Recall

- Solve Dirichlet problem $\begin{cases} \Delta u = 0 & \text{in } B_R \\ u = \varphi & \text{on } \partial B_R \end{cases}$

  Green's function $\rightsquigarrow$ Poisson kernel.

- Properties of harmonic functions

  (1) $u \in C^2 \rightsquigarrow$ mean value property

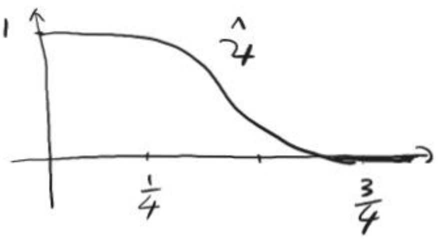
  (2) $u \in C^\infty \rightsquigarrow \sup_{B_r(x_0)} |\nabla^k u| \le \dfrac{c(n,k)}{(R-r)^k} \sup_{B_R(x_0)} |u|$

Theorem. If $u \in C^2(\Omega)$ is harmonic, then $u \in C^\infty(\Omega)$

Proof. Step 1. Mollifier

Choose one-variable function $\hat\psi \in C^\infty([0,+\infty))$

$$\begin{cases} \hat\psi \ge 0 & \text{in } [0,+\infty) \\ \hat\psi = 1 & \text{in } [0,\frac14] \\ \hat\psi = 0 & \text{in } [\frac34,+\infty) \end{cases}$$



Suppose $n\omega_n \int_0^1 r^{n-1}\hat\psi(r)\,dr = c^{-1}$. Set $\psi = c\hat\psi$, then

$$\begin{cases} \psi \ge 0 & \text{in } [0,+\infty) \\ \psi = c & \text{in } [0,\frac14] \\ \psi = 0 & \text{in } [\frac34,+\infty) \\ n\omega_n \int_0^1 r^{n-1}\psi(r)\,dr = 1 \end{cases}$$

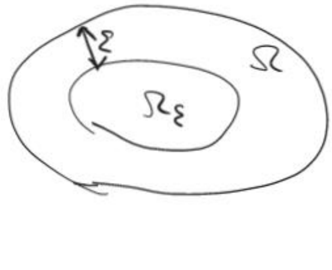
Define $\eta(x) = \psi(|x|)$. $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$

$\Rightarrow \operatorname{supp}\eta \subset B_1$. $\operatorname{supp}\eta_\varepsilon \subset B_\varepsilon$

$\Rightarrow \int_{B_1} \eta(y)\,dy = \int_0^1 \int_{\partial B_r} \eta(y)\,dS_y\,dr = \int_0^1 n\omega_n r^{n-1}\cdot\psi(r)\,dr = 1$

$\Rightarrow \int_{B_\varepsilon} \eta_\varepsilon(y)\,dy = 1$

Step 2. Approximation



Set $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) > \varepsilon\}$

$\forall x \in \Omega_\varepsilon$ define

$$u_\varepsilon(x) = \int_\Omega u(y)\, \eta_\varepsilon(y-x)\, dy \Rightarrow u_\varepsilon \in C^\infty(\Omega_\varepsilon)$$

$$(\operatorname{supp}\eta_\varepsilon \subset B_\varepsilon) \quad = \int_{B_\varepsilon(x)} u(y)\, \eta_\varepsilon(y-x)\, dy$$

$$\left(\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)\right) \quad = \frac{1}{\varepsilon^n}\int_{B_\varepsilon(x)} u(y)\, \eta\left(\frac{y-x}{\varepsilon}\right) dy$$

$$\left(w = \frac{y-x}{\varepsilon}\right) \quad = \int_{B_1} u(x+\varepsilon w)\, \eta(w)\, dw$$

$$(w \to y) \quad = \int_{B_1} u(x+\varepsilon y)\, \eta(y)\, dy$$

$$\Rightarrow \Delta u_\varepsilon(x) = \int_{B_1} \Delta u(x+\varepsilon y)\, \eta(y)\, dy = 0$$

($\uparrow$ u is harmonic in Ω.)

$\Rightarrow u_\varepsilon$ is a smooth harmonic function in Ω_ε.

Step 3. $u \in C^\infty(\Omega)$

Goal: $\forall B_R(x_0) \subset\subset \Omega$. $u \in C^\infty(B_R(x_0))$.

$\exists d > 0$ s.t. $B_R(x_0) \subset B_{R+3d}(x_0) \subset \Omega$.

$\Rightarrow B_{R+2d}(x_0) \subset\subset \Omega_\varepsilon$ for $\varepsilon \ll 1$.

Recall $\int_{B_1} \eta(y)\, dy = 1$. $\Rightarrow u(x) = \int_{B_1} u(x)\, \eta(y)\, dy$

$$u_\varepsilon(x) = \int_{B_1} u(x+\varepsilon y)\, \eta(y)\, dy \quad \forall x \in B_{R+d}(x_0)$$

$$|u_\varepsilon(x) - u(x)| \le \int_{B_1} |u(x+\varepsilon y) - u(x)| \cdot \eta(y)\, dy$$

$$(\text{Mean value theorem}) \quad \le \|\nabla u\|_{L^\infty(B_{R+2d}(x_0))} \cdot \int_{B_1} |\varepsilon y| \cdot \eta(y)\, dy$$

$$= \varepsilon \cdot \|\nabla u\|_{L^\infty(B_{R+2d}(x_0))} \int_{B_1} |y|\, \eta(y)\, dy$$

$$\to 0 \quad \text{as } \varepsilon \to 0.$$

$$\Rightarrow u_\varepsilon \to u \text{ in } C^0\left(\overline{B_{R+d}(x_0)}\right)$$

$$\Rightarrow \sup_{\overline{B}_{R+d}(x_0)} |u_\varepsilon| \le \sup_{\overline{B}_{R+d}(x_0)} |u| + 1 \qquad \varepsilon << 1$$

u_ε is smooth harmonic in Ω_ε

$$\forall k \in \mathbb{Z}_+ , \Rightarrow \|u_\varepsilon\|_{C^{k+1}(\overline{B}_R(x_0))} \le \frac{C(n,k)}{d^{k+1}} \sup_{\overline{B}_{R+d}(x_0)} |u_\varepsilon|$$

$$\le \frac{C(n,k)}{d^{k+1}} \left(\sup_{\overline{B}_{R+d}(x_0)} |u| + 1 \right)$$

Arzela-Ascoli theorem $\Rightarrow \exists\, \varepsilon_i \to 0$ s.t. $u_{\varepsilon_i} \to v$ in $C^k(\overline{B_{R+d}(x_0)})$

$\Rightarrow u = v$ in $\overline{B_{R+d}(x_0)}$

$\Rightarrow u \in C^k(\overline{B_{R+d}(x_0)})$

$k \in \mathbb{Z}_+$ is arbitrary $\Rightarrow u \in C^\infty(B_R(x_0))$. #

Lecture 3. Maximum Principle (I)

1. Notations & Definitions

$\Omega \subseteq \mathbb{R}^n$: bounded domain

a_{ij}, b_i, c bounded continuous functions in Ω with $a_{ij} = a_{ji}$

Define the operator $L = \sum_{ij=1}^{n} a_{ij}\partial_{ij} + \sum_{i=1}^{n} b_i \partial_i + c$.

i.e., $\forall u \in C^2(\Omega)$

$$Lu = \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{i=1}^{n} b_i \partial_i u + cu \quad \text{in } \Omega$$

Definition. (1) L is said to be strictly elliptic if $\exists\, \lambda > 0$. s.t.

$$(a_{ij}(x)) \ge \lambda I_n \Leftrightarrow \sum_{ij} a_{ij}(x)\xi_i\xi_j \ge \lambda |\xi|^2 \quad \forall x \in \Omega \ \forall \xi \in \mathbb{R}^n.$$

(2) L ———————— uniformly elliptic if $\exists\, \lambda, \Lambda > 0$. s.t.

$\lambda I_n \le (a_{ij}(x)) \le \Lambda I_n \iff \lambda|\xi|^2 \le \sum_{i,j} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \forall x\in\Omega \ \forall \xi \in \mathbb{R}^n$

Definition $f \in C(\Omega)$. Linear 2nd order PDE $Lu = f$. (*)

(1) If $u \in C^2(\Omega)$ & $Lu \ge f$ then u is called a subsolution of (*)

(2) ——— $Lu \le f$ ——— supersolution of (*)

Remark. When $a_{ij} = \delta_{ij}$, $b_i = 0$, $c = 0$.

L becomes Laplace's operator $L = \Delta$.

In addition, if $f = 0$ then (*) is Laplace's equation.

Subsolution ⟶ subharmonic

Supersolution ⟶ superharmonic.

2. Weak maximum principle

Theorem. Ω: bounded domain in $\mathbb{R}^n$.

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$

Suppose $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$ $\quad (a_{ij}(x)) \ge \lambda I_n \ (\lambda > 0)$

$c \le 0$.

If $u \in C^2(\Omega) \cap C(\bar\Omega)$ satisfies $Lu \ge 0$ in Ω, then

$$\max_{\bar\Omega} u \le \max_{\partial\Omega} u^+$$

where $u^+ = \max(u, 0)$.

Proof. · Special case. $Lu > 0$ in Ω

$u \in C(\bar\Omega) \Rightarrow \exists x_0 \in \bar\Omega$ s.t. $u(x_0) = \max_{\bar\Omega} u$

Subcase 1: $x_0 \in \partial\Omega$ or $u(x_0) \le 0 \Rightarrow u(x_0) \le \max_{\partial\Omega} u^+$ ✓

Subcase 2. $x_0 \notin \partial\Omega$ and $u(x_0) > 0$.

$\Rightarrow x_0$ is an interior point of Ω

$\Rightarrow \partial_i u(x_0) = 0$. $\quad (\partial_{ij}u(x_0)) \le 0$.

$$\Rightarrow 0 < Lu(x_0) = \underbrace{\sum_{i,j} a_{ij}(x_0)\partial_{ij}u(x_0)}_{\leq 0} + \underbrace{\sum_i b_i(x_0)\partial_i u(x_0)}_{= 0} + \underbrace{c\,u(x_0)}_{\leq 0\ (c \leq 0)} \leq 0$$

$$(a_{ij}(x_0)) > 0 \Rightarrow (a_{ij}(x_0)) = P^T P$$

$$\sum_{i,j} a_{ij}(x_0)\partial_{ij}u(x_0) = tr\left((a_{ij}(x_0)) \cdot (\partial_{ij}u(x_0)) \right)$$

$$= tr\left(P^T \cdot P (\partial_{ij}u(x_0)) \right)$$

$$= tr\left(P \cdot (\partial_{ij}u(x_0)) \cdot P^T \right) \leq 0.$$

$\Rightarrow$ Contradiction.

· General case. $Lu \geq 0$ in Ω.

$\forall \varepsilon > 0$. define $v(x) = u(x) + \varepsilon e^{Ax_1}$.

where $A > 0$ is a large constant to be determined.

$$Lv = Lu + \varepsilon L(e^{Ax_1})$$

$$= Lu + \varepsilon e^{Ax_1}(a_{11}A^2 + b_1 A + c)$$

$(a_{ij}) \geq \lambda I_n \Rightarrow a_{11} \geq \lambda > 0.$ $\quad b_1, c \in L^\infty(\Omega)$.

$\exists A >> 1$. s.t. $a_{11}A^2 + b_1 A + c \geq \lambda A^2 - \|b_1\|_{L^\infty(\Omega)} A - \|c\|_{L^\infty(\Omega)} > 0$

$\Rightarrow Lv > 0$ in Ω

Special case $\Rightarrow \max_{\bar{\Omega}} v \leq \max_{\partial\Omega} v^+$

Recall $v = u + \varepsilon e^{Ax_1} \Rightarrow$

$$\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} v \leq \max_{\partial\Omega} v^+ \leq \max_{\partial\Omega} u^+ + \varepsilon \max_{\partial\Omega} e^{Ax_1}$$

Ω bounded $\Rightarrow \max_{\partial\Omega} e^{Ax_1} \neq \infty$

Let $\varepsilon \to 0$. $\Rightarrow \max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$ #

Remark. If $c = 0$, then stronger inequality holds:

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u$$

Remark. Assumptions "Ω is bounded" and "$c \le 0$" are necessary.

Counterexample (Ω is unbounded)

$\Omega = \{x \in \mathbb{R}^n \mid x_n > 0\}$ $\quad u = x_n$

$$\Rightarrow \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{But } \sup_{\Omega} u = +\infty > \sup_{\partial\Omega} u^+ = 0$$

Counterexample (c is not non-positive)

$\Omega = (0,\pi) \times \cdots \times (0,\pi) \subset \mathbb{R}^n$. $\quad u = \prod_{i=1}^{n} \sin x_i$

$$\Rightarrow \begin{cases} \Delta u + n u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{But } \max_{\overline{\Omega}} u = 1 > \sup_{\partial\Omega} u^+ = 0.$$

Assumption ($\bigstar$): Ω: bounded domain in $\mathbb{R}^n$

$L = \sum_{i,j=1}^{n} a_{ij}\partial_{ij} + \sum_{i=1}^{n} b_i \partial_i + c$ satisfies

$a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$. $\quad (a_{ij}) \ge \lambda I_n$

$c \le 0$.

Corollary. Under assumption ($\bigstar$), if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{cases} Lu \ge 0 & \text{in } \Omega \\ u \le 0 & \text{on } \partial\Omega \end{cases}$$

then $u \le 0$ in Ω

Proof. Trivial.

Corollary (Comparison principle). Under assumption (*), if $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\begin{cases} Lu \geq Lv & \text{in } \Omega \\ u \leq v & \text{on } \partial\Omega \end{cases}$$

then $u \leq v$ in Ω.

Proof. Define $w = u - v$. $\Rightarrow \begin{cases} Lw \geq 0 & \text{in } \Omega \\ w \leq 0 & \text{on } \partial\Omega \end{cases}$

the above corollary $\Rightarrow w \leq 0$ in Ω

$\Rightarrow u \leq v$ in Ω #

Corollary. Under assumption (*). $f \in C(\Omega)$. $\varphi \in C(\partial\Omega)$. there exists at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of $\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$

Proof. Suppose that there exist two solutions u_1 and u_2

$$\begin{cases} Lu_1 = f = Lu_2 & \text{in } \Omega \\ u_1 = u_2 & \text{on } \partial\Omega \end{cases}$$

Comparison principle $\Rightarrow u_1 \leq u_2$. $u_2 \leq u_1$ in Ω.

$\Rightarrow u_1 = u_2$ in Ω #

Corollary. Under assumption (*). Suppose $u, v, w \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\begin{cases} Lu \geq f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \quad \begin{cases} Lv = f & \text{in } \Omega \\ v = \varphi & \text{on } \partial\Omega \end{cases} \quad \begin{cases} Lw \leq f & \text{in } \Omega \\ w = \varphi & \text{on } \partial\Omega \end{cases}$$

then $u \leq v \leq w$ in Ω.

Proof. Trivial.

Remark. the above corollary implies

subsolution $\leq$ solution $\leq$ supersolution.

3. Hopf Lemma

Theorem $B = B_R(y_0)$ $\quad x_0 \in \partial B$

$L = \sum_{ij=1}^{n} a_{ij}\partial_{ij} + \sum_{i=1}^{n} b_i \partial_i + c \quad (a_{ij}) \geq \lambda I_n \quad c \leq 0 \quad a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$

If $u \in C^2(B) \cap C^1(\bar{B})$ satisfies

(1) $Lu \geq 0$ in B. (2) $u(x) < u(x_0)$ $\forall x \in B$ (3) $u(x_0) \geq 0$.

then $\frac{\partial u}{\partial \vec{n}}(x_0) > 0$. where $\vec{n}$ is the unit outward normal to ∂B.

Remark. $\frac{\partial u}{\partial \vec{n}}(x_0) \geq 0$ is trivial.

The key part is strict positivity.

Proof. After translation assume $B = B_R = B_R(0)$

Define $V(x) = u(x) - u(x_0) + \varepsilon W(x)$.

$W(x) = e^{-A|x|^2} - e^{-AR^2}$

where ε and A are constants to be determined.

Write $D = B_R \setminus \overline{B}_{R/2}$

Claim choose $A \gg 1$. $\varepsilon \ll 1$. $\begin{cases} LV \geq 0 & \text{in } D \\ V \leq 0 & \text{on } \partial D \end{cases}$

Claim + max. prin. $\Rightarrow$ $V \leq 0$ in D.

$V(x_0) = u(x_0) - u(x_0) + \varepsilon W(x_0) = 0 \qquad x_0 \in \partial B \quad |x_0| = R$

$\Rightarrow V(x_0) = \max_{\bar{D}} V \Rightarrow \frac{\partial V}{\partial \vec{n}}(x_0) \geq 0$

$\Rightarrow \frac{\partial u}{\partial \vec{n}}(x_0) + \varepsilon \frac{\partial W}{\partial \vec{n}}(x_0) \geq 0$

$\Rightarrow \frac{\partial u}{\partial \vec{n}}(x_0) \geq -\varepsilon \sum_{i=1}^{n} (-2A) e^{-A|x_0|^2} x_0 \frac{x_0}{R}$

$$= 2\varepsilon A R e^{-AR^2} > 0.$$

Proof of claim.

- $Lv \geq 0$ in D.

$$W = e^{-A|x|^2} - e^{-AR^2} \Rightarrow \partial_i W = -2A\, e^{-A|x|^2} x_i$$

$$\Rightarrow \partial_{ij} W = 4A^2 e^{-A|x|^2} x_i x_j - 2A\, e^{-A|x|^2} \delta_{ij}$$

$$\Rightarrow LW = e^{-A|x|^2}\left(4A^2 \sum_{ij} a_{ij} x_i x_j - 2A \sum_i a_{ii} - 2A \sum_i b_i x_i + c\right) - c\, e^{-AR^2}$$

$$(c \leq 0) \quad \geq e^{-A|x|^2}\left(4A^2 \sum_{ij} a_{ij} x_i x_j - 2A \sum_i a_{ii} - 2A \sum_i b_i x_i + c\right)$$

$$\left((a_{ij}) \geq \lambda I_n\right) \quad \geq e^{-A|x|^2}\left(4A^2 \lambda |x|^2 - 2A \sum_i a_{ii} - 2A \sum_i b_i x_i + c\right)$$

$$\left(\frac{R}{2} \leq |x| \leq R\right) \quad \geq e^{-A|x|^2}\left(\lambda A^2 R^2 - 2A \sum_i a_{ii} - 2AR \sum_i |b_i| + c\right)$$

$$\geq 0 \quad (\text{choose } A \gg 1)$$

$$\Rightarrow Lv = Lu - c\,u(x_0) + \varepsilon LW \geq 0 \quad (u(x_0) \geq 0,\ c \leq 0)$$

in D.

- $v \leq 0$ on ∂D.

$$\partial D = \partial B_{R/2} \cup \partial B_R$$

Recall $u(x) < u(x_0)$ in B_R.

On $\partial B_{R/2}$ $\exists \varepsilon > 0$. s.t. $u(x) < u(x_0) - \varepsilon$ on $\partial B_{R/2}$

$$W(x) = e^{-A|x|^2} - e^{-AR^2} \leq e^{-A|x|^2} \leq 1.$$

$$\Rightarrow v(x) = u(x) - u(x_0) + \varepsilon W(x) \leq -\varepsilon + \varepsilon = 0.$$

On ∂B_R. $W(x) = 0 \Rightarrow v(x) = u(x) - u(x_0) \leq 0.$

#

Remark. If $c=0$. then assumption $u(x_0)\geq 0$ can be removed.