Introduction to K-stability @ Nankai by Yuchen Liu.

Lecture 4. K-moduli theory II.

• Recall. Boundedness: Fix $n = \dim X$, $V = (-K_X)^n$

Then $\{X$ $\mathbb{Q}$-Fano variety, K-semistable$\}$ form a bounded collection.

In other words, $\exists\, m = m(n, V)$ s.t.

$|-mK_X| : X \hookrightarrow \mathbb{P}^N$, N has an upper bound

($|-mK_X|$ very ample.)

Openness: In a $\mathbb{Q}$-Gorenstein family $\mathcal{X} \to T$ of $\mathbb{Q}$-Fano varieties,

$$\{ t \in T \mid \mathcal{X}_t \text{ is K-semistable} \} \subseteq T \text{ is Zariski open.}$$

Boundedness + Openness $\Rightarrow$ $\exists$ K-moduli stack $\underline{\underline{\mathcal{M}^{Kss}_{n,V}}}$

parametrize <u>K-semistable</u> $\mathbb{Q}$-Fano varieties X with $\dim X = n$ & $(-K_X)^n = V$.

$\mathcal{M}^{Kss}_{n,V}$ is an Artin stack of finite type (in fact, it is a quotient stack)

In order to construct K-moduli space, we need 3 more steps.

- Separatedness (Hausdorff)
- Properness (Compactness)
- Projectivity.

Ex: moduli of curves $\overline{M}_g$ $(g \geq 2)$: stack $\overline{\mathcal{M}}_g$ is separated

Deligne-Mumford stack. Every object has finite stabilizer (finite Automorphisms).

C smooth $|\underset{\parallel}{G}\ \mathrm{Aut}(C)| \leq 84(g-1)$. Hurwitz bound. C/G orbifold of general type.

For K-moduli of Fano varieties, Aut could be infinite.

<u>Ex</u>. • $\mathbb{P}^n$ is K-polystable.

$$\mathrm{Aut}(\mathbb{P}^n) = PGL(n+1)$$

• cubic surfaces: every smooth cubic surface is K-stable.

$\exists$ singular cubic surface $X_0 = \{xyz + w^3 = 0\} \subseteq \mathbb{P}^3$ which is K-polystable.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & T \\ & & \Psi \\ X_0 & \longmapsto & 0 \end{array}$$

$\mathcal{X}_t$ smooth cubic surface $(t \neq 0)$

X_0 singular cubic surface as above.

$\mathrm{Aut}(\mathcal{X}_t)$ is finite, $\mathrm{Aut}(X_0) \supseteq (\mathbb{C}^*)^2$. infinite.

- Mukai-Umemura 3-fold. (smooth)

$\mathrm{Aut}^0(X_{MU}) \cong PGL(2)$. X_{MU} is K-polystable ($\exists$ KE)

$\exists$ smoothing (X_t) of $X_{MU} = X_0$ s.t. $\mathrm{Aut}(X_t)$ is finite for $t \neq 0$.

[Tian'97] $\exists$ smoothing s.t. X_t is K-semistable not K-polystable.

X_t doesn't admit KE, & $\mathrm{Aut}(X_t)$ is finite.

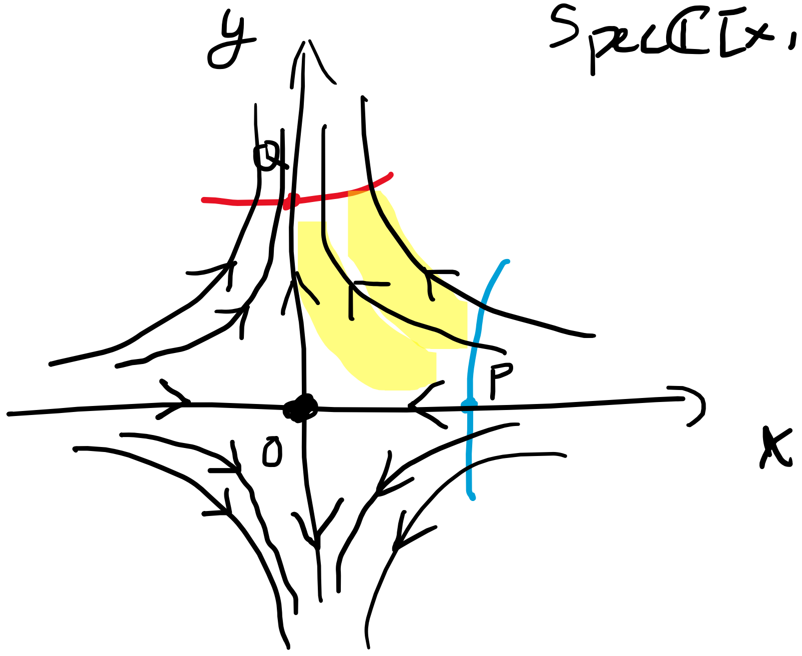
In dim 2, X smooth + $\mathrm{Aut}(X)$ finite $\Rightarrow$ X admits KE.

Consider the affine GIT :

$xy \mapsto (tx)(t^{-1}y) = xy$.

<u>Ex</u>. $\mathbb{C}^* \curvearrowright \mathbb{C}^2$ by $t\cdot(x,y) = (tx, t^{-1}y)$.

$\mathbb{C}^2 = \operatorname{Spec}\mathbb{C}[x,y]$



two pts $P, Q \in \mathbb{C}^2$ are **S-equivalent** if $\overline{O}_P \cap \overline{O}_Q \neq \phi$.

In this picture, $O_P = \mathbb{C}_x \setminus \{0\}$

$O_Q = \mathbb{C}_y \setminus \{0\}$.

$\overline{O}_P \cap \overline{O}_Q = \{0\}$.

GIT quotient

$$\mathbb{C}^2 /\!/ \mathbb{C}^* = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{C}^*} = \operatorname{Spec} \mathbb{C}[xy] \cong \mathbb{C}.$$

$\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2/\!/\mathbb{C}^* \cong \mathbb{C}$ is given by $\pi(x,y) = xy$.

$\pi(P) = \pi(Q) \iff P$ and Q are S-equivalent.

If $z \in \mathbb{C}^2 /\!/ \mathbb{C}^*$ is non-zero,

$$\pi^{-1}(z) = \{ (x,y) \in \mathbb{C}^2 \mid xy = z \}$$

$= \text{single orbit} = O_{(z,1)}$.

If $z = 0$, $\pi^{-1}(0) = \{ (x,y) \in \mathbb{C}^2 \mid xy = 0 \}$

$$= O_{(1,0)} \sqcup O_{(0,1)} \sqcup O_{(0,0)}.$$

polystable

$\Updownarrow$

$O_{(0,0)}$: unique closed orbit

3 orbits here are S-equivalent.

Affine GIT

$X = \operatorname{Spec} R$, G reductive gp.

$G \curvearrowright X$.

$X /\!/ G = \operatorname{Spec} R^G$. ← finitely generated

$\pi : X \longrightarrow X /\!/ G$.

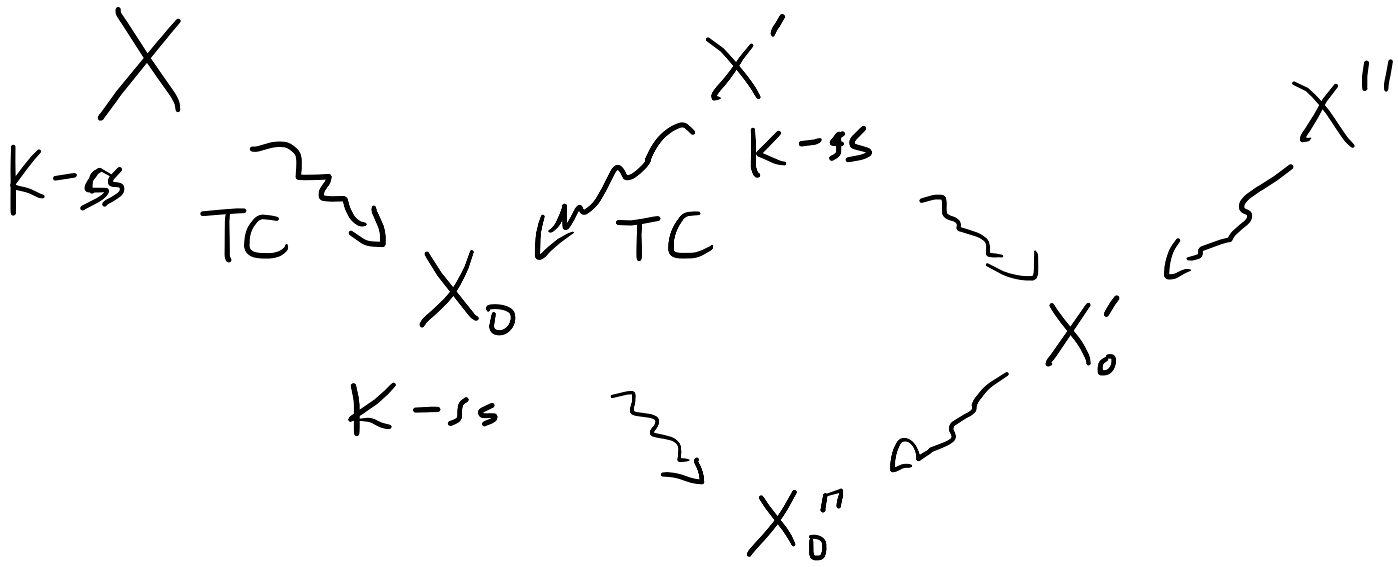
$P, Q \in X$.

$\pi(P) = \pi(Q) \iff P$ and Q are S-equivalent.

In every S-equiv. class, $\exists!$ minimal orbit.

Def. X, X' : 2 K-ss $\mathbb{Q}$-Fano var.

We say X and X' are S-equivalent if they admit a common K-semistable degeneration



Thm (Li-Wang-Xu '18 arXiv, '21 published).

- S-equiv is an equivalence relation for K-ss Fano.
- In each S-equiv class, ∃! K-polystable Fano.

Thm (Blum-Xu '18 arXiv, '19 publ.) [Uniqueness of K-polystable degeneration].

Suppose $X^0 \longrightarrow B^0$ is a family of K-ss $\mathbb{Q}$-Fano over a punctured curve $B^0 = B \setminus \{0\}$.

Suppose we have two (filling) extension $X \to B$, $X' \to B$ s.t. X_0, X'_0 are K-ss. Then X_0 & X'_0 are S-equivalent.

In particular, if X_0, X_0' are K-polystable,

then $X_0 \cong X_0'$.

Later, ABHLX shows that, based the above results, (and Alper-Halpern-Leistner-Heinloth)

$\exists$ separated algebraic space $M^{Kps}_{n,V}$ ← K-moduli space parametrize

K-polystable $\mathbb{Q}$-Fano X with $\dim X = n$, $(-K_X)^n = V$.

(S-equiv classes of K-ss)

Moreover, $\exists$ $\mathcal{M}^{Kss}_{n,V} \to M^{Kps}_{n,V}$ is a good moduli space morphism.

good moduli space (Alper): abstraction of GIT quotient.

§ Properness.

In terms of stacks:

$$\begin{array}{ccc} B^{o} & \longrightarrow & \mathcal{M}^{Kss}_{n,V} \\ \downarrow & \nearrow_{\exists} & \\ B & & \end{array}$$

To show $M^{Kps}_{n,V}$ is proper, it suffices to show

for every family $X^{o} \longrightarrow B^{o}$ with K-semistable fibers

over $B^{o} = B \setminus \{0\}$ a punctured curve,

$\exists$ an extension (after a finite base change)

$X \longrightarrow B$ s.t. X_0 is also K-semistable.

Langton's algorithm. [Huybrechts-Lehn].

It comes from properness of moduli of semistable bundles/C

Suppose $\mathcal{E}^0 \longrightarrow C \times B^0$ family of semistable vector bundles over C.

Step 1. Extend $\mathcal{E}^0$ to a flat family $\mathcal{E} \longrightarrow C \times B$

Usually $\mathcal{E}_0$ is not semistable.

$\exists$ a maximal destabilising subbundle $F \subseteq \mathcal{E}_0$.

$$0 \longrightarrow F \longrightarrow \mathcal{E}_0 \longrightarrow G \longrightarrow 0.$$

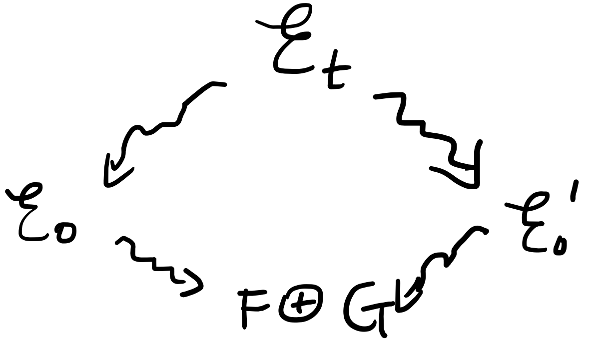
Step 2. Construct $\mathcal{E}' \longrightarrow C \times B$

s.t. $0 \longrightarrow G \longrightarrow \mathcal{E}'_0 \longrightarrow F \longrightarrow 0$ and $\mathcal{E}'|_{B\setminus\{0\}} \cong \mathcal{E}|_{B\setminus\{0\}}$

Then the stability of $\mathcal{E}'$ improves.

$$\mathcal{E}' = \ker\left(\mathcal{E} \longrightarrow \mathcal{E}_0 \longrightarrow G\right)$$

Step 3. Repeat the above process, it terminates, we get a semistable bundle as limit.



For Fano varieties,

$X^\circ \longrightarrow B^\circ$ every fiber X_t is K-semistable $\mathbb{Q}$-Fano var.

(B°: puncture curve)

<u>Step 1</u>. $\exists$ extension $X \longrightarrow B$ s.t. X_0 is also $\mathbb{Q}$-Fano.

[Li-Xu '11 arXiv, '14 published].

<u>Step 2</u>. <u>Q</u>. What is analogue for maximal destab. bundle? in Fano varieties.

A. Minimizer of δ-invariant.

Recall. X $\mathbb{Q}$-Fano variety.

E div over X (i.e. $E \underset{\text{div}}{\subset} Y \xrightarrow[\mu]{\text{bir}} X$).

$$A_X(E) = 1 + \text{coeff}_E(K_Y - \mu^* K_X).$$

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^{+\infty} \text{vol}(\mu^*(-K_X) - tE)\, dt.$$

$$\delta(X) = \inf_{E \text{ div}/X} \frac{A_X(E)}{S_X(E)}.$$

X is K-ss $\iff$ $\delta(X) \geq 1$.

Conj. (Optimal destabilization) (ODC)

If X is K-unstable, i.e. $\delta(X) < 1$.

Then $\exists E$ divisor over X s.t. $\delta(X) = \frac{A_X(E)}{S_X(E)}$.

Thm (Blum-L.-Zhou).

$$R = \bigoplus_{m=0}^{\infty} H^0(X, -mK_X)$$

Assume ODC holds and $\delta(X) < 1$.

Then E is a dreamy divisor, i.e. $gr_E R$ is f.g.

E induces a TC $\mathcal{X}$ of X s.t. $\delta(X) = \delta(X_0)$.

↖ optimal destabilization.

Halpern-Leistner : Θ - stratification

This is a package that encodes all information for optimal destabilization.

For bundles, Θ - stratif = HN filtration

For proj. GIT, Θ - stratif. = Kempf's optimal destab.

For K-moduli, Θ - stratif = δ - minimizer.

B-HL-L-X : ODC $\Longrightarrow$ step 2 of Langton works.

Boundedness $\Longrightarrow$ step 3 also works.

Thm (L.-Xu-Zhuang '21).

ODC holds for all $\mathbb{Q}$-Fano varieties.

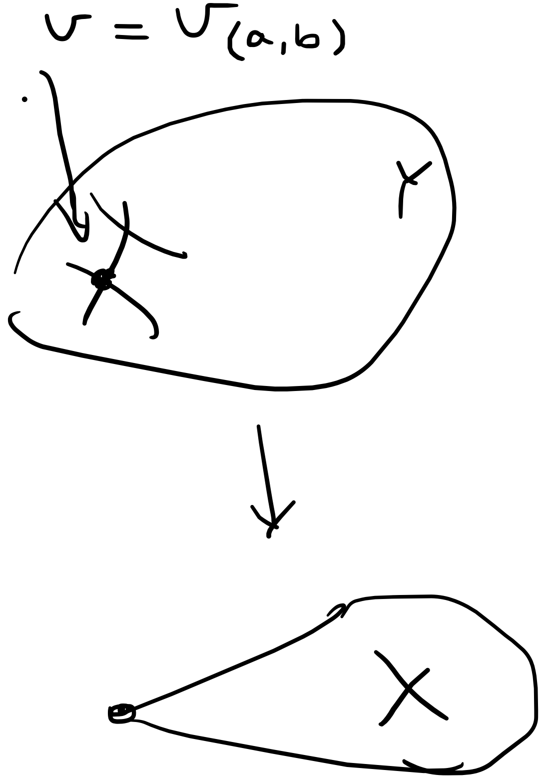
$\Rightarrow$ **Cor.** $M^{Kps}_{n,V}$ is proper.

+ [Xu-Zhuang] and projective (scheme).

Cor. YTD conj for $\mathbb{Q}$-Fano (or log Fano pairs) holds.
[L.-Xu-Zhuang].
[Berman-Boucksom-Jonsson '21] + [Li-Tian-Wang] + [Li '22].

Goal. Find E computes $\delta(X)$.

$\exists E_i$ s.t. $\lim_{i\to\infty} \frac{A_X(E_i)}{S_X(E_i)} = \delta(X)$.



① [Blum-Jonsson]
$\exists$ valuation v as E_i's limit.

s.t. $\frac{A_X(v)}{S_X(v)} = \delta(X)$.

② [Blum-L.-Xu].

We can choose v as a quasi-monomial valuation.

$\sum D_j = D \subseteq Y \longrightarrow X$ s.t. v on Y is monomial in charts defined by D_j.

③ [LXZ] Show $\mathrm{gr}_v R$ is finitely generated

(Higher rank finite generation).

Perturb weights of v to v_k (rat'l weights).

$\mathrm{gr}_{v_k} R$ is finitely generated.

Moreover, $X_k = \mathrm{Proj}\, \mathrm{gr}_{v_k} R$ is bounded & $\mathbb{Q}$-Fano.