Lecture 8

Moduli spaces and Hitchin morphisms

§1. Moduli spaces

$k = \mathbb{C}$

X smooth irreducible projective variety $/\mathbb{C}$

$(\mathrm{Sch}/\mathbb{C})$: category of $\mathbb{C}$-schemes

(Set) : category of sets

Def 1.1 We introduce the following 3 moduli functors: $(n \in \mathbb{Z}_{>0})$

(1) Betti moduli functor:

$$\widetilde{M}_B(X,n) : (\mathrm{Sch}/\mathbb{C})^{op} \to (\mathrm{Set})$$

via

- $\widetilde{M}_B(X,n)(S) := \left\{ E \to X \times S \;\; \begin{array}{l} \cdot\, E \text{ locally constant sheaf} \\ \quad \text{of } \mathbb{C}\text{-v.s. } \mathrm{rk} = n \\ \cdot\, \text{flat over } S \end{array} \right\} \Big/ \sim_S$

where $\sim_S$ means isomorphism:

$$E \sim_S E' \iff E \simeq E' \otimes \pi_S^* L \quad \text{for } L \to S \text{ line bundle}$$

$$\pi_S : X \times S \to S$$

- $f : S \to T$

$$\widetilde{M}_B(X,n)(f) := \underset{\substack{\parallel \\ (\mathrm{id}_X \times f)^*}}{f^*} : \widetilde{M}_B(X,n)(T) \to \widetilde{M}_B(X,n)(S)$$

(2) De Rham moduli functor:

$$\widetilde{M}_{dR}(X,n) : (\mathrm{Sch}/\mathbb{C})^{op} \to (\mathrm{Set})$$

via

- $\widetilde{M}_{dR}(X,n)(S) := \left\{ (\mathcal{E}, \nabla) \to X \times S : \begin{array}{l} \cdot\, \mathcal{E} \text{ vector bundle of } \mathrm{rk}\, n \\ \cdot\, \text{flat over } S \\ \cdot\, \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{X \times S} \text{ integrable conn.} \end{array} \right\} \Big/ \sim_S$

(3) Dolbeault moduli functor:

$$\widetilde{M}_{Dol}(X,n) : (\mathrm{Sch}/\mathbb{C})^{op} \to (\mathrm{Set})$$

via

- $\widetilde{M}_{Dol}(X, n)(S) := \left\{ (\mathcal{E}, \varphi) \to X \times S \;\middle|\; \begin{array}{l} \bullet\ (\mathcal{E}, \varphi)\ p\text{-semistable Higgs bundles of rk } n \\ \bullet\ \text{flat over } S \\ \bullet\ \forall s \in S(\mathbb{C}).\ c_i(\mathcal{E}_s) = 0 \text{ in } H^2 \quad (\mathcal{E}_s = \mathcal{E}|_{X \times \{s\}}) \\ \bullet\ P(\mathcal{E}) := P(\mathcal{E}(m)) = nP(\mathcal{O}_X) \end{array} \right\} \Big/ \sim_S$

- p-stability: $\dfrac{P(\mathcal{E}(m))}{a_d}$
- $c_i = 0$. p-semi-stability $=$ μ-semistability
  torsion free $\Rightarrow$ locally free.

Def 1.2 We also introduce the following 3 framed moduli functors:

(1) framed Betti ... :

$$\widetilde{R}_B(X, x, n) : (Sch/\mathbb{C})^{op} \to (Set)$$

via

- $\widetilde{R}_B(X, x, n)(S) := \left\{ (E, \beta) : \begin{array}{l} \bullet\ x \in X \\ \bullet\ E \cdots \\ \bullet\ \beta : E|_{\{x\} \times S} \simeq \mathcal{O}_S^n \end{array} \right\} \Big/ \sim_S$

(2) framed deRham ... :

$$\widetilde{R}_{dR}(X, x, n) : (Sch/\mathbb{C})^{op} \to (Set)$$

via

- $\widetilde{R}_{dR}(X, x, n)(S) := \left\{ (\mathcal{E}, \nabla, \beta) : \begin{array}{l} \bullet\ x \in X \\ \bullet\ (\mathcal{E}, \nabla) \cdots \\ \bullet\ \beta : \mathcal{E}|_{\{x\} \times S} \simeq \mathcal{O}_S^n \end{array} \right\} \Big/ \sim_S$

(3) framed Dolbeault ... :

$$\widetilde{R}_{Dol}(X, x, n) : (Sch/\mathbb{C})^{op} \to (Set)$$

via

- $\widetilde{R}_{Dol}(X, x, n)(S) := \left\{ (\mathcal{E}, \varphi, \beta) : \begin{array}{l} \cdots \\ \cdots \\ \cdots \end{array} \right\} \Big/ \sim_S$

Def 1.3: Let $\mathcal{M} : (Sch/\mathbb{C})^{op} \to (Set)$ moduli functor i.e.

$S \mapsto$ Iso. classes of S-flat family of ...

$f \mapsto f^* = (id \times f)^*$

A <u>fine moduli space</u> of $\mathcal{M}$ is a scheme $M \in Sch/\mathbb{C}$ s.t. M <u>represents</u> $\mathcal{M}$

A <u>coarse moduli space</u> of $\mathcal{M}$ is $M \in Sch/\mathbb{C}$ s.t. M <u>corepresents</u> $\mathcal{M}$

- represent: functor iso. $\eta : \mathcal{M} \xrightarrow{\sim} h_M := Hom(-, M)$
    - if exists, is unique up to iso.
    - $\exists$ universal family. element corresponds to $id_M \in h_M(M)$

$$\begin{array}{ccc} \mathcal{M}(M) & \overset{\eta_M}{\underset{\sim}{\longleftrightarrow}} & h_M(M) = Hom(M, M) \\ \mathcal{U} = \eta_M^{-1}(id_M) & \longleftarrow & id_M \end{array}$$

- corepresent: • bijective: $\mathcal{M}(\operatorname{spec}\mathbb{C}) \xrightarrow{\eta_{\operatorname{spec}\mathbb{C}}} h_M(\operatorname{spec}\mathbb{C})$

    i.e. $\mathbb{C}$-pts in $M \Leftrightarrow$ iso. classes of ...

    • universal property: $\forall$ natural trans. $\nu : \mathcal{M} \to h_T = Hom(-, T)$

    uniquely factors through $\eta : \mathcal{M} \to h_M$

$$\begin{tikzcd} & h_M \arrow[d, "h_f"] \\ \mathcal{M} \arrow[ur, "\eta"] \arrow[r, "\nu"'] & h_T \end{tikzcd} \quad \exists !\, f : M \to T$$

if exists, is unique up to iso.

Ex

(1) Quot scheme.

$X \in Sch/\mathbb{C}$, $\mathcal{F} \to X$ coherent sheaf

$\underline{Quot}_X(\mathcal{F}) : (Sch/\mathbb{C}) \to (Set)$

$S \mapsto \{ q_S : \pi_S^* \mathcal{F} \twoheadrightarrow \mathcal{E} \} / \sim_S$

$\exists$ fine moduli space, $Quot_X(\mathcal{F})$.

(2) Hilbert scheme: $X \in Sch/\mathbb{C}$

$\underline{Hilb}_X : (Sch/\mathbb{C}) \to (Set)$

$S \mapsto \{ Y \subset X \times S$ closed subscheme flat over $S \} / \sim$

$\exists$ fine moduli space. Hilb_X

But, the pathological behaviours s.t. no existence of moduli spaces:

(1) unboundedness: $\nexists F \in \mathcal{M}(S)$ s.t. $F|_{X \times \{s\}} \simeq \cdots$

(2) jump phenomenon: $\exists F \in \mathcal{M}(S)$ s.t. $\cdot\ F|_{X \times \{s_1\}} \sim F|_{X \times \{s_2\}}$ $\forall s_1, s_2 \neq s_0$

$\cdot\ F|_{X \times \{s\}} \not\sim F|_{X \times \{s_0\}}$ $\forall s \neq s_0$

one direction: add stability s.t. family becomes "small".

Thm 1.4 (Simpson)

(1) $\exists$ fine moduli spaces $R_B(X, x, n)$, $R_{dR}(X, x, n)$, $R_{Dol}(X, x, n)$ for the functors $\tilde{R}_B(X, x, n)$, $\tilde{R}_{dR}(X, x, n)$, $\tilde{R}_{Dol}(X, x, n)$. resp.

they are quasi-proj.

(2) $\exists$ coarse moduli spaces $M_B(X, n)$, $M_{dR}(X, n)$, $M_{Dol}(X, n)$ for the functors $\tilde{\mathcal{M}}_B(X, n)$, $\tilde{\mathcal{M}}_{dR}(X, n)$, $\tilde{\mathcal{M}}_{Dol}(X, n)$. resp.

they are quasi-proj.

Moreover, $M_B(X, n)$ is exactly the one we constructed before via affine GIT.

(3) $GL(n, \mathbb{C}) \curvearrowright R_B(X, x, n)$, $R_{dR}(X, x, n)$, $R_{Dol}(X, x, n)$

s.t.

$$R_B(X, x, n) \to M_B(X, n)$$
$$R_{dR}(X, x, n) \to M_{dR}(X, n)$$
$$R_{Dol}(X, x, n) \to M_{Dol}(X, n)$$

are GIT quotients. independent of $x \in X$

(4) By GIT, closed pts of M_B parametrizes closed orbits in R_B

M_{dR} — — M_{dR}

M_{Dol} — — M_{Dol}

i.e. iso. classes of polystable ...

or S-equivalent classes of semistable ...

Thm 1.5 (Nonabelian Hodge correspondence, moduli version. Simpson)

(1) Complex analytic iso (i.e. iso. as $\mathbb{C}$-analytic spaces)

$$M_B^{(an)}(X,n) \simeq M_{dR}^{(an)}(X,n)$$

(2) Real analytic iso. (i.e. homeor as top. spaces)

$$M_{dR}^{(top)}(X,n) \simeq M_{Dol}^{(top)}(X,n)$$

pf (sketch)

(1)

$$\begin{tikzcd} R_B^{(an)}(X,x,n) \arrow[r] \arrow[d, "//GL(n,\mathbb{C})"'] \arrow[dr, phantom, "\circlearrowright"] & R_{dR}^{(an)}(X,x,n) \arrow[d, "//GL(n,\mathbb{C})"] \\ M_B(X,n) \arrow[r] & M_{dR}(X,n) \end{tikzcd}$$

Key step: show $R_B^{(an)}(X,x,n)$ and $R_{dR}^{(an)}(X,x,n)$ represent the same analytic moduli functor:

$$R^{(an)}(X,x,n) : (\mathrm{Aan})^{op} \to (\mathrm{Set})$$

$$S^{an} \mapsto \{(\mathcal{F},\beta) : \cdot\ \mathcal{F} \to X^{an}\times S^{an} \text{ locally free sheaf of } \mathcal{O}_{X^{an}\times S^{an}}\text{-modules}, \ \cdot\ \beta : \mathcal{F}|_{\{x\}^{an}\times S} \simeq \mathcal{O}_{S^{an}}^n\}/\sim$$

$\cdot\ x \in X$

$$\Rightarrow R_B^{(an)}(X,x,n) \simeq R_{dR}^{(an)}(X,x,n)$$

$$\Rightarrow M_B^{(an)}(X,n) \simeq M_{dR}^{(an)}(X,n)$$

(2) Let $R_{Dol}^J(X,x,n) \subset R_{DR}(X,x,n)$ consists of:

- (E,φ,β) : $\cdot\ (E,\varphi) \to X$ Higgs bundles with pluri-harmonic metric h

  $\cdot\ \beta : E_x \simeq \mathbb{C}^n$

- $\beta(h) \simeq J$ ← standard Hm. product on $\mathbb{C}^n$.

i.e. consists of Higgs bundles admits pluri-harmonic metric compatible with the frame.

Similarly, define $R^J_{dR}(X,x,n) \subset R_{dR}(X,x,n)$

$$\begin{tikzcd}
R^J_{Dol}(X,x,n) \arrow[r] \arrow[d, "/U(n)"'] & R^J_{dR}(X,x,n) \arrow[d, "/U(n)"] \\
M^{(top)}_{Dol}(X,n) \arrow[r] & M^{(top)}_{dR}(X,n)
\end{tikzcd}$$

- Nonabelian Hodge of categorical version provides homeo.

$$R^J_{Dol}(X,x,n) \simeq R^J_{dR}(X,x,n)$$

- show $R^J_{Dol/dR}(X,x,n) \to M^{(top)}_{Dol/dR}(X,n)$ proper.

Dol: $(E_i, \bar\partial_{E_i}, \varphi_i, \beta_i)$ sequence of compatible frame Higgs bundles lying over some cpt $K \subset M_{Dol}$.

show. it converges.

- $\Rightarrow M^{(top)}_{Dol}(X,n) \simeq M^{(top)}_{dR}(X,n)$

$\boxtimes$

Ex ($n=1, m=1$) rk$=1$, dim $=1$ X cpt. R.S. $g \geq 2$. $x \in X$

- $M_B(X,1) = \mathrm{Hom}(\pi_1(X,x), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{2g}$ affine
- $M_{Dol}(X,1) = \{ (L,\varphi) : L \in \mathrm{Jac}(X),\ \varphi \in H^0(X,K_X) \}$

$\simeq \mathrm{Jac}(X) \times H^0(X,K_X)$

$\simeq T^*\mathrm{Jac}(X)$

- $M_{dR}(X,1) \to \mathrm{Jac}(X)$ affine bundle. i.e. fibers are affine spaces modelled $H^0(X,K_X)$

twisted cotangent bundle

$\mathbb{C}^* \simeq \mathbb{R}^{>0} \times S^1$

§2. $\mathbb{C}^*$-action and Hitchin morphism.

$M_{Dol}(X,n)$ admits an algebraic action of $\mathbb{G}_m = \mathbb{C}^*$:

$$\mathbb{C}^* \times M_{Dol}(X,n) \longrightarrow M_{Dol}(X,n)$$

$$(t, (\mathcal{E},\varphi)) \longmapsto (\mathcal{E}, t\varphi)$$

does not change the stability.

Denote by $M_{Dol}(X,n)^{\mathbb{C}^*}$ the set of $\mathbb{C}^*$-fixed pts.

<u>Lem 2.1</u> $(E, \bar\partial_E, \varphi) \in M_{Dol}(X,n)^{\mathbb{C}^*} \iff (E, \bar\partial_E, \varphi)$ is a system of Hodge bundles

i.e. $\left((E, \bar\partial_E) = \bigoplus_{i=1}^{\ell} (E_i, \bar\partial_{E_i}),\ \varphi = \begin{pmatrix} 0 & & & \\ \varphi_1 & 0 & & \\ & \ddots & \ddots & \\ & & \varphi_{\ell-1} & 0 \end{pmatrix} \right)$

$$\varphi_i : \mathcal{E}_i \to \mathcal{E}_{i+1} \otimes_{\mathcal{O}_X} \Omega^1_X$$

pf $[(E, \bar\partial_E, \varphi)] = [(E, \bar\partial_E, t\varphi)]$ for some $t \in \mathbb{C}^* \setminus U(1)$
not root of unity

$\iff (E, \bar\partial_E, \varphi)$ system of Hodge bundles

"$\Leftarrow$" If $(E, \bar\partial_E, \varphi)$ is, then

$$g_t = \begin{pmatrix} t^a \mathrm{Id}_{E_1} & & & \\ & t^{a+1}\mathrm{Id}_{E_2} & & \\ & & \ddots & \\ & & & t^{a+\ell-1}\mathrm{Id}_{E_\ell} \end{pmatrix}, \quad \forall a \in \mathbb{Z}$$

$$\Rightarrow \begin{cases} g_t \circ \bar\partial_E \circ g_t^{-1} = \bar\partial_E \\ g_t \circ \varphi \circ g_t^{-1} = t\varphi \end{cases}$$

"$\Rightarrow$" If $[(E, \bar\partial_E, \varphi)] = [(E, \bar\partial_E, t\varphi)]$

$\Rightarrow \exists\, g_t$ s.t. $\begin{cases} g_t \circ \bar\partial_E \circ g_t^{-1} = \bar\partial_E \\ g_t \circ \varphi \circ g_t^{-1} = t\varphi \end{cases} \Rightarrow \bar\partial_E(g_t) = 0$ i.e. g_t holo.

consider the $\det(\lambda \mathrm{Id} - g_t)$. coefficents are holo. functions on X

$\rightsquigarrow$ coefficents are constant.

$\rightsquigarrow$ eigenvalues of g_t are constant.

$$\Rightarrow \mathcal{E} = \bigoplus_\lambda \mathcal{E}_\lambda \quad \text{for } \mathcal{E}_\lambda := \ker\left((g_t - \lambda \mathrm{Id})^n\right)$$

$$g_t \circ \varphi \circ g_t^{-1} = t\varphi \iff g_t \circ \varphi = t\varphi \circ g_t$$

$$\Rightarrow (g_t - t\lambda \mathrm{Id})^n \circ \varphi = t^n \varphi \circ (g_t - \lambda \mathrm{Id})^n$$

$$\Rightarrow \varphi : \mathcal{E}_\lambda \to \mathcal{E}_{t\lambda}$$

i.e. $\lambda, \ t\lambda, \ t^2\lambda, \dots$

$\# \ t \in \mathbb{C}^* \setminus U(1)$. $\lambda, t\lambda, \dots, t^{\ell-1}\lambda$ are eigenvalues

but $t^{-1}\lambda, \ t^\ell \lambda$ not eigenvalues

$$\rightsquigarrow (\mathcal{E}, \varphi) = \left(\bigoplus_{i=1}^{\ell} \mathcal{E}_i, \ \varphi = \begin{pmatrix} 0 & & & \\ \varphi_1 & \ddots & & \\ & \ddots & \ddots & \\ & & \varphi_{\ell-1} & 0 \end{pmatrix} \right)$$

$\blacksquare$

From now on. work on $m=1$. i.e. X cpt. R.S. $g \geq 2$

Def 2.2 The affine space $A := \bigoplus_{i=1}^{n} H^0(X, \mathrm{Sym}^i K_X) = \bigoplus_{i=1}^{n} H^0(X, K_X^{\otimes i})$ is called the Hitchin base.

Hitchin map $h : \mathcal{M}_{Dol}(X, n) \longrightarrow A$

$$(\mathcal{E}, \varphi) \mapsto \text{coefficients of } \det(\lambda \mathrm{Id} - \varphi)$$

namely.

$$(\mathcal{E}, \varphi) \mapsto (a_1, \dots, a_n) \qquad a_i = (-1)^i \operatorname{Tr}(\wedge^i \varphi)$$

Rmk. (1) $\dim A \overset{\text{Riemann-Roch}}{=} n^2(g-1) + 1 = \frac{1}{2} \dim \mathcal{M}_{Dol}(X, n)$

(2) h can be defined for X has $m > 1$

$$h : M_{Dol}(X,n) \to A := \bigoplus_{i=1}^{n} H^0(X, \mathrm{Sym}^i \Omega_X^1)$$

(3) Motivation for h in Lie theory (especially for G)

G reductive gp $/\mathbb{C}$ $\qquad \mathfrak{g} = \mathrm{Lie}(G)$

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalg.

W Weyl group associated to $\mathfrak{h}$ $\qquad W \curvearrowright \mathfrak{h}$

$G \curvearrowright \mathfrak{g}$

Chevalley's restriction thm:

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\simeq} \mathbb{C}[\mathfrak{h}]^W$$

$\rightsquigarrow$

$\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\!\!/G \simeq \mathfrak{h}/W$ (chevalley morphism)

chose homog. poly. $p_1, \dots p_k$ of $d_1 \dots . d_k$ as deg. for $\mathbb{C}[\mathfrak{h}]^W$

as basis

$\rightsquigarrow$ iso $\mathfrak{h}/W \otimes K_X \simeq \bigoplus_{i=1}^{k} K_X^{\otimes d_i}$

$\rightsquigarrow$ induces a map $\quad M_{Dol}(X,n) \to \bigoplus_{i=1}^{k} H^0(X, K_X^{\otimes d_i})$

$(\mathcal{E}, \varphi) \mapsto (p_1(\varphi), \dots p_k(\varphi))$

Thm 2.3 (Hitchin $(n=1)$, Simpson $(n>1)$, Nitsure $(n=1$, L-twisted$)$)

$\varphi : \mathcal{E} \to \mathcal{E} \otimes L$

h is proper

Pf $\quad (\mathcal{E}, \bar\partial_E, \varphi)$ harmonic bundle. h harmonic metric

Lem 2.4 Given C_1. $\exists\, C_2$ s.t. all eigenvalue of φ is bounded

$$|\lambda_i|_g \le C_1$$

$$\Rightarrow \quad |\varphi|_{g,h} \le C_2$$

To show h proper. it suffices to show any sequence of harmonic bundles $(\mathcal{E}_i, \varphi_i, h_i)$ lying in the inverse of some cpt $K \subset A$.

it has a convergent subsequence.

By lemma. $\Rightarrow$ $|\varphi_i| \le C_2$

$$\underset{F_{D_{h_i}}}{\overset{\shortparallel}{F_{h_i}}} + [\varphi_i, \varphi_i^{*h_i}] = 0 \Rightarrow F_{h_i} = -[\varphi_i, \varphi_i^{*h_i}]$$

$$|F_{h_i}| \le C'$$

By Uhlenbeck's weak compactness thm.

$\Rightarrow$ $\exists$ unitary connection $\partial_h + \bar\partial$, subsequence $\{i'\}$, C^∞-auto. $g_{i'}$, φ

s.t.

$$g_{i'}^*(h_i) = h$$

$$g_{i'}^*(\bar\partial_{i'}) - \bar\partial, \quad g_{i'}^*(\partial_{h_{i'}}) - \partial_h, \quad g_{i'}^*(\varphi_{i'}) - \varphi.$$

$\to 0$ weakly in $W^{1,p}$

$\to 0$ strongly in L^p

$\Rightarrow$ $\bar\partial\varphi = 0$

$(\varphi \wedge \varphi = 0)$

$\bar\partial\varphi = 0$

$\Rightarrow$ $(\bar\partial, \varphi)$ Higgs bundle. h is harmonic metric for $(\bar\partial, \varphi)$

$$\nabla_h^2 = (\partial_h + \bar\partial + \varphi + \varphi^{*h})^2 = 0$$

i.e. $(\bar\partial_{h_i}, \varphi_i, h_i) \to (\bar\partial, \varphi, h)$ weakly ~~strongly~~ in $W^{1,p}$

strongly in L^p.

Elliptic regularity $\Rightarrow$ C^∞ convergent. ☒

Cor 2.5. h is surjective.

Rmk. Not true for $n > 1$.

pf (Beauville - Narasimhan - Ramanan)

$$T^*N^s_{(X,n)} \longrightarrow A \quad \text{dominant.}$$

$\cap \leftarrow$ open dense.

$M_{Dol}(X,n)$ $\qquad N^s(X,n)$ stable vect. b. moduli space

~~h~~ proper $\Rightarrow$ h is surjective.

$\blacksquare$

Rmk: h is $\mathbb{C}^*$-equivariant in the sense that:

$$h(\mathcal{E}.\varphi) = (a_1, \dots, a_n)$$

$$h(t.(\mathcal{E}.\varphi)) = (ta_1, \dots, t^n a_n)$$

$\mathbb{C}^* \curvearrowright \mathbb{A}$ weighted action

Cor 2.6 $\forall\, (\mathcal{E}.\varphi) \in M_{Dol}(X,n)$. $\lim_{t\to 0} t.(\mathcal{E}.\varphi)$ exists in $M_{Dol}(X,n)$

$\forall\, a \in \mathbb{A}$

$\lim_{t\to 0} t.a = 0$ $\quad$ h is $\mathbb{C}^*$-equiv. & proper $\Rightarrow$ $\lim_{t\to 0} t.(\mathcal{E}.\varphi) \in h^{-1}(0)$

As $\lim_{t\to 0} t.(\mathcal{E}.\varphi)$ is $\mathbb{C}^*$-fixed pt.

$\overset{\text{lem}}{\Longrightarrow}$ $M_{Dol}(X,n)^{\mathbb{C}^*}$ systems of Hodge bundles.

$$\forall\, (\mathcal{E}.\varphi) \in M_{Dol}^{\mathbb{C}^*} \quad \lim_{t\to 0} t(\mathcal{E}.\varphi) = (\mathcal{E}.\varphi)$$

$$\Rightarrow \quad M_{Dol}(X,n)^{\mathbb{C}^*} = \left\{ \lim_{t\to 0} t.(\mathcal{E}.\varphi) \;:\; (\mathcal{E}.\varphi) \in M_{Dol}(X,n) \right\}$$

Bialynicki-Birula:
get stratification of $M_{Dol}(X,n)$ into locally closed subsets

§3. Spectral correspondence (BNR correspondence)

As h is surjective.

Q: What is $h^{-1}(a)$ for general $a \in \mathbb{A}$?

Def 3.1 $\forall\, a = (a_1, \dots, a_n) \in \mathbb{A}$. the associated spectral curve is

$$\Sigma_a : \lambda^n + \sum_{i=1}^{n} a_i \lambda^{n-i} = 0 \quad \subset T^*X$$

In particular. $\Sigma_{(\mathcal{E},\varphi)} := \Sigma_{h(\mathcal{E},\varphi)}$

n-sheeted cover:

Rmk: Σ_a may be singular, reducible, non-reduced.

BNR: set $U := \{a \in \mathcal{A} : \Sigma_a \text{ smooth}\} \subset \mathcal{A}$ dense open

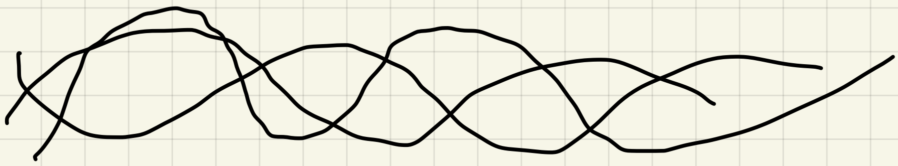
generic spectral curve is smooth.

Smoothness can be characterized via

$$\Delta_a := \prod_{i>j} (\lambda_i - \lambda_j)^2 \in H^0(X, K_X^{n(n-1)})$$

zeros of Δ_a

if Δ_a has only simple zero, then Σ_a is smooth.



Prop 3.2 For a generic Σ_a. it has $2n(n-1)(g-1)$ ramification pts

and genus $g(\Sigma_a) = n^2(g-1)+1 = \frac{1}{2}\dim M_{Dol}(X,n)$

pf.

$$\# = \deg(K_X^{n(n-1)}) = 2n(n-1)(g-1)$$

Riemann-Hurwitz formula:

$\pi_a : \Sigma_a \to \Sigma$ n-sheeted cover. ramif. index 2

$$\chi(\Sigma_a) = n \cdot \chi(X) - \#$$

$$\underset{2-2g(\Sigma_a)}{\shortparallel} \quad = n\cdot(2-2g) - 2n(n-1)(g-1)$$

$$\Rightarrow g(\Sigma_a) = n^2(g-1)+1.$$

∎

Thm 3.7 (BNR corresp. generic)

For generic Σ_a. we have the following bijective corresp.

$$\mathrm{Pic}^d(\Sigma_a) \longleftrightarrow h^{-1}(a)$$

$\uparrow$ iso. classes of line bundles of deg $d = n(n-1)(g-1)$

Higgs bundles in $h^{-1}(a)$ are stable

pf:

$L \in \mathrm{Pic}^d(\Sigma_a)$

$E := \pi_{a*}L \to X$ rk n bundle.

$$\pi_a : \Sigma_a \to \overset{K_X \atop \downarrow}{X} \qquad \pi_a^* K_X \qquad \pi^* K_X \text{ for } \pi : \mathrm{Tot}(K_X) \to X$$

tautological section σ. $\sigma(y) = y$

$\sigma_a := \sigma|_{\Sigma_a} \in \pi_a^* K_X$

$$L \xrightarrow{\cdot\sigma_a} L \otimes_{\mathcal{O}_{\Sigma_a}} \pi_a^* K_X \rightsquigarrow \pi_{a*}L \xrightarrow{\varphi} \pi_{a*}L \otimes_{\mathcal{O}_X} K_X$$

(E, φ)

Riemann-Roch.

$$\chi(X, \underset{\overset{\|}{E}}{\pi_{a*}L}) = \chi(\Sigma_a, L)$$

$$\deg(E) + n(1-g) \qquad\qquad d + (1-g)$$

$$\Rightarrow d = n(n-1)(g-1)$$

if (E,φ) not stable. the $\det(\lambda \mathrm{Id} - \varphi)$ reducible

$\Rightarrow \Sigma_a$ reducible. ↯

∎

Thm 3.4 (BNR)

Σ_a integral curve (irreducible + reduced) (can singular)

$$\overline{\mathrm{Pic}^d(\Sigma_a)} \longleftrightarrow h^{-1}(a)$$

$\uparrow$

iso. classes of torsion-free rk 1 sheaves on Σ_a

compactification of $\mathrm{Pic}^d(\Sigma_a)$ by adding torsion-free but not invertible sheaves of rk 1.