Introduction to K-stability @ Nankai by Yuchen Liu.

Lecture 2. α-invariants and stability thresholds.

§ Singularity of pairs.

A pair (X, D) : X normal variety (quasi-proj) $\Leftrightarrow$ R_1 (smooth away from codim ≥ 2) + S_2 (Hartog)

$D \geq 0$ $\mathbb{Q}$-divisor

s.t. $K_X + D$ is $\mathbb{Q}$-Cartier, i.e. $\exists\, m \in \mathbb{N}_{>0}$

s.t. $m(K_X + D)$ is Cartier.

<u>Ex</u>. If X is smooth (or $\mathbb{Q}$-factorial), then (X, D) is a pair.

Log discrepancy : $E \overset{\text{div}}{\subseteq} Y \xrightarrow[\mu]{\text{bir}} X$.

$$A_X(E) := 1 + \mathrm{coeff}_E(K_Y - \mu^* K_X).$$

For a pair (X,D), $E \overset{\text{div}}{\subseteq} Y \xrightarrow[\mu]{\text{bir}} X$,

$$A_{X,D}(E) := 1 + \mathrm{coeff}_E(K_Y - \mu^*(K_X + D)).$$

If K_X and D are both $\mathbb{Q}$-Cartier, then

$$A_{X,D}(E) = A_X(E) - \mathrm{ord}_E(D).$$

<u>Def.</u> We say a pair (X,D) is

prime divisor over X

- klt if $A_{X,D}(E) > 0$ for $\forall E$ div/X.

Kawamata log terminal

- log canonical if $A_{X,D}(E) \geq 0$ for $\forall E$ div/X.

By remark next slide, it suffices to choose one Y that is a log resolution.

Ex. ① X smooth, $\mathrm{Supp}\, D$ is simple normal crossing.

(e.g. $X = \mathbb{A}^2$, $D = a\{x=0\} + b\{y=0\}$, $\mathrm{Supp}\, D = \{xy=0\}$).

(X, D) is klt $\iff$ all coeff.s $D < 1$.

lc $\iff$ all coeff.s $D \leq 1$.

* If $E \overset{\text{div}}{\subseteq} X$, then $A_{X,D}(E) = 1 - \mathrm{coeff}_E(D)$.

Rmk. In checking (X, D) is klt/lc or not,

we only need a log resolution $Y \to (X, D)$

and check all divisors E on Y.

Terminal: log discrepancy > 1

canonical: log discrepancy ≥ 1.

② If $\dim X = 2$, $D = 0$.

Then X is klt $\Longleftrightarrow$ X has quotient singularities.

i.e. $\forall x \in X$ $\exists\, G_x \le GL(2,\mathbb{C})$ finite group acting linearly on $\mathbb{C}^2$ s.t. $(x \in X)^{an} \cong 0 \in \mathbb{C}^2/G_x$.

canonical 2-dim'l sing.

$\Updownarrow$

e.g. $\boxed{\text{ADE singularity}}$: $G \leq SL(2,\mathbb{C})$.

The canonical divisor of $\mathbb{C}^2/G$ is Cartier

($\forall g \in G$, $g^*(dx \wedge dy) = \det(g)\, dx \wedge dy = dx \wedge dy$.

i.e. $\omega_{\mathbb{C}^2}$ has trivial monodromy under G).

If $\dim X \geq 3$, $\{\text{quotient sing}\} \subsetneq \{\text{klt sing.}\}$.

klt, not ADE :

$d \geq 3$

$\underset{u,\,v,\,w}{\mathbb{P}(1,1,d)}$ = projective cone over rational normal curve of deg d in $\mathbb{P}^d$.

singular pt $[0,0,1]$.

Affine chart $(w=1) \cong \mathbb{C}^2/(\mathbb{Z}/d\mathbb{Z})$.

$\forall [k] \in \mathbb{Z}/d\mathbb{Z}$, $[k]\cdot(x,y) = \left(e^{\frac{2\pi i k}{d}}x,\ e^{\frac{2\pi i k}{d}}y\right)$.

$\det([1]) = e^{\frac{4\pi i}{d}} \neq 1$ as $d \geq 3$.

Let E be the exc. div. on the blow-up, $0 < A_X(E) = \frac{2}{d} < 1$.

Def. (log canonical threshold)

X : klt variety. $D \geq 0$ $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

$$\mathrm{lct}(X; D) := \overset{\text{max.}}{\sup}\{c \mid (X, cD) \text{ is log canonical}\}.$$

Ex ① $X = \mathbb{A}^2$, $D = a\{x=0\} + b\{y=0\}$. $0 < a \leq b$.

$$(X, cD) \text{ is lc} \iff ca \leq 1 \text{ and } cb \leq 1.$$

$$\iff c \leq \frac{1}{b}.$$

$$\mathrm{lct}(X, D) = \frac{1}{b}.$$

Prop. If $\mu : Y \to (X, D)$ is a log resolution.

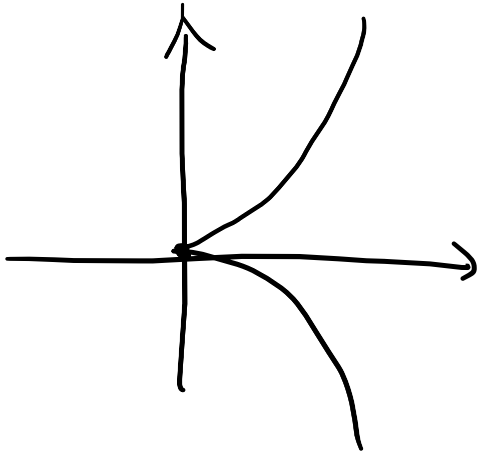
Then
$$\mathrm{lct}(X;D) = \min_{E_i} \frac{A_X(E_i)}{\mathrm{ord}_{E_i}(D)}$$

where $\{E_i\} = \{\text{irred. comp. of } D\} \cup \mathrm{Exc}(\mu)$

exceptional divisors.

Ex. $X = \mathbb{A}^2$, $D = (y^2 - x^3 = 0)$.

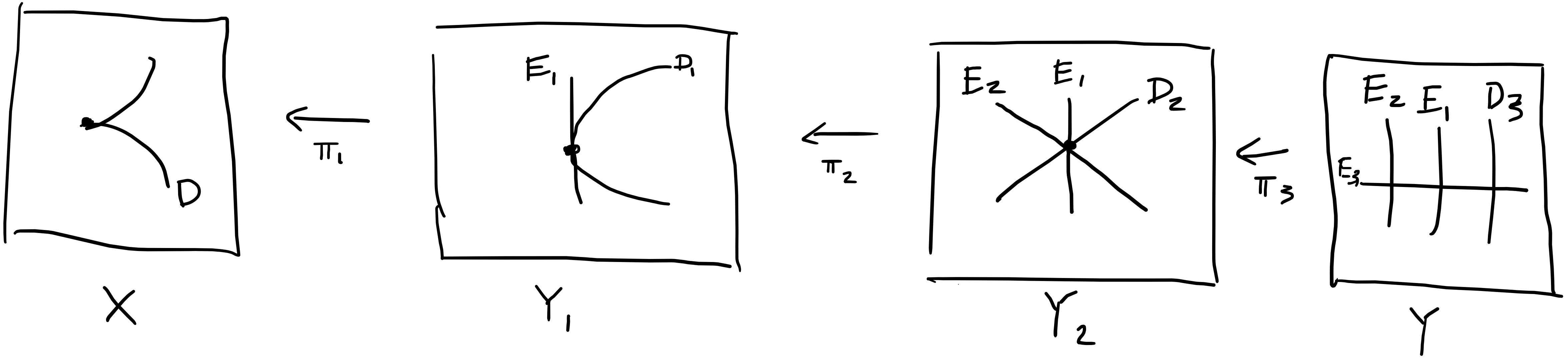
Compute $\mathrm{lct}(X;D)$.



Log resolution : 3 blow-ups

$\tilde{\pi}_2 = \pi_1 \circ \pi_2$, $\tilde{\pi}_3 = \tilde{\pi}_2 \circ \pi_3$

$(x_1, x_1 y_1) \longleftarrow (x_1, y_1)$



$A_X(E_1) = 2$.

$\mathrm{ord}_{E_1}(D) = 2$

$A_X(E_2) = 3$

$\mathrm{ord}_{E_2}(D) = 3$

$A_X(E_3) = 5$

$\mathrm{ord}_{E_3}(D) = 6$

$K_{Y_1} = \pi_1^* K_X + E_1$

$K_{Y_2} = \pi_2^* K_{Y_1} + E_2$

$= \tilde{\pi}_2^* K_X + E_1 + 2E_2$.

$K_{Y_3} = \pi_3^* K_{Y_2} + E_3$

$= \tilde{\pi}_3^* K_X + E_1 + 2E_2 + 4E_3$

Apply prop. $\Rightarrow$ $\boxed{\mathrm{lct}(X; D) = 5/6}$.

$\pi_1^* D = D_1 + 2E_1$

$\tilde{\pi}_2^* D = \pi_2^*(\pi_1^* D)$

$= D_2 + 2E_1 + 3E_2$

$\tilde{\pi}_3^* D = D_3 + 2E_1 + 3E_2 + 6E_3$.

Def [Tian'90]. X : klt Fano variety ($\mathbb{Q}$-Fano).

$m \in \mathbb{N}$.

min

We define $\alpha_m(X) := \inf\limits_{mD \in |-mK_X|} \operatorname{lct}(X; D)$.

$\exists\, s \in H^0(X, -mK_X)$ s.t. $D = \frac{1}{m}(s=0)$.

The <u>α-invariant</u> of X is defined as

$$\alpha(X) = \inf_{m \in \mathbb{N}} \alpha_m(X) = \lim_{m \to \infty} \alpha_m(X).$$

<u>Def</u>. [Fujita-Odaka '16]. X : $\mathbb{Q}$-Fano variety. $m \in \mathbb{N}$.

An m-basis type divisor D on X is

$$D = \frac{1}{m N_m}\left((s_1 = 0) + \cdots + (s_{N_m} = 0) \right) \sim_{\mathbb{Q}} -K_X.$$

where $N_m = \dim H^0(-mK_X)$, $(s_1, \ldots, s_{N_m})$ basis of $H^0(-mK_X)$.

We define $$\delta_m(X) = \inf_{D:\, m\text{-basis type}} \operatorname{lct}(X; D).$$

The <u>stability threshold</u> (δ-invariant) of X is

$$\delta(X) = \lim_{m \to \infty} \delta_m(X).$$

<u>Prop</u>.

$$\alpha(X) = \inf_{E \text{ div}/X} \frac{A_X(E)}{T_X(E)} \qquad \sup\{t \mid \mu^*(-K_X) - tE \text{ is big}\}$$

$$\delta(X) = \inf_{E \text{ div}/X} \frac{A_X(E)}{S_X(E)} \qquad \frac{1}{(-K_X)^n}\int_0^{T_X(E)} \mathrm{vol}(\mu^*(-K_X) - tE)\,dt$$

★<u>Thm</u> (Fujita-Odaka, Blum-Jonsson).

X is K-semistable <u>iff</u> $\delta(X) \geq 1$.

(Valuative criteria : K-ss $\Longleftrightarrow$ $\beta_X(E) \geq 0 \quad \forall E$ div/X

$\Longleftrightarrow$ $A_X(E) \geq S_X(E) \quad \forall E$

$\overset{\text{Prop}}{\Longleftrightarrow}$ $\delta(X) \geq 1$)

Pf of prop. By definition $\alpha_m(X) = \inf_{mD \in |-mK_X|} \operatorname{lct}(X;D)$.

$$\operatorname{lct}(X;D) = \min_{E \text{ div}/X} \frac{A_X(E)}{\operatorname{ord}_E(D)}.$$

$$\Rightarrow \quad \alpha_m(X) = \inf_{mD \in |-mK_X|} \left(\min_{E \text{ div}/X} \frac{A_X(E)}{\operatorname{ord}_E(D)} \right)$$

$$= \inf_{E \text{ div}/X} \left(\inf_{mD \in |-mK_X|} \frac{A_X(E)}{\operatorname{ord}_E(D)} \right) = \inf_{E \text{ div}/X} \frac{A_X(E)}{\sup_{mD \in |-mK_X|} \operatorname{ord}_E(D)}.$$

Not hard to show $\lim_{m \to \infty} \overbrace{\sup_{mD \in |-mK_X|} \operatorname{ord}_E(D)}^{T_m(E)} = T_X(E)$. (uniform convergence)

$$\Rightarrow \quad \alpha(X) = \inf_{E \text{ div}/X} \frac{A_X(E)}{T_X(E)}.$$

By similar arguments, in order to show $\delta(X) = \inf_E \frac{A_X(E)}{S_X(E)}$

it suffices to show

$$\lim_{m\to\infty} \sup_{D:\, m\text{-basis type}} \mathrm{ord}_E(D) = S_X(E). \quad \text{(uniform)}$$

$$\mathrm{ord}_E(D) = \frac{1}{mN_m} \sum_{i=1}^{N_m} \mathrm{ord}(s_i).$$

#

See [Blum-Jonsson (2020), Adv. in Math.].

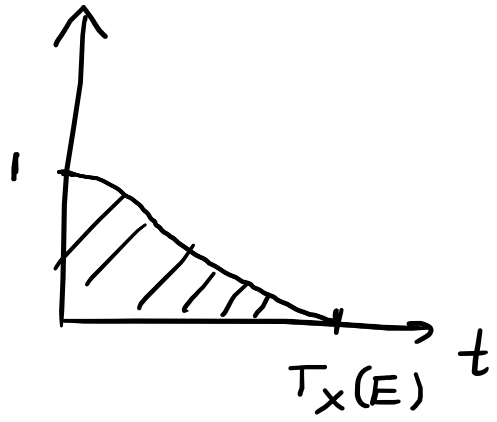
__Lemma__. $\frac{1}{n+1}\delta(X) \leq \alpha(X) \leq \frac{n}{n+1}\delta(X)$, $n = \dim X$.

Pf. $\alpha(X) = \inf_E \frac{A_X(E)}{T_X(E)}$, $\delta(X) = \inf_E \frac{A_X(E)}{S_X(E)}$

It suffices to show $\frac{1}{n+1} T_X(E) \leq S_X(E) \leq \frac{n}{n+1} T_X(E)$.

$$S_X(E) = \int_0^{T_X(E)} \frac{\mathrm{vol}(-K_X - tE)}{\mathrm{vol}(-K_X)}\, dt.$$

$f(t)$.



Brunn-Minskowski inequality

$\Longrightarrow$ $\mathrm{vol}(-K_X - tE)^{1/n}$ is decreasing concave in t.

Okounkov body [Lazarsfeld-Mustață].

#

Theorem (Tian's criterion)

If $\alpha(X) \geq \frac{n}{n+1}$, then X is K-semistable.

$ >$ K-stable.

Theorem (Fujita–Odaka)

If X is K-semistable, then $\alpha(X) \geq \frac{1}{n+1}$.

Ex. $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$. ($\mathbb{P}^n$ is K-polystable)

① Take $D = (n+1) \cdot (x_0 = 0) \in |-K_{\mathbb{P}^n}| = |\mathcal{O}(n+1)|$.

$\operatorname{lct}(\mathbb{P}^n; D) = \frac{1}{n+1} \geq \alpha(\mathbb{P}^n)$.

② Lower semicontinuity of lct. $\forall D \in \frac{1}{m}|-mK_X|$

By torus action, $\exists$ isotrivial degeneration $D \rightsquigarrow D_0$ toric divisor.

so $D_0 = \sum_{i=0}^{n} c_i (x_i = 0) \sim_{\mathbb{Q}} -K_X \Rightarrow \sum_{i=0}^{n} c_i = n+1$

$$\mathrm{lct}(X; D) \geq \mathrm{lct}(X; D_0)$$

$$= \frac{1}{\max_{0 \leq i \leq n} \{c_i\}} \geq \frac{1}{n+1}.$$

i.e. $\alpha(\mathbb{P}^n) \geq \frac{1}{n+1}$.

① + ② $\Rightarrow \alpha(\mathbb{P}^n) = \frac{1}{n+1}$.

Degeneration. $D \in |\mathcal{O}(n+1)|$. $\quad D = (f=0)$

$$f = \sum C_{\underline{m}} x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n} \qquad \sum_i m_i = n+1.$$

$$\begin{aligned} x_0 &\longmapsto t^n x_0 \\ x_1 &\longmapsto t^{n-1} x_1 \\ &\vdots \\ x_n &\longmapsto x_n. \end{aligned}$$

let $t \to 0$.

If we choose generic weight, only 1 monomial remains.

$\qquad\qquad\qquad\qquad \parallel$

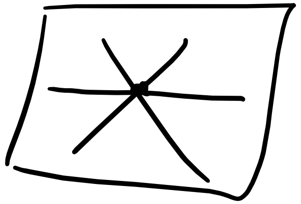
$\qquad\qquad\qquad\qquad f_0$.

Ex. (Cheltsov)

$X \subseteq \mathbb{P}^3$ smooth cubic surface.

then $$\alpha(X) = \begin{cases} \frac{3}{4} & \text{if } X \text{ has no Eckardt pt} \Rightarrow \text{K-stable} \\ \frac{2}{3} & \text{if } X \text{ has Eckardt pt.} \Rightarrow \text{K-semistable} \end{cases}$$

Eckardt pt: $\exists$ 3 coplanar line in X through the same pt



L.-Zhuang: singular not true

Fujita: If X is smooth Fano,

$$\alpha(X) = \frac{n}{n+1} \Rightarrow X \text{ is K-stable}$$

<u>Cheltsov-Park</u>. $X \subseteq \mathbb{P}^{n+1}$ degree $n+1$ smooth hypersurface

$$\Rightarrow \quad \alpha(X) \geq \frac{n}{n+1}. \quad \overset{\text{prop}}{\Longrightarrow} \quad \delta(X) \geq 1$$

$\overset{\text{Fujita}}{\Longrightarrow}$ X is K-stable. ([Kento Fujita: Jussieu '19]).

[Abban-Zhuang '20, '21]. index 2, index $\leq n^{\frac{1}{3}}$.

[L.-Xu '19, L. '20] cubic 3-folds, cubic 4-folds

Ex. a singular cubic surface:

$$X = (xyz + w^3 = 0) \subseteq \mathbb{P}^3.$$

$$X \cong \mathbb{P}^2 / (\mathbb{Z}/3\mathbb{Z}). \qquad [x_0, x_1, x_2] \mapsto [x_0, \xi x_1, \xi^2 x_1]$$

$$\xi = e^{2\pi i/3}.$$

Berman, Robert: explicit KE metrics.