Recall: · Interpolation inequality.

· Extremely special case of Schauder estimate

Proposition. $A = (a_{ij})$ constant symmetric matrix.

$\lambda I_n \le A \le \Lambda I_n$ $(\lambda, \Lambda > 0)$ If $u \in L^\infty(B_R) \cap C^2(B_R)$

satisfies $\sum_{i,j} a_{ij}\partial_{ij} u = f$ in B_R

for some $f \in C^\alpha(\overline{B_R})$. then $u \in C^{2,\alpha}(B_{R/2})$ and

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_R)} + R^2\|f\|^*_{C^\alpha(B_R)}\right).$$

· Goal: Generalize the above result to $Lu = f$.

3. Special case (small oscillation)

Proposition. $L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$. $a_{ij}, b_i, c \in C^\alpha(\overline{B_R})$ $\alpha \in (0,1)$

$\lambda I_n \le (a_{ij}) \le \Lambda I_n$ $(\lambda, \Lambda > 0)$

$\|a_{ij}\|^*_{C^\alpha(B_R)}$, $R\|b_i\|^*_{C^\alpha(B_R)}$, $R^2\|c\|^*_{C^\alpha(B_R)} \le \Lambda$

Suppose that $u \in L^\infty(B_R) \cap C^2(B_R)$ satisfies

$Lu = f$ in B_R

for some $f \in C^\alpha(\overline{B_R})$. Then $\exists\, \varepsilon(\lambda, \Lambda, \alpha, n)$ s.t.

if $\sup_{x', x'' \in B_R} |a_{ij}(x') - a_{ij}(x'')| \le \varepsilon$. then $u \in C^{2,\alpha}(B_{R/2})$ and

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_R)} + R^2\|f\|^*_{C^\alpha(B_R)}\right)$$

Proof. By scaling method. assume $R = 1$.

[Actually define $a^R_{ij}(x) = a_{ij}(Rx)$ $b^R_i = R b_i(Rx)$. $c^R(x) = R^2 c(Rx)$.

$u_R(x) = u(Rx)$. $f_R(x) = R^2 f(Rx)$

$L_R = \sum_{i,j} a^R_{ij}\partial_{ij} + \sum_i b^R_i\partial_i + c^R$

$Lu = f$ in $B_R \iff L_R u_R = f_R$ in B_1]

Goal: $[\nabla^2 u]_{C^\alpha(B_{1/2})} \le C(\lambda, \Lambda, n, \alpha)\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right)$

Goal + interpolation inequality $\Rightarrow$ Proposition.

$$Lu = f \Rightarrow \sum_{i,j} a_{ij}(0)\partial_{ij} u = \tilde{f} \quad (\star)$$

where $\tilde{f} = f - cu - \sum_i b_i\partial_i u - \sum_{i,j}(a_{ij} - a_{ij}(0))\partial_{ij} u$.

Apply extremely special case to ($\bigstar$) in $B_r \subset B_1$

$$r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2})} \le C(\lambda,\Lambda,\alpha,n)\left(\|u\|_{L^\infty(B_r)} + r^2\|\tilde f\|_{L^\infty(B_r)} + r^{2+\alpha}[\tilde f]_{C^\alpha(B_r)}\right)$$

Compute

$$\|\hat f\|_{L^\infty(B_r)} \le \|f\|_{L^\infty(B_r)} + \|c\|_{L^\infty(B_r)}\|u\|_{L^\infty(B_r)} + \sum_i \|b_i\|_{L^\infty(B_r)}\|\nabla u\|_{L^\infty(B_r)}$$
$$+ \sum_{i,j}\|a_{ij}-a_{ij}(0)\|_{L^\infty(B_r)}\|\nabla^2 u\|_{L^\infty(B_r)}$$

Note that $\forall\, V, W \in C^\alpha(\overline{B_r})$ $\quad [VW]_{C^\alpha(B_r)} \le [V]_{C^\alpha(B_r)}\|W\|_{L^\infty(B_r)} + \|V\|_{L^\infty(B_r)}[W]_{C^\alpha(B_r)}$

$$[\hat f]_{C^\alpha(B_r)} \le [f]_{C^\alpha(B_r)} + [c]_{C^\alpha(B_r)}\|u\|_{L^\infty(B_r)} + \|c\|_{L^\infty(B_r)}[u]_{C^\alpha(B_r)}$$
$$+ \sum_i [b_i]_{C^\alpha(B_r)}\|\nabla u\|_{L^\infty(B_r)} + \sum_i \|b_i\|_{L^\infty(B_r)}[\nabla u]_{C^\alpha(B_r)}$$
$$+ \sum_{i,j}[a_{ij}-a_{ij}(0)]_{C^\alpha(B_r)}\|\nabla^2 u\|_{L^\infty(B_r)} + \sum_{i,j}\|a_{ij}-a_{ij}(0)\|_{L^\infty(B_r)}[\nabla^2 u]_{C^\alpha(B_r)}$$

Assumption $\Rightarrow$ $\|a_{ij}\|_{C^\alpha(B_r)}, \|b_i\|_{C^\alpha(B_r)}, \|c\|_{C^\alpha(B_r)} \le \Lambda$

$\qquad\qquad\qquad \|a_{ij}-a_{ij}(0)\|_{L^\infty(B_r)} \le \varepsilon$

$$\Rightarrow r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2})} \le C(\lambda,\Lambda,\alpha,n)\left(r^2\|f\|^*_{C^\alpha(B_r)} + \|u\|^*_{C^2(B_r)} + \varepsilon r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_r)}\right)$$

Interpolation inequality (cf. Lecture 7) $\Rightarrow$

$$\|u\|^*_{C^2(B_r)} \le \varepsilon r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_r)} + C(\varepsilon,\alpha,n)\|u\|_{L^\infty(B_r)}.$$

$$\Rightarrow r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2})} \le C\left(r^2\|f\|^*_{C^\alpha(B_r)} + \varepsilon r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_r)} + C_\varepsilon\|u\|_{L^\infty(B_r)}\right)$$

$C = C(\lambda,\Lambda,\alpha,n)$ $\qquad \le C\varepsilon r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_r)} + C\|f\|_{C^\alpha(B_1)} + C_\varepsilon\|u\|_{L^\infty(B_1)}$

$C_\varepsilon = C(\varepsilon,\lambda,\Lambda,\alpha,n)$.

By the same argument, this holds for any $B_r(x_0) \subset B_1$

$$(*)\quad r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2}(x_0))} \le C\varepsilon r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_r(x_0))}$$
$$+ C\|f\|_{C^\alpha(B_1)} + C_\varepsilon\|u\|_{L^\infty(B_1)}$$

Define $K = \sup\left\{ r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2}(x_0))} \mid B_r(x_0) \subset B_1 \right\}$

For any $B_r(x_0) \subset B_1$. $\exists N(n)$ $y_1, \cdots, y_N \in B_{r/2}(x_0)$ s.t.

$$B_{r/2}(x_0) \subset \bigcup_{i=1}^{N} B_{r/16}(y_i)$$

Apply $(*)$ to $B_{r/4}(y_i) \subset B_1$.

$$(1) \quad r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/8}(y_i))} \le C\varepsilon\left(\frac{r}{4}\right)^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/4}(y_i))} + C\|f\|_{C^\alpha(B_1)} + C_\varepsilon\|u\|_{L^\infty(B_1)}$$

$B_{r/2}(x_0) \subset \bigcup_{i=1}^{N} B_{\frac{r}{16}}(y_i)$ Similar argument of claim in the extremely special case $\Rightarrow$

$$(2) \quad [\nabla^2 u]_{C^\alpha(B_{\frac{r}{2}}(x_0))} \le C\sum_{i=1}^{N}\left([\nabla^2 u]_{C^\alpha(B_{\frac{r}{8}}(y_i))} + r^{-2-\alpha}\|u\|_{L^\infty(B_{\frac{r}{8}}(y_i))}\right)$$

$$(1),(2) \Rightarrow r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2}(x_0))} \le C\varepsilon\sum_{i=1}^{N}\left(\frac{r}{4}\right)^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{\frac{r}{4}}(y_i))} + C\|f\|_{C^\alpha(B_1)} + C_\varepsilon\|u\|_{L^\infty(B_1)}$$

Definition of K & $B_{\frac{r}{2}}(y_i) \subset B_1$.

$$\Rightarrow \left(\frac{r}{4}\right)^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/4}(y_i))} \le K$$

$$\Rightarrow r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2}(x_0))} \le C\varepsilon K + C\|f\|_{C^\alpha(B_1)} + C_\varepsilon\|u\|_{L^\infty(B_1)}$$

Take supremum over all $B_r(x_0) \subset B_1$.

$$\Rightarrow K \le C\varepsilon K + C\|f\|_{C^\alpha(B_1)} + C_\varepsilon\|u\|_{L^\infty(B_1)}$$

choose $\varepsilon = \frac{1}{2C}$ $\quad C = C(\lambda, \Lambda, \alpha, n) \Rightarrow \varepsilon = \varepsilon(\lambda, \Lambda, \alpha, n)$

$$\Rightarrow K \le C\|f\|_{C^\alpha(B_1)} + C\|u\|_{L^\infty(B_1)}$$

Definition of $K \Rightarrow \forall r \in (0,1)$ $B_r \subset B_1$

$$r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2})} \le K \le C\|f\|_{C^\alpha(B_1)} + C\|u\|_{L^\infty(B_1)}$$

Let $r \to 1$. $\Rightarrow$ Goal. #

4. General Case

Theorem. $L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$. $a_{ij}, b_i, c \in C^{\alpha}(\overline{B_R})$, $\alpha \in (0,1)$

$\lambda I_n \le (a_{ij}) \le \Lambda I_n$ $(\lambda, \Lambda > 0)$ $\|a_{ij}\|^*_{C^{\alpha}(B_R)}, R\|b_i\|^*_{C^{\alpha}(B_R)}, R^2\|c\|^*_{C^{\alpha}(B_R)} \le \Lambda$

If $u \in L^{\infty}(B_R) \cap C^2(B_R)$ satisfies $Lu = f$ in B_R

for some $f \in C^{\alpha}(\overline{B_R})$, then $u \in C^{2,\alpha}(B_{R/2})$ and

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^{\infty}(B_R)} + R^2\|f\|^*_{C^{\alpha}(B_R)}\right)$$

Proof. By scaling method, assume $R=1$.

$\forall B_{r_0}(x_0) \subset B_1$. $\sup_{x', x'' \in B_r(x_0)} |a_{ij}(x') - a_{ij}(x'')| \le [a_{ij}]_{C^{\alpha}(B_1)} |x'-x''| \le \Lambda (2r_0)^{\alpha}$

choose $r_0 \in (0,1)$ s.t. $\Lambda (2r_0)^{\alpha} = \varepsilon$ (ε is the constant in special case)

$\varepsilon = \varepsilon(\lambda, \Lambda, \alpha, n) \Rightarrow r_0 = r_0(\lambda, \Lambda, \alpha, n)$.

Apply special case to $B_{r_0}(x_0)$.

$$(*)\ r_0^{2+\alpha}[\nabla^2 u]_{C^{\alpha}(B_{\frac{r_0}{2}}(x_0))} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^{\infty}(B_{r_0}(x_0))} + r_0^2\|f\|^*_{C^{\alpha}(B_{r_0}(x_0))}\right)$$

Goal: $[\nabla^2 u]_{C^{\alpha}(B_{1/2})} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^{\infty}(B_1)} + \|f\|_{C^{\alpha}(B_1)}\right)$

Goal + interpolation inequality $\Rightarrow$ Theorem.

For $B_{1/2}$. $\exists N(n, r_0)$ $y_1, \cdots, y_N \in B_{1/2}$ s.t.

$$B_{1/2} \subset \bigcup_{i=1}^{N} B_{\frac{r_0}{8}}(y_i).$$

Similar covering argument as before

$$\Rightarrow [\nabla^2 u]_{C^{\alpha}(B_{1/2})} \le C\sum_{i=1}^{N}\left([\nabla^2 u]_{C^{\alpha}(B_{\frac{r_0}{4}}(y_i))} + r_0^{-2-\alpha}\|u\|_{L^{\infty}(B_{\frac{r_0}{4}}(y_i))}\right)$$

$$(*) \Rightarrow \quad \le C\|u\|_{L^{\infty}(B_1)} + C\|f\|_{C^{\alpha}(B_1)} \qquad \#$$

Lecture 8. Schauder theory (II)

1. $C^{2,\alpha}$ regularity and estimate in domains

Theorem. Ω: bounded domain in $\mathbb{R}^n$.

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$ $\quad a_{ij}, b_i, c \in C^\alpha(\bar{\Omega})$. $\alpha \in (0,1)$

$\lambda I_n \le (a_{ij}) \le \Lambda I_n$ $\quad (\lambda, \Lambda > 0)$.

$\|a_{ij}\|_\alpha, \|b_i\|_\alpha, \|c\|_\alpha \le \Lambda$

If $u \in L^\infty(\Omega) \cap C^2(\Omega)$ satisfies $Lu = f$ in Ω

for some $f \in C^\alpha(\bar{\Omega})$ then $u \in C^{2,\alpha}(\Omega)$

and $\forall\ \Omega' \subset\subset \Omega$.

$$\|u\|_{C^{2,\alpha}(\Omega')} \le C(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)}) \quad (*)$$

where $C' = C(\Omega', \mathrm{dist}(\Omega', \partial\Omega), \lambda, \Lambda, \alpha, n)$.

Proof. Set $d = \mathrm{dist}(\Omega', \partial\Omega)$. $\exists\ N(\Omega', d, n)$

$y_1, \dots, y_N \in \Omega'$ s.t. $\Omega' \subset \bigcup_{i=1}^N B_{d/4}(y_i)$ $\quad (1)$

Apply Schauder estimate to $B_d(y_i)$.

$$\|u\|^*_{C^{2,\alpha}(B_{d/2}(y_i))} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_d(y_i))} + d^2 \|f\|^*_{C^\alpha(B_d(y_i))}\right)$$

$$\Rightarrow \|u\|_{C^{2,\alpha}(B_{d/2}(y_i))} \le C(d, \lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)}\right) \quad (2)$$

(1), (2). Similar covering argument as before

$\Rightarrow$ $(*)$

Ω' is arbitrary $\Rightarrow$ $u \in C^{2,\alpha}(\Omega)$. #

2. Weighted Hölder norms

Ω: bounded domain in $\mathbb{R}^n$. For $x, y \in \Omega$, define

$d_x = \mathrm{dist}(x, \partial\Omega)$. $\quad d_{x,y} = \min(d_x, d_y)$.

$k \in \mathbb{Z}_{\geq 0}$. $\alpha \in (0,1)$. $\sigma \in \mathbb{R}$. define.

$$[u]_{C^k_\sigma(\Omega)} = \sup_{x\in\Omega} d_x^{k+\sigma} |\nabla^k u(x)| \qquad \|u\|_{C^k_\sigma(\Omega)} = \sum_{i=0}^{k} [u]_{C^i_\sigma(\Omega)}$$

$$\|u\|_{C_\sigma(\Omega)} = \|u\|_{C^0_\sigma(\Omega)}$$

$$[u]_{C^{k,\alpha}_\sigma(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} d_{x,y}^{k+\alpha+\sigma} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}$$

$$\|u\|_{C^{k,\alpha}_\sigma(\Omega)} = \|u\|_{C^k_\sigma(\Omega)} + [u]_{C^{k,\alpha}_\sigma(\Omega)}$$

$$\|u\|_{C^\alpha_\sigma(\Omega)} = \|u\|_{C^{0,\alpha}_\sigma(\Omega)}$$

$$C^k_\sigma(\Omega) = \{u \in C^k(\Omega) \mid \|u\|_{C^k_\sigma(\Omega)} < \infty\}$$

$$C^{k,\alpha}_\sigma(\Omega) = \{u \in C^{k,\alpha}(\Omega) \mid \|u\|_{C^{k,\alpha}_\sigma(\Omega)} < \infty\}$$

Remark. (1) σ may be negative.

(2) $(C^k_\sigma(\Omega), \|\cdot\|_{C^k_\sigma(\Omega)})$ and $(C^{k,\alpha}_\sigma(\Omega), \|\cdot\|_{C^{k,\alpha}_\sigma(\Omega)})$ are Banach spaces

(3). If $\sigma \geq 0$ then

$$\|wv\|_{C^\alpha_\sigma(\Omega)} \leq \|w\|_{C^\alpha_0(\Omega)} \|v\|_{C^\alpha_\sigma(\Omega)}.$$

3. Schauder estimate revisited

Theorem. Ω: bounded domain in $\mathbb{R}^n$. $\alpha \in (0,1)$ $\beta \in (0,1)$

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$. $a_{ij}, b_i, c \in C^\alpha(\bar{\Omega})$.

$\lambda I_n \leq (a_{ij}) \leq \Lambda I_n$. $(\lambda, \Lambda > 0)$.

$\|a_{ij}\|_{C^\alpha(\Omega)}, \|b_i\|_{C^\alpha(\Omega)}, \|c\|_{C^\alpha(\Omega)} \le \Lambda$.

If $u \in C^2(\Omega)$ satisfies $Lu = f$ in Ω

for some $f \in C^\alpha(\Omega)$ with

$\|u\|_{C_{-\beta}(\Omega)}, \|f\|_{C^\alpha_{2-\beta}(\Omega)} < \infty$

then $\|u\|_{C^{2,\alpha}_{-\beta}(\Omega)} \le C(\beta, \lambda, \Lambda, \alpha, n)(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^\alpha_{2-\beta}(\Omega)})$

Proof. $\forall x \in \Omega$, $d_x = \operatorname{dist}(x, \partial\Omega)$. set $d = d_x/2$.

Apply Schauder estimate (in balls) to $B_d(x) \subset\subset \Omega$.

$$d|\nabla u(x)| + d^2|\nabla^2 u(x)| + d^{2+\alpha} \sup_{\substack{y \in B_{d/2}(x) \\ y \ne x}} \frac{|\nabla^2 u(x) - \nabla^2 u(y)|}{|x-y|^\alpha}$$

$$\le \|u\|^*_{C^{2,\alpha}(B_{d/2}(x))}$$

$$\le C\Big(\sup_{y \in B_d(x)} |u(y)| + d^2 \sup_{y \in B_d(x)} |f(y)| + d^{2+\alpha} \sup_{\substack{y,z \in B_d(x) \\ y \ne z}} \frac{|f(y) - f(z)|}{|y-z|^\alpha}\Big)$$

$\forall y, z \in B_d(x) \Rightarrow d \le d_y, d_z \le 3d$. $\quad d = d_x/2$

Multiplying $d^{-\beta}$ on both sides

$$d_x^{1-\beta}|\nabla u(x)| + d_x^{2-\beta}|\nabla^2 u(x)| + \sup_{\substack{y \in B_{d/2}(x) \\ y \ne x}} \frac{|\nabla^2 u(x) - \nabla^2 u(y)|}{|x-y|^\alpha}$$

$$\le C\Big(\sup_{y \in B_d(x)} d_y^{-\beta}|u(y)| + \sup_{y \in B_d(x)} d_y^{2-\beta}|f(y)| + \sup_{\substack{y,z \in B_d(x) \\ y \ne z}} d_{y,z}^{2+\alpha-\beta} \frac{|f(y) - f(z)|}{|y-z|^\alpha}\Big)$$

$$\leq C\left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^{\alpha}_{2-\beta}(\Omega)}\right) \qquad (1)$$

$$\Rightarrow \quad \|u\|_{C^2_{-\beta}(\Omega)} \leq C\left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^{\alpha}_{2-\beta}(\Omega)}\right) \qquad (2)$$

To control $[u]_{C^{2,\alpha}_{-\beta}(\Omega)}$ $\forall\, x, y \in \Omega$

Case 1. If $|x-y| < d/2$, then (1) $\Rightarrow$

$$d_{x,y}^{2+\alpha-\beta} \frac{|\nabla^2 u(x) - \nabla^2 u(y)|}{|x-y|^{\alpha}} \leq C\left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^{\alpha}_{2-\beta}(\Omega)}\right)$$

Case 2. If $|x-y| \geq d/2$ $\qquad d = d_x/2$

$$d_{x,y}^{2+\alpha-\beta} \frac{|\nabla^2 u(x) - \nabla^2 u(y)|}{|x-y|^{\alpha}} \leq d_{x,y}^{2+\alpha-\beta} \frac{|\nabla^2 u(x)| + |\nabla^2 u(y)|}{(d/2)^{\alpha}}$$

$$\leq d_{x,y}^{2+\alpha-\beta} \frac{|\nabla^2 u(x)| + |\nabla^2 u(y)|}{(d_x/4)^{\alpha}}$$

$$\leq d_{x,y}^{2-\beta} \cdot d_{x,y}^{\alpha} \cdot \frac{|\nabla^2 u(x)| + |\nabla^2 u(y)|}{d_x^{\alpha}} \cdot 4^{\alpha}$$

$$\leq 4^{\alpha} d_{x,y}^{2-\beta} \left(|\nabla^2 u(x)| + |\nabla^2 u(y)|\right)$$

$$\leq 4^{\alpha}\left(d_x^{2-\beta} |\nabla^2 u(x)| + d_y^{2-\beta} |\nabla^2 u(y)|\right)$$

$$\leq 2 \times 4^{\alpha} \|u\|_{C^2_{-\beta}(\Omega)}$$

$$\overset{(2)}{\leq} C\left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^{\alpha}_{2-\beta}(\Omega)}\right)$$

$$\Rightarrow [u]_{C^{2,\alpha}_{-\beta}} \leq C\left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^{\alpha}_{2-\beta}(\Omega)}\right) \qquad (3)$$

(2). (3) $\Rightarrow$ Theorem. #

Corollary $\alpha, \beta \in (0,1)$. $L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$, $B = B_R(x_0)$

$a_{ij}, b_i, c \in C^{\alpha}(\bar{B})$. $\lambda I_n \leq (a_{ij}) \leq \Lambda I_n$ $c \leq 0$.

$\|a_{ij}\|_{C^\alpha(B)}, \|b_i\|_{C^\alpha(B)}, \|c\|_{C^\alpha(B)} \le \Lambda.$

If $u \in C(\bar{B}) \cap C^2(B)$ satisfies

$$\begin{cases} Lu = f & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases}$$

for some $f \in C^\alpha(B)$ with $\|f\|_{C^\alpha_{2-\beta}(B)} < \infty.$

then $\|u\|_{C^{2,\alpha}_{-\beta}(B)} \le C \|f\|_{C^\alpha_{2-\beta}(B)}$

where $C = C(\lambda, \Lambda, \alpha, \beta, R, n)$

Proof. Previous theorem $\Rightarrow$

$$\|u\|_{C^{2,\alpha}_{-\beta}(B)} \le C\left(\|u\|_{C_{-\beta}(B)} + \|f\|_{C^\alpha_{2-\beta}(B)}\right) \quad (1).$$

cf. Lecture 4 $\Rightarrow$ $\sup\limits_{B} d_x^{-\beta} |u(x)| \le C \sup\limits_{x \in B} d_x^{2-\beta} |f(x)|$

$$\Rightarrow \quad \|u\|_{C_{-\beta}(B)} \le C \|f\|_{C_{2-\beta}(B)} \quad (2)$$

(1) (2) $\Rightarrow$ Corollary.

#