infty(B_R)} + R^2\|f\|_{L^\infty(B_R)} + R^{2+\alpha}[f]_{C^\alpha(B_R)}\right)$.

#

Corollary. Ω bounded domain in $\mathbb{R}^n$. $f\in C^\alpha(\bar\Omega)$. $\alpha\in(0,1)$

If $u\in L^\infty(\Omega)\cap C^2(\Omega)$. Satisfies $\Delta u=f$ in Ω

Then $u\in C^{2,\alpha}(\Omega)$.

Proof. For $\Omega' \subset\subset \Omega$ set $d=\operatorname{dist}(\Omega',\partial\Omega)>0$.

Goal: $\displaystyle\sup_{\substack{x',x''\in\Omega' \\ x'\neq x''}} \frac{|\partial_{ij}u(x')-\partial_{ij}u(x'')|}{|x'-x''|^\alpha} < +\infty$

Case 1. $|x'-x''|\ge \frac{d}{2}$.

$$\frac{|\partial_{ij}u(x')-\partial_{ij}u(x'')|}{|x'-x''|^\alpha} \le \frac{2\|\nabla^2 u\|_{L^\infty(\Omega')}}{\left(\frac{d}{2}\right)^\alpha}$$

Case 2. $|x'-x''|<\frac{d}{2}$

$x''\in B_{\frac{d}{2}}(x') \subset B_d(x') \subset\subset \Omega$

Previous corollary $\Rightarrow$

$$\frac{|\partial_{ij}u(x')-\partial_{ij}u(x'')|}{|x'-x''|^\alpha} \le [\nabla^2 u]_{C^\alpha(B_{d/2}(x'))}$$

$$\le \frac{C(\alpha,n)}{d^{2+\alpha}}\left(\|u\|_{L^\infty(B_d(x'))} + d^2\|f\|_{L^\infty(B_d(x'))} + d^{2+\alpha}[f]_{C^\alpha(B_d(x'))}\right)$$

$$\le \frac{C(\alpha,n)}{d^{2+\alpha}}\left(\|u\|_{L^\infty(\Omega)} + d^2\|f\|_{L^\infty(\Omega)} + d^{2+\alpha}[f]_{C^\alpha(\Omega)}\right).$$

#

2. Dirichlet Problem

Theorem. $\alpha \in (0,1)$. $f \in C^{\alpha}(\overline{B_R(x_0)})$. $\varphi \in C(\partial B_R(x_0))$.

$\exists$ Unique solution $u \in C(\overline{B_R(x_0)}) \cap C^{2,\alpha}(B_R(x_0))$

of the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } B_R(x_0) \\ u = \varphi & \text{on } \partial B_R(x_0) \end{cases}$$

Proof. After translation. assume $x_0 = 0$

Special case: $R = 1$.

W: Newtonian potential of f in B_1.

$\Rightarrow \Delta W = f$ in B_1 & $W \in C^1(\overline{B_1}) \cap C^{2,\alpha}(B_1)$.

$\exists$ Solution $V \in C(\overline{B_1}) \cap C^2(B_1)$ (cf. Lecture 2)

of the Dirichlet problem $\begin{cases} \Delta V = 0 & \text{in } B_1 \\ V = \varphi - W & \text{on } \partial B_1 \end{cases}$

$\Rightarrow V \in C^{\infty}(B_1)$ (cf. Lecture 3) Set $u = V + W$. then

$$\begin{cases} \Delta u = \Delta V + \Delta W = f & \text{in } B_1 \\ u = \varphi - W + W = \varphi & \text{on } \partial B_1 \end{cases}$$

$u \in C(\overline{B_1}) \cap C^{2,\alpha}(B_1)$.

Uniqueness of solutions follows from maximum principle (cf. Lecture 3)

General case: Scaling method.

Define $u_R(x) = u(Rx)$. $f_R(x) = f(Rx)$ $\varphi_R(x) = \varphi(Rx)$. $x \in B_1$

$$\begin{cases} \Delta u = f & \text{in } B_R \\ u = \varphi & \text{on } \partial B_R \end{cases} \quad (*_1) \iff \begin{cases} \Delta u_R = R^2 f_R & \text{in } B_1 \\ u_R = \varphi_R & \text{on } \partial B_1 \end{cases} \quad (*_2)$$

Special case $\Rightarrow \exists\, u_R$ Solves $(*)_2$. Then $u(x) = u_R(\frac{x}{R})$

Solves $(*_1)$.

maximum principle $\Rightarrow$ uniqueness of solutions. #