Lecture 6

Dolbeault spaces

Recall:

the categorical correspondence given by the Riemann-Hilbert correspondence:

$$\mathcal{C}_{Loc}(X, n) \simeq \mathcal{C}_{Flat}(X, n)$$

Today: define a new category so that it is equivalent to a full subcategory of $\mathcal{C}_{Flat}(X, n)$

$$\begin{array}{ccc} & & \subsetneq \mathcal{C}_{Flat}(X,n) \\ \mathcal{C}_{dR}(X,n) & \simeq & \mathcal{C}_{Dol}(X,n) \end{array}$$

Setting: (X, ω) cpt Kähler mfd $\dim_{\mathbb{C}} X = m$

In particular. smooth proj. mfd.

§1. Higgs bundles and harmonic metric

Def 1.1 $\mathcal{E} := (E, \bar\partial_E)$ be a holo. vector bundle. a Higgs field is a morphism of holo. bundles (sheaf morphism)

$$\varphi : \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1$$

s.t.

$$\varphi \wedge \varphi = 0$$

in $End(\mathcal{E}) \otimes \Omega_X^2$.

$$\mathcal{E} \xrightarrow{\varphi} \mathcal{E} \otimes \Omega_X^1 \xrightarrow{\varphi \otimes id_{\Omega_X^1}} \mathcal{E} \otimes \Omega_X^1 \otimes \Omega_X^1 \xrightarrow{\wedge} \mathcal{E} \otimes \Omega_X^2$$

$(\mathcal{E}, \varphi)$ or $(E, \bar\partial_E, \varphi)$ is called a Higgs bundle.

Rmk. A Higgs field φ is a holomorphic section $\varphi \in H^0(X, End(\mathcal{E}) \otimes \Omega_X^1)$ s.t. $\varphi \wedge \varphi = 0$.

The above definition has a C^∞ interpretation:

E smooth complex vector bundle. define

$$D'' : A^0(X, E) \to A^1(X, E)$$

satisfies

(1) $\bar{\partial}$-Leibniz rule:

$$D''(fs) = \bar{\partial} f \otimes s + f D''(s) \quad \forall f \in C^\infty(X, \mathbb{C}),\ s \in A^0(X, E)$$

(2) integrability condition:

$$(D'')^2 = 0$$

under natural extension

$$D'': A^k(X, E) \to A^{k+1}(X, E)$$

Indeed, decompose D'' into different types:

$$D'' := \varphi + \bar{\partial}_E$$

then

$$(D'')^2 = 0 \iff \begin{cases} (\bar{\partial}_E)^2 = 0 \\ \bar{\partial}_E \varphi = 0 \quad \leftarrow \bar{\partial}_E \circ \varphi + \varphi \circ \bar{\partial}_E \\ \varphi \wedge \varphi = 0 \end{cases}$$

(1) & (2) $\iff$ $(E, \bar{\partial}_E, \varphi)$ is Higgs bundle.

Ex.: $n = 1$ i.e. X cpt. R.S. $g \geq 2$

(1) $K_X := T^*_{1,0} X$ holo. cotangent bundle $\quad K_X = \Omega^1_X = \Lambda^1(T^*_{1,0}X) = T^*_{1,0}X$

line bundle.

choose a square root $K_X^{\frac{1}{2}}$ of K_X $\quad$ (2^{2g} choices)

then the following is a Higgs bundle.

$$\left(E = K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}},\ \varphi = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix} \right)$$

$$1: K_X^{\frac{1}{2}} \to K_X^{-\frac{1}{2}} \otimes K_X$$

$$q_2: K_X^{-\frac{1}{2}} \to K_X^{\frac{1}{2}} \otimes K_X$$

i.e. $q_2 \in H^0(X, K_X^2)$ holo. quadratic differential

Hitchin: space of q_2, i.e. $H^0(X, K_X^2)$ parametrizes the Teichmüller space of the underlying oriented smooth surface S of X

(2) $\left\{ \mathcal{E} = K_X \oplus \mathcal{O}_X \oplus K_X^{-1},\ \varphi = \begin{pmatrix} 0 & 0 & q_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$

q_2 q_2

$$q_3 : K_X^{-1} \to K_X \otimes K_X \quad \text{i.e. } q_3 \in H^0(X, K_X^3)$$

holo. cubic differential

(3) In general. $\mathcal{E} = \mathrm{Sym}^{n-1}(K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}}) = K_X^{\frac{n-1}{2}} \oplus K_X^{\frac{n-3}{2}} \oplus \cdots \oplus K_X^{-\frac{n-1}{2}}$. rk n

$$\varphi = \begin{pmatrix} 0 & q_2 & q_3 & \cdots & q_n \\ 1 & 0 & q_2 & \cdots & q_{n-1} \\ 0 & 1 & 0 & \cdot & \ddots \\ \vdots & \cdot & \ddots & 1 & 0 \end{pmatrix}$$

$$q_i : K_X^{\frac{n-2i+1}{2}} \to K_X^{\frac{n-1}{2}} \otimes K_X \quad \text{i.e. } q_i \in H^0(X, K_X^i)$$

holo. i-th differential

Hitchin: if we regard $q := (q_2, \dots q_n) \in \bigoplus_{i=2}^{n} H^0(X, K_X^i)$, then the Higgs bundles $(\mathcal{E}, \varphi)$ considered above are located in the image of a section of Hitchin morphism, called a Hitchin section.

$$\Updownarrow$$

$$\rho : \pi_1(X, x) \to SL(n, \mathbb{R})$$

Now. $(E, \bar{\partial}_E, \varphi)$ a Higgs bundle. h a given hermition metric. recall

$$g_{\mathbb{C}} : A^k(X, \mathbb{C}) \times A^k(X, \mathbb{C}) \to \mathbb{C}$$

$$(\alpha, \beta) \mapsto g_{\mathbb{C}}(\alpha, \beta)\,\mathrm{Vol}_g := \alpha \wedge * \bar{\beta}$$

$\rightarrow$ induces an inner product on bundle-valued k-forms:

$$\langle\ \rangle_{g,h} : A^k(X, E) \times A^k(X, E) \to A^0(X, \mathbb{C})$$

$$(\alpha \otimes u,\ \beta \otimes v) \longmapsto g_\alpha(\alpha, \beta) \cdot h(u, v)$$

$$\underset{A^k(X,\mathbb{C})}{\cap} \quad \underset{A^0(X,E)}{\cap}$$

Chern connection:

$$\bar\partial_E \ \& \ h \ \rightsquigarrow \ \partial_h \qquad \partial\text{-type operator}$$

∂-Leibniz rule.

$$\varphi \ \& \ h \ \rightsquigarrow \ \varphi^{*h} \quad \text{via}$$

$$h(\varphi(u), v) = h(u, \varphi^{*h}(v))$$

adjoint operator of φ

locally. Choose $\{z_1, \dots z_m\}$. $\varphi = \sum_{\alpha=1}^{m} \varphi_\alpha dz_\alpha$

$$\Rightarrow \varphi^{*h} = \sum_{\alpha=1}^{m} \varphi_\alpha^{*h} d\bar z_\alpha$$

where $\varphi_\alpha^{*h} = \bar H^{-1} \cdot \bar\varphi_\alpha^{T} \cdot \bar H$ for H the Hermitian matrix corresponds to h

Introduce

$$\nabla_h := \underset{\substack{\| \\ (\partial_h + \bar\partial_E)}}{D_h} + \varphi + \varphi^{*h}$$

It's easy to check ∇_h is C^∞ connection on E.

$$\nabla_h : A^0(X,E) \to A^1(X,E)$$

$$\text{s.t.} \quad \nabla_h(fs) = df \otimes s + f\nabla_h(s)$$

Def 2.2 h on $(E, \bar\partial_E, \varphi)$ is a pluri-harmonic metric if the induced C^∞ connection ∇_h is flat, i.e. $\nabla_h^2 = 0$

$\rightsquigarrow (E, \bar\partial_E, \varphi, h)$ harmonic Higgs bundle.

Rmk: By definition

$$\nabla_h^2 = 0 \iff (D_h + \varphi + \varphi^{*h})^2 = 0$$

$$\Longleftrightarrow \begin{cases} D_h^2 + [\varphi, \varphi^{*h}] = 0 \\ \partial_h \varphi = 0 \left(= \bar\partial_E (\varphi^{*h}) \right) \end{cases} \qquad (*)$$

since $(\partial_h \varphi)^{*h} = \bar\partial_E(\varphi^{*h})$

$(*)$ is called the Hitchin's self-duality equation.

$$\underset{\substack{\shortparallel \\ \partial_h + \bar\partial_E}}{(D_h + \varphi + \varphi^*)^2} = D_h^2 + \varphi\wedge\varphi + \varphi^{*h}\wedge\varphi^{*h} + [\varphi, \varphi^{*h}]$$
$$+ \partial_h(\varphi) + \partial_h(\varphi^{*h}) + \bar\partial_E(\varphi) + \bar\partial_E(\varphi^{*h})$$

$$\# \quad D_h^2 = (\partial_h + \bar\partial_E)^2 = \partial_h^2 + \bar\partial_E^2 + [\partial_h, \bar\partial_E]$$

In particular, if $m=1$, then

$$(*) \Longleftrightarrow D_h^2 + [\varphi, \varphi^{*h}] = 0$$

Def 2.3 h is harmonic if $\Lambda_\omega F_h = 0$ for $F_h := \nabla_h^2$

h is Hermite-Einstein if $\Lambda_\omega F_h = \lambda \cdot \mathrm{Id}_E$

Rmk: Easy to check

$$\lambda = -2\pi\sqrt{-1}\,\frac{\deg E}{\mathrm{rk}(E)\cdot \mathrm{vol}_g(X)}$$

Rmk: By definition:

pluri-harmonic $\Rightarrow$ harmonic $\Rightarrow$ Hermite-Einstein

Hermite-Einstein & $c_1(E)=0 \Rightarrow$ harmonic.

Q: When is a harmonic metric pluri-harmonic?

Recall total chern class of E can be defined via any connection ∇

$$\det\left(\mathrm{Id} + \tfrac{\sqrt{-1}}{2\pi} F_\nabla\right) = 1 + c_1(E) + c_2(E) + \cdots$$

$c_i(E) \in H^{2i}(X, \mathbb{Z})$

Prop 2.4 A harmonic metric is pluri-harmonic if and only if $ch_2(E) = 0$.

pf. "$\Leftarrow$" ✓

"$\Rightarrow$" express $ch_2(E)$ in terms F_h as

$$-\int_X ch_2(E) \wedge \frac{\omega^{m-2}}{(m-2)!} = \int_X \left(c_2(E) - \frac{1}{2} c_1^2(E)\right) \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$= -\int_X \mathrm{Tr}\left(\left(\frac{\sqrt{-1}}{2\pi} F_h\right)^2\right) \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$= \frac{1}{8\pi^2} \int_X \mathrm{Tr}(F_h \wedge F_h) \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$F_h = \mathcal{D}_h^2 = (D_h + \varphi + \varphi^{*h})^2$$

$$= (D_h^2 + [\varphi, \varphi^{*h}]) + (\partial_h \varphi + \bar{\partial}_E(\varphi^{*h}))$$

$$=: F_1 + F_2$$

$\Rightarrow$ $(F_1)^{*h} = -F_1$ since $(D_h^2)^{*h} = -D_h^2$

$(F_2)^{*h} = F_2$ $\quad ([\varphi, \varphi^{*h}])^{*h} = -[\varphi, \varphi^{*h}]$

$(\partial_h(\varphi))^{*h} = \bar{\partial}_E(\varphi^{*h})$

$F_1 =: F_{01} + F_{11}$

for F_{01} primitive part of F_1, i.e. $F_{01} = F_1 - \frac{\omega}{m} \Lambda_\omega F_1$

$(\Lambda_\omega F_{01} = 0)$

$$= \left(F_1 - \frac{\omega}{m} \Lambda_\omega F_1\right) + \frac{\omega}{m} \Lambda_\omega F_1$$

orthogonal decomposition

Apply $*$:

$$*F_{01} = -F_{01} \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$*F_{11} = F_{11} \wedge \frac{\omega^{m-2}}{(m-1)!}$$

$$*F_2 = F_2 \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$\Rightarrow$

$$*F_h^{*h} = *(-F_{01} - F_{11} + F_2)$$

$$= F_h \wedge \frac{\omega^{m-2}}{(m-2)!} - \frac{m}{m-1} F_{11} \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$= F_h \wedge \frac{\omega^{m-2}}{(m-2)!} - m * F_{11}^{*h}$$

$$\Rightarrow \int_X \mathrm{Tr}(F_h \wedge F_h) \wedge \frac{\omega^{m-2}}{(m-2)!} = \|F_h\|_{L^2}^2 - m\|F_{11}\|_{L^2}^2$$

$$= \|F_h\|_{L^2}^2 - \|\Lambda_\omega F_h\|_{L^2}^2$$

$\Rightarrow$ $\Lambda_\omega F_h = 0$ & $ch_2(E) = 0$ $\Longleftrightarrow$ $F_h = 0$

i.e. h harmonic & $ch_2(E) = 0$ $\Longleftrightarrow$ h pluri-harmonic. $\boxed{\text{Q.E.D.}}$

Cor 2.5 $m = 1$

pluri-harmonic = harmonic

Lem 2.6 (Kähler identities for Higgs bundles)

$(E, \bar\partial_E, \varphi)$ Higgs bundle, h hermitian metric

let $D_h' := \partial_h + \varphi^{*h}$

$D_h'' := \bar\partial_E + \varphi$

$$\Rightarrow (D_h')^* = \sqrt{-1}[\Lambda_\omega, D_h'']$$

$$(D_h'')^* = -\sqrt{-1}[\Lambda_\omega, D_h']$$

§2. Flat bundles and pluri-harmonic metrics

(E, ∇) C^∞ flat bundle, h hermitian metric.

$\Rightarrow$ ∇ decomposes uniquely as

$$\nabla = D_h + \Psi_h$$

for
- D_h unitary connection w.r.t. h
- $\Psi_h \in A^1(X, \mathrm{End}(E))$ is self-adjoint, i.e. $\Psi_h^{*h} = \Psi_h$

Indeed. Φ_h is defined as

$$h(\Phi_h(u),v) = \frac{1}{2}\Big(h(\nabla(u),v) + h(u,\nabla(v)) - dh(u,v)\Big)$$

$$\forall\ u,v \in A^0(X,E)$$

$\Rightarrow$

$$h(\Phi_h(v),u) = \frac{1}{2}\Big(h(\nabla(v),u) + h(v,\nabla(u)) - dh(v,u)\Big)$$

$\Rightarrow$ $$h(u,\Phi_h(v)) = \frac{1}{2}\Big(h(u,\nabla(v)) + h(\nabla(u),v) - dh(u,v)\Big)$$

$\Rightarrow$ $\Phi_h^{*h} = \Phi_h$

Similary, one checks

$$h(D_h(u),v) + h(u,D_h(v)) = dh(u,v)$$

Decomposes D_h & Φ_h furthermore into different types

$$D_h := \partial_h + \bar\partial_h \qquad \Phi_h := \varphi_h + \varphi_h^{*h}$$

introduce

$$D_h' = \partial_h + \varphi_*^{*h} \qquad D_h'' = \bar\partial_h + \varphi_h$$

$\Rightarrow$ one checks

$$D_h'' : A^0(X,E) \to A^1(X,E) \quad \text{is a } \bar\partial\text{-operator.}$$

$$D_h''(fs) = \bar\partial f \otimes s + f D_h''(s).$$

define $G_h := (D_h'')^2$. then G_h is $C^\infty(X,\mathbb{C})$-linear operator

i.e. $\mathrm{End}(E)$-valued 2-form

called the pseudo-curvature.

Def 2.1 A hermitian metric h on a flat bundle (E,∇) is called a

harmonic metric if

$$\Lambda_\omega G_h = 0$$

pluri-harmonic metric if

$$G_h = 0$$

i.e. $(D_h'')^2 = 0 \iff (\bar\partial_h + \varphi_h)^2 = 0$

$\iff (E, \bar\partial_h, \varphi_h)$ defines a Higgs bundle.

When h pluri-harmonic, we call (E, ∇, h) a harmonic flat bundle.

Lem 2.2 (Kähler identities for flat bundles)

(E, ∇) flat bundle. h hermitian metric

define $D_h^c := D_h'' - D_h'$

$$= (\bar\partial_h + \varphi_h) - (\partial_h + \varphi_*^{*h})$$

$\Rightarrow$

$$(D_h^c)^* = -\sqrt{-1}[\Lambda_\omega, \nabla]$$

$$(\nabla)^* = \sqrt{-1}\,[\Lambda_\omega, D_h^c].$$

Q: When is a harmonic metric on (E, ∇) pluri-harmonic ?

Prop 2.3 When the base (X, ω) cpt Kähler mfld.

then h is pluri-harmonic $\iff$ h is harmonic

i.e. $G_h = 0 \iff \Lambda_\omega G_h = 0$

pf: Claim: $\overset{=(\partial_h + \varphi_*^{*h})}{(D_h')^2} = -(D_h'')^2$

Indeed, decompose ∇ into different types

$$\nabla = d' + d''$$

$h \rightsquigarrow$ d' determines a δ_h'' s.t. $d' + \delta_h''$ unitary

d'' determines a δ_h' s.t. $d'' + \delta_h'$ unitary

$\Rightarrow$ $\partial_h = \frac{1}{2}(d' + \delta_h')$ $\quad$ $\bar\partial_h = \frac{1}{2}(d'' + \delta_h'')$

$\varphi_h = \frac{1}{2}(d' - \delta_h')$ $\quad$ $\varphi_*^{*h} = \frac{1}{2}(d'' - \delta_h'')$

then $(D_h')^2 = -(D_h'')^2$ follows.

$$\Rightarrow \quad 0 = \nabla^2 = (D_h' + D_h'')^2 = (D_h')^2 + (D_h'')^2 + D_h' \circ D_h'' + D_h'' \circ D_h'$$
$$= D_h' D_h'' + D_h'' D_h'$$

$$\Rightarrow \quad (D_h^c)^2 = (D_h'' - D_h')^2$$
$$= -(D_h'' D_h' + D_h' D_h'')$$
$$= 0$$

$$\left\{ \begin{array}{l} \nabla = D_h' + D_h'' \\ D_h^c = D_h'' - D_h' \end{array} \right.$$

$$\Rightarrow \quad G_h = (D_h'')^2 = \frac{1}{4}(\nabla + D_h^c)^2$$
$$= \frac{1}{4}(D_h^c \nabla + \nabla D_h^c)$$

$\Rightarrow$ Bianchi identities for G_h:

$$D_h^c G_h = [D_h^c, G_h] = 0$$
$$\nabla G_h = [\nabla, G_h] = 0$$

Apply Kähler identities for flat bundles (Lem 2.2)

$$\Rightarrow \quad (\nabla)^* G_h = \sqrt{-1}\,[\Lambda_\omega, D_h^c]\, G_h$$
$$= 0 \qquad (\because \ \Lambda_\omega G_h = 0)$$

On the other hand, one checks

$$D_h^c = d'' - d' + 2(\varphi_h - \varphi_h^{*h})$$

$$\Rightarrow$$

$$4G_h = D_h^c \nabla + \nabla D_h^c$$
$$= 2\nabla(\varphi_h - \varphi_h^{*h})$$

$$\nabla = d' + d''$$
$$\nabla^2 = 0 \Leftrightarrow \left\{ \begin{array}{l} (d')^2 = 0 = (d'')^2 \\ d'd'' + d''d' = 0 \end{array} \right.$$

i.e.

$$G_h = \frac{1}{2}\nabla(\varphi_h - \varphi_h^{*h})$$

Consequently,

$$\Rightarrow \quad \|G_h\|_{L^2}^2 = \int_X |G_h|_{g,h}^2 \frac{\omega^m}{m!}$$

$$= \int_X \langle G_h, G_h \rangle_{g,h} \frac{\omega^m}{m!}$$

$$= \frac{1}{2} \int_X \langle \nabla(\varphi_h \cdot \varphi_h^{*_h}), G_h \rangle_{g,h} \frac{\omega^m}{m!}$$

$$= \frac{1}{2} \int_X \langle \varphi_h \cdot \varphi_h^{*_h}, (\nabla)^* G_h \rangle_{g,h} \frac{\omega^m}{m!}$$

$$= 0$$

$$\Rightarrow G_h = 0$$

$\square$

<u>Rmk</u>: The Kählerian condition can be relaxed.

In conclusion, the flatness condition $\nabla^2 = 0$ is more rigid constraint than the Higgs condition $(\bar{\partial}_E + \varphi)^2 = 0$.

§3. Equivalence of categories

Fix $n \in \mathbb{Z}_{>0}$.

$\mathcal{C}_{Dol}(X, n)$: category of harmonic Higgs bundles of rk n

$\mathcal{C}_{dR}(X, n)$: category of harmonic flat bundles of rk n

<u>Thm 3.1</u>. There is one-to-one correspondence

$$\Xi_X : \mathcal{C}_{Dol}(X, n) \xrightarrow{\sim} \mathcal{C}_{dR}(X, n)$$

$$(E, \bar{\partial}_E, \varphi, h) \mapsto (E, \nabla_h := D_h + \varphi + \varphi^{*_h}, h)$$

Moreover, such equivalence of categories preserves direct sums, duals, and tensor products, and it is functorial, i.e. $f: Y \to X$ morphism of proj. mflds (or Kähler mflds), then f^* induces

$$\Xi_Y := f^*\Xi_X : \mathcal{C}_{Dol}(Y,n) \to \mathcal{C}_{dR}(Y,n)$$

that is.

$$\Xi_Y \circ f^*(E, \bar\partial_E, \varphi, h) = f^* \circ \Xi_X(E, \bar\partial_E, \varphi, h).$$

pf.

Ξ_X is well-defined.: $(E, \bar\partial_E, \varphi, h)$ harmonic Higgs

$\Rightarrow$ ∇_h flat & h is a harmonic metric for (E, ∇_h)

To show the equivalence of cats. need to show Ξ_X maps morphisms in $\mathcal{C}_{Dol}(X,n)$ to morphisms in $\mathcal{C}_{dR}(X,n)$

$$(E, \bar\partial_E, \varphi, h) \xmapsto{\Xi_X} (E, \nabla_h, h)$$

$H^0_{Dol}(E, D'')$ space of D''-flat sections of E i.e. $s \in A^0(X, E)$ with $D''(s) = 0$

$H^0_{dR}(E, \nabla_h)$. space of ∇_h flat sections of E.

lem 2.2 $D''(s) = 0 \iff \nabla_h(s) = 0$

i.e.

$$H^0_{Dol}(E, D'') \simeq H^0_{dR}(E, \nabla_h)$$

pf.

"$\Rightarrow$" $D''s = 0$. to show $\nabla_h(s) = 0$. we need to show $D'_h(s) = 0$

Indeed. use Kähler identities for Higgs bundles.

$\Rightarrow$

$$(D'_h)^* D'_h(s) = \sqrt{-1}\Lambda_\omega D'' D'_h(s) = -\sqrt{-1}\Lambda_\omega D'_h D''_h(s) = 0$$

$\Rightarrow$

$$\|D'_h(s)\|^2_{L^2} = \int_X \langle (D'_h)^*(D'_h(s)), s\rangle_{g,h} \frac{\omega^m}{m!}$$

$$= 0$$

"$\Leftarrow$" $\nabla_h(s) = 0$. $(\nabla_h = D'_h + D'')$

use Kähler identities for flat bundles.

$$\Rightarrow \|D_h^c(s)\|_{L^2}^2 = \int_X \langle (D_h^c)^* D_h^c(s), s\rangle_{g,h} \frac{\omega^m}{m!}$$

$$= \int_X \langle -\sqrt{-1}\,\Lambda_\omega \nabla_h D_h^c(s), s\rangle_{g,h} \frac{\omega^m}{m!}$$

$$= 0$$

the last equality is due to.

$$\begin{aligned}
\nabla_h \circ D_h^c &= (D'' + D_h') \circ (D'' - D_h') \\
&= (D'')^2 - (D_h')^2 - D'' \circ D_h' + D_h' \circ D'' \\
&= -D'' \circ D_h' + D_h' \circ D'' \\
&= -(D'' - D_h') \circ (D'' + D_h') \\
&= -D_h^c \circ \nabla_h
\end{aligned}$$

$$\Rightarrow D''(s) = \frac{1}{2}(\nabla_h + D_h^c)(s) = 0$$

$\blacksquare$

So Ξ_X maps morphisms in $\mathcal{C}_{ol}(X,n)$ to morphisms in $\mathcal{C}_{ onR}(X,n)$

as long as we show.

dual of harmonic is still harmonic

tensor product of harmonic is still harmonic.

<u>dual</u>:

$(E, \bar\partial_E, \varphi)$ Higgs. $D_E'' = \bar\partial_E + \varphi$

dual Higgs bundle. E^*.

- $\bar\partial$-type operator $D_{E^*}'' : A^0(X, E^*) \to A^1(X, E^*)$ defined as

$$D_{E^*}''(u)(v) := -u(D_E''(v)) + \bar\partial(u(v))$$

$$u \in A^0(X, E^*)$$

$$v \in A^0(X, E)$$

extends $\quad D''_{E^*} : A^k(X, E^*) \to A^{k+1}(X, E^*)$.

satisfies $\quad (D''_{E^*})^2 = 0$.

$\rightsquigarrow$ dual Higgs bundle.

holo. str. $\quad \bar\partial_{E^*}(u)(v) = -u(\bar\partial_E(v)) + \bar\partial(u(v))$

$$\varphi_{E^*}(u)(v) = -u(\varphi_E(v))$$

given h on $(E, \bar\partial_E, \varphi)$. $\rightsquigarrow$ h^* on $(E^*, \bar\partial_{E^*}, \varphi_{E^*})$ as

$$h^*(u_1, u_2) := u_1\left((u_2)_h^+\right)$$

for $(u_2)_h^+ \in A^0(X, E)$ determined by u_2 & h via

$$u_2(v) = h(v, (u_2)_h^+)$$

Lem 3.3. $(1,0)$-part of the Chern connection of $\bar\partial_{E^*}$ is related to ∂_h as

$$\partial_{h^*}(u)(v) = -u(\partial_h(v)) + \partial(u(v))$$

Lem 3.4 $(\varphi_{E^*})^{*h^*}$ satisfies

$$(\varphi_{E^*})^{*h^*}(u)(v) = -u\left((\varphi_E)^{*h}(v)\right)$$

Prop 3.5 Dual of harmonic Higgs bundle is still a harmonic Higgs bundle.

pf.

$$F_{h^*}(u)(v) = \nabla_{h^*} \circ \nabla_{h^*}(u)(v)$$

$$= \nabla_{h^*}(u)(\nabla_h(v)) + d(\nabla_{h^*}(u)(v))$$

$$= -u(\underbrace{\nabla_h \circ \nabla_h}(v)) + d[u(\nabla_h(v))] + d(-u(\nabla_h(v)) + d(u(v)))$$

$$= -u F_h(v)$$

$$= 0$$

$\blacksquare$

define inverse functor

$$\Xi'_X : \mathcal{C}_{dR}(X, n) \to \mathcal{C}_{Dol}(X, n)$$
$$(E, \nabla, h) \mapsto (E, \bar{\partial}_h, \varphi_h, h)$$

satisfies

$$\Xi_X \circ \Xi'_X = \mathrm{id}_{\mathcal{C}_{dR}}, \quad \Xi'_X \circ \Xi_X = \mathrm{id}_{\mathcal{C}_{Dol}}.$$

$\boxtimes$

Rmk. From now on, we will directly call harmonic bundles

$$(E, \bar{\partial}_E, \varphi, h). \quad \text{or} \quad (E, \nabla, D'', h)$$

Q: Under which conditions that a Higgs bundle (resp. flat bundle) admit a pluri-harmonic metric?

A: Need stability!