Recall: · Weak maximum principle

Ω: bounded domain in $\mathbb{R}^n$. $L = \sum_{ij} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$

$a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$ $(a_{ij}) \geq \lambda I_n$. $c \leq 0$.

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω, then

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$$

where $u^+ = \max(u, 0)$

Remark. If $c \equiv 0$, then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u$.

Some corollaries: Comparison principle, uniqueness of solutions.

· Hopf lemma:

$B = B_R(y_0)$. $x_0 \in \partial B$. L: elliptic operator

$a_{ij}, b_i, c \in L^\infty(B) \cap C(B)$. $(a_{ij}) \geq \lambda I_n$. $c \leq 0$.

If $u \in C^2(B) \cap C^1(\bar{B})$ satisfies

(1) $Lu \geq 0$ in B. (2) $u(x) < u(x_0)$ $\forall x \in B$. (3) $u(x_0) \geq 0$.

then $\frac{\partial u}{\partial \vec{n}}(x_0) > 0$, where $\vec{n}$ is the unit outward normal to ∂B.

Remark. If $c \equiv 0$, "$u(x_0) \geq 0$" can be removed.

Lecture 4. Maximum Principle (II).

1. Strong maximum principle

Theorem. Ω: bounded domain in $\mathbb{R}^n$

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$. $(a_{ij}) \geq \lambda I_n$. $c \leq 0$.

$a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$. $u \in C^2(\Omega) \cap C(\bar{\Omega})$. $Lu \geq 0$ in Ω

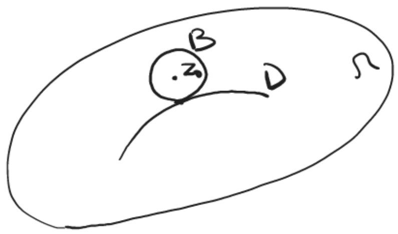
If $\exists\, y_0 \in \Omega$ s.t. $u(y_0) = \max_{\bar{\Omega}} u^+$, then u is constant.

Proof. Define $M = \max_{\bar{\Omega}} u^+ \geq 0$ and $D = \{x \in \Omega \mid u(x) = M\}$

$y_0 \in D \Rightarrow D \neq \phi$ and D is closed.

Goal: u is constant $\Leftrightarrow D = \Omega$

Argue by contradiction. If $D \neq \Omega$, then $\Omega \setminus D \neq \phi$ is open.



choose $z_0 \in \Omega \setminus D$ s.t.

$\operatorname{dist}(z_0, D) < \operatorname{dist}(z_0, \partial\Omega)$

Set $d_0 = \operatorname{dist}(z_0, D)$ and $B = B_{d_0}(z_0)$

Then
- $B \subset \Omega \setminus D$
- $\partial B \cap D \neq \phi$ assume $x_0 \in \partial B \cap D$

(1) $Lu \geq 0$ in B. (2) $u(x) < M = u(x_0)$ $\forall x \in B$. (3) $u(x_0) = M \geq 0$.

Hopf lemma $\Rightarrow \dfrac{\partial u}{\partial \vec{n}}(x_0) > 0$.

However, x_0 is an interior maximum point of u.

$\Rightarrow \nabla u(x_0) = 0 \Rightarrow \dfrac{\partial u}{\partial \vec{n}}(x_0) = 0 \Rightarrow$ Contradiction. #

Remark. When $c = 0$, the conclusion becomes

If $\exists y_0 \in \Omega$ s.t. $u(y_0) = \max_{\bar{\Omega}} u$, then u is constant.

Theorem. Ω: bounded domain in $\mathbb{R}^n$.

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$. $(a_{ij}) \geq \lambda I_n$. $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ and $u \leq 0$ in Ω,

then either $u < 0$ in Ω or $u \equiv 0$ in Ω.

Remark. No restriction on c. But we need $u \leq 0$ in Ω.

Proof. $c^+ = \max(c, 0)$, $c^- = \max(-c, 0)$. $\Rightarrow c = c^+ - c^-$

$Lu \geq 0 \Rightarrow \sum_{i,j} a_{ij}\partial_{ij} u + \sum_i b_i \partial_i u - c^- u \geq -c^+ u \underset{\substack{\uparrow \\ u \leq 0}}{\geq} 0$

Define $\tilde{L} = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + (-c^-)$

$\Rightarrow \tilde{L}u \geq 0$. $u \leq 0$ in Ω

$\Rightarrow \max_{\bar{\Omega}} u^+ = 0$

Case 1. $u(y) < 0 \quad \forall y \in \Omega$

Case 2. $\exists y_0 \in \Omega$ s.t. $u(y_0) = 0$

Strong maximum principle $\Rightarrow u \equiv 0$ in Ω #

2. A priori estimates

Motivation: We apply the continuity method to solve PDE.

$\rightsquigarrow$ Construct a sequence of function $\{u_i\}$

Suppose that $\{u_{i_j}\}$ is a convergent subsequence

Solution $\approx$ "limit" of $\{u_{i_j}\}$

A priori estimate: $\|u_i\|_{C^k} \leq C$ (independent of i)

Arzela-Ascoli theorem guarantees the existence of

Convergent subsequence.

Theorem. Ω: bounded domain in $\mathbb{R}^n$

L: elliptic operator. $(a_{ij}) \geq \lambda I_n$. $c \leq 0$. $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq f$ in Ω for $f \in L^\infty(\Omega) \cap C(\Omega)$

Then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+ + C \max_{\Omega} f^-$

where $C = C(n, \lambda, \operatorname{diam}(\Omega), \|b_i\|_{L^\infty(\Omega)})$

Proof. Set $d = \operatorname{diam}(\Omega)$. after translation. assume $\Omega \subset \{0 < x_1 < d\}$

Write $M = \max_{\partial\Omega} u^+ \geq 0$. $F = \max_{\Omega} f^- \geq 0$.

$$\Rightarrow \begin{cases} Lu = f \geq -F & \text{in } \Omega \\ u \leq M & \text{on } \partial\Omega \end{cases}$$

Define $V(x) = M + d^2 F(e^A - e^{\frac{Ax_1}{d}})$

Where $A > 0$ is a constant to be determined.

Compute $LV = -(a_{11}A^2 + db_1 A)Fe^{\frac{Ax_1}{d}} + cM + cd^2F(e^A - e^{\frac{Ax_1}{d}})$

$(c \leq 0)$ $\qquad \leq -(a_{11}A^2 + db_1 A)Fe^{\frac{Ax_1}{d}}$

$(a_{11} \geq \lambda)$ $\qquad \leq -(\lambda A^2 + db_1 A)Fe^{\frac{Ax_1}{d}}$.

Choose $A = A(\lambda, d, \|b_1\|_{L^\infty(\Omega)})$ s.t. $\lambda A^2 + db_1 A \geq 1$.

$\Rightarrow \quad LV \leq -Fe^{\frac{Ax_1}{d}} \underset{(x_1 > 0)}{\leq} -F \quad \text{in } \Omega$.

$\Omega \subset \{0 < x_1 < d\} \Rightarrow \quad V(x) = M + d^2F(e^A - e^{\frac{Ax_1}{d}}) \geq M \quad \text{in } \overline{\Omega}$

$\Rightarrow \quad V(x) \geq M \quad \text{on } \partial\Omega$

$$\Rightarrow \begin{cases} LV \leq -F \leq f = Lu & \text{in } \Omega \\ V \geq M \geq u & \text{on } \partial\Omega \end{cases}$$

Comparison Principle $\Rightarrow \quad u \leq V \quad \text{in } \Omega$

$\Rightarrow \quad u \leq M + d^2F(e^A - e^{\frac{Ax_1}{d}}) \quad \text{in } \Omega$

$\Rightarrow \quad \max_{\overline{\Omega}} u \leq M + d^2Fe^A$.

$M = \max_{\partial\Omega} u^+ \quad F = \max_{\Omega} f^- \quad \Rightarrow \quad \max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C \cdot \max_{\Omega} f^-$

where $C = d^2 e^A$ #

Corollary. Ω: bounded domain in $\mathbb{R}^n$. L: elliptic operator.

$(a_{ij}) \geq \lambda I_n$. $c \leq 0$. $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$.

If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$

where $f \in L^\infty(\Omega) \cap C(\Omega)$ and $\varphi \in C(\partial\Omega)$. Then

$$\max_{\bar\Omega} |u| \le \max_{\partial\Omega} |\varphi| + C \cdot \max_{\Omega} |f|$$

where $C = C(n, \lambda, \mathrm{diam}(\Omega), \|b_i\|_{L^\infty})$

Proof. Apply the above theorem to u and $(-u)$.

Proposition. $B = B_R(y_0) \subset \mathbb{R}^n$. L: elliptic operator

$(a_{ij}) \ge \lambda I_n$. $c \le 0$. $a_{ij}, b_i, c \in L^\infty(B) \cap C(B)$

$\beta \in (0,1)$. $f \in C(B)$ $\quad \sup_{x \in B} d_x^{2-\beta} |f(x)| < +\infty$. $\quad d_x = \mathrm{dist}(x, \partial B)$

If $u \in C^2(B) \cap C(\bar B)$ satisfies $\begin{cases} Lu = f & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$

then $\quad \sup_{x \in B} d_x^{-\beta} |u(x)| \le c \sup_{x \in B} d_x^{2-\beta} |f(x)|$.

where $C = C(n, \beta, \lambda, R, \|b_i\|_{L^\infty})$

Proof. After translation, assume $B = B_R(0) = B_R$

Set $F = \sup_{x \in B_R} d_x^{2-\beta} |f(x)|$. Note that $d_x = R - |x|$.

Goal: $\sup_{x \in B_R} d_x^{-\beta} |u(x)| \le C \cdot F \iff \forall x_0 \in B_R,\ |u(x_0)| \le C \cdot F \cdot d_{x_0}^{\beta}$

Fix $x_0 \in B_R$. after rotation, assume $x_0 = (r_0, 0, \cdots, 0)$ $\quad (r_0 \ge 0)$

$\Rightarrow d_{x_0} = R - r_0$.

Define $\quad W_1(x) = R^{-\beta} (R^2 - |x|^2)^\beta \quad W_2(x) = R^\beta (e^A - e^{\frac{Ax_1}{R}})$

$$W = \gamma_1 W_1 + \gamma_2 W_2$$

claim: choose A, γ_1, γ_2 suitably, s.t.

$$\begin{cases} LW \le -(R - |x|)^{\beta - 2} & \text{in } B_R \\ W \ge 0 & \text{on } \partial B_R \end{cases}$$

$F = \sup_{x \in B_R} d_x^{2-\beta} |f(x)|$

$\Rightarrow |f(x)| = d_x^{2-\beta} |f(x)| \cdot d_x^{\beta-2} \le F \cdot d_x^{\beta-2} = F (R - |x|)^{\beta-2}$.

$$\Rightarrow \begin{cases} L(\pm u) = \pm f \geq -F(R-|x|)^{\beta-2} \geq L(FW) & \text{in } B_R \\ \pm u = 0 \leq FW & \text{on } \partial B_R \end{cases}$$

Comparison principle $\Rightarrow$ $|u| \leq FW$ in B_R.

$$\Rightarrow |u(x_0)| \leq F W(x_0) \qquad x_0 = (r_0, 0, \cdots, 0) \quad (r_0 \geq 0)$$

$$\Rightarrow |u(x_0)| \leq \underbrace{F\gamma_1 R^{-\beta}(R^2-r_0^2)^{\beta}}_{I_1} + \underbrace{F\gamma_2 R^{\beta}(e^{A} - e^{\frac{Ar_0}{d}})}_{I_2}$$

$$I_1 = F\cdot\gamma_1\cdot R^{-\beta}(R+r_0)^{\beta}(R-r_0)^{\beta} \leq F\gamma_1\cdot R^{-\beta}(2R)^{\beta} d_{x_0}^{\beta} \leq CF d_{x_0}^{\beta}$$

$$I_2 = F\gamma_2 R^{\beta} e^{\frac{Ar_0}{R}}\left(e^{A(1-\frac{r_0}{R})} - 1\right) \underset{\uparrow}{\leq} F\gamma_2 R^{\beta} e^{\frac{Ar_0}{R}} A e^{A}\left(1-\frac{r_0}{R}\right)$$

$$\left(e^{Ax} - 1 \leq Ae^{A}\cdot x \quad x\in[0,1]\right)$$

$$\leq F\cdot\gamma_2 R^{\beta} e^{A}\cdot A\cdot e^{A}\cdot\frac{1}{R}(R-r_0) \leq CF d_{x_0}^{\beta}$$

$$\Rightarrow |u(x_0)| \leq CF d_{x_0}^{\beta} \quad \Rightarrow \text{Goal.}$$

Recall Claim: $W_1(x) = R^{-\beta}(R^2-|x|^2)^{\beta}$ $\quad W_2(x) = R^{\beta}(e^{A} - e^{\frac{Ax_1}{d}})$

$W = \gamma_1 W_1 + \gamma_2 W_2$ after choosing A, γ_1, γ_2 suitably.

$$\begin{cases} LW \leq -(R-|x|)^{\beta-2} & \text{in } B_R \\ W \geq 0 & \text{on } \partial B_R \end{cases}$$

Proof of Claim.

Step 1. Term W_1.

$$\partial_i W_1(x) = -2\beta R^{-\beta}(R^2-|x|^2)^{\beta-1} x_i$$

$$\partial_{ij} W_1(x) = 4\beta(\beta-1)R^{-\beta}(R^2-|x|^2)^{\beta-2} x_i x_j - 2\beta R^{-\beta}(R^2-|x|^2)^{\beta-1}\delta_{ij}$$

$$LW_1(x) = \beta R^{-\beta}(R^2-|x|^2)^{\beta-2}\Big[\underbrace{4(\beta-1)\sum_{i,j} a_{ij}x_i x_j} - \underbrace{2(R^2-|x|^2)\sum_i a_{ii}}_{\leq 0} - \underbrace{2(R^2-|x|^2)\sum_i b_i x_i}\Big] + \underbrace{cR^{-\beta}(R^2-|x|^2)^{\beta}}_{\leq 0}$$

$$\leq \beta R^{-\beta}(R^2-|x|^2)^{\beta-2}\left[-4(1-\beta)\lambda|x|^2 + 2nM_b R(R^2-|x|^2)\right]$$

$$M_b = \max_i \|b_i\|_{L^\infty(B_R)}$$

choose $\theta \in (0,1)$ (close to 1) s.t. $2nM_b R(1-\theta^2) \leq 3(1-\beta)\lambda\theta^2$

If $\theta R \leq |x| < R$ then

$$\begin{aligned}
LW_1(x) &\leq \beta R^{-\beta}(R^2-|x|^2)^{\beta-2}\left[-(1-\beta)\lambda|x|^2 - 3(1-\beta)\lambda\theta^2 R^2 \right.\\
&\qquad\left. + 2nM_b R(1-\theta^2)R^2\right] \\
&\leq -\beta R^{-\beta}(R^2-|x|^2)^{\beta-2}(1-\beta)\lambda|x|^2 \\
&= -\beta R^{-\beta}(R-|x|)^{\beta-2}(R+|x|)^{\beta-2}(1-\beta)\lambda|x|^2 \\
&\leq -\beta R^{-\beta}(R-|x|)^{\beta-2}(2R)^{\beta-2}(1-\beta)\lambda\theta^2 R^2 \\
&\leq -C_1^{-1}(R-|x|)^{\beta-2}
\end{aligned}$$

If $|x| < \theta R$. $\quad LW_1(x) \leq \beta R^{-\beta}(R^2-|x|^2)^{\beta-2} \cdot 2n M_b R(R^2-|x|^2)$

$$\leq C_2(R-|x|)^{\beta-2}$$

$$\Rightarrow \quad LW_1(x) \leq \begin{cases} -C_1^{-1}(R-|x|)^{\beta-2} & \theta R \leq |x| < R \\ C_2(R-|x|)^{\beta-2} & |x| < \theta R \end{cases}$$

Step 2. Term W_2

$$W_2(x) = R^\beta(e^A - e^{\frac{Ax_1}{d}})$$

$$\Rightarrow \quad LW_2(x) = -(a_{11}A^2 + b_1 RA)R^{\beta-2}e^{\frac{Ax_1}{R}} + cR^\beta(e^A - e^{\frac{Ax_1}{d}})$$

(choose $A \gg 1$) $\quad \leq -R^{\beta-2}e^{\frac{Ax_1}{R}} \leq -e^{-A}R^{\beta-2}$

If $\theta R \leq |x| < R$, $\quad LW_2(x) \leq 0$.

If $|x| \leq \theta R$. $\quad LW_2(x) \leq -e^{-A}(1-\theta)^{2-\beta}(R-\theta R)^{\beta-2}$

$$\begin{aligned}
&\leq -e^{-A}(1-\theta)^{2-\beta}(R-|x|)^{\beta-2} \\
&\leq -C_3^{-1}(R-|x|)^{\beta-2}
\end{aligned}$$

$$\Rightarrow \quad LW_2(x) \le \begin{cases} 0 & \theta R \le |x| < R \\ -C_3^{-1}(R-|x|)^{\beta-2} & |x| < \theta R. \end{cases}$$

choose $\gamma_1 = C_1$. $\gamma_2 = (C_1 C_2 + 1) C_3$.

$W = \gamma_1 W_1 + \gamma_2 W_2$ satisfies

$$\begin{cases} LW \le -(R-|x|)^{\beta-2} & \text{in } B_R \\ W \ge 0 & \text{on } \partial B_R \end{cases} \Rightarrow \text{claim.} \qquad \#$$