Lecture 1 — PART I Introduction to nonabelian Hodge

§1. 0th Motivation (Example: mflds with 2 diff. alg. structures)

X smooth proj. curve $/\mathbb{C}$ genus g

cpt. Riemann surface

$g=1$ $E = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$ elliptic curve

$$\begin{array}{ccccc} T^*E \simeq E\times\mathbb{C} & \not\simeq & (\mathbb{C}^*)^2 \simeq \mathrm{Hom}(\pi_1(E),\mathbb{C}^*) =: M_B(E,1) \\ \| \quad\quad \wr\| \quad \wr\| & & \\ M_{Dol}(X,1) \quad \mathbb{C}/\mathbb{Z}\oplus\mathbb{C}/\mathbb{Z} \quad \mathbb{R}^2 & & \\ \wr\| & & \\ (S^1)^2 & & \end{array}$$

affine variety

$(S^1)^2 \xrightarrow{\text{fibers proj}} \mathbb{C}$

$g\geq 2$ $K_X := \Omega^1_X$ canonical line bundle

$$M_{Dol}(X,1) =: \underbrace{\mathrm{Pic}^0(X)\times H^0(X,K_X)}_{\cong\ \mathbb{C}^g \simeq \mathbb{R}^{2g}} \underset{\mathrm{quasi\text{-}proj}}{\not\simeq} (\mathbb{C}^*)^{2g} \simeq \mathrm{Hom}(\pi_1(X),\mathbb{C}^*) = M_B(X,1)$$

affine

$\mathrm{Pic}^0(X)$ iso classes of line bundles over X of deg 0

$$\simeq H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$$

$$\simeq H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$$

$$\simeq (S^1)^{2g}$$

"Higgs bundles" $\longleftrightarrow$ "local systems"

§2. 1st Motivation: nonabelian analogue of Hodge theory

X smooth cplx mfd. $\dim_{\mathbb{C}} X = n$

$\rightsquigarrow$ 3 information:

(1) topological world:

X^{top} : underlying top. space by forgetting all extra str.

(2) differential world:

$X_{\mathbb{R}}$: forgetting complex str. looking as a real mfd of dim $2n$

(3) holo./analytic world:

X : picking up all str.

$\Rightarrow$ 3 kinds of coho. gps.

(1) Betti coho.

$$H^k_B(X, \mathbb{Z}) := H^k(X^{top}, \mathbb{Z})$$

$\rightsquigarrow$ k-th singular coho. gp.

(2) de Rham coho.

$$H^k_{dR}(X, \mathbb{R}) := H^k(X_{\mathbb{R}}, \mathbb{R})$$
$$:= H^k(C^\bullet_{dR})$$

$$C^\bullet_{dR} : 0 \to A^0(X, \mathbb{R}) \xrightarrow{d} A^1(X, \mathbb{R}) \xrightarrow{d} A^2(X, \mathbb{R}) \to \cdots$$

(3) Dolbeault coho.

$$H^k_{Dol}(X, \mathbb{C}) := \bigoplus_{p+q=k} H^{p,q}(X)$$

$$:= \bigoplus_{p+q=k} H^q(C^{p,\bullet}_{Dol})$$

$$C^{p,\bullet}_{Dol} : 0 \to A^{p,0}(X) \xrightarrow{\bar{\partial}} A^{p,1} \to \cdots$$

<u>Thm</u> (Main thms in Hodge theory)

(1) $\forall k$ $$H^k_B(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R} \simeq H^k_{dR}(X,\mathbb{R})$$

(2) $\forall p,q$ $$H^{p,q}(X) \simeq H^q(X,\underbrace{\Omega^p_X}_{\text{sheaf of holo. p-forms on } X})$$

(3) X cpt. kähler $\forall k$
$$H^k_{dR}(X,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} \simeq H^k_{Dol}(X,\mathbb{C})$$
$$= \bigoplus_{p+q=k} H^{p,q}(X)$$

Taking $k=1$:

$$H^1_{Dol}(X,\mathbb{C}) = H^1(X,\mathcal{O}_X)\oplus H^0(X,\Omega^1_X)$$
$$\wr\!\wr$$
$$H^1_{dR}(X,\mathbb{C}) \simeq H^1_B(X,\mathbb{C}) \simeq \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C})$$
$$\simeq \mathrm{Hom}(\pi_1(X)^{ab},\mathbb{C})$$
$$= \mathrm{Hom}(\pi_1(X),\mathbb{C})$$

$\Rightarrow$

$$H^1(X,\mathcal{O}_X)\oplus H^0(X,\overset{\omega_X}{K_X}) \sim \underline{\mathrm{Hom}(\pi_1(X),\mathbb{C})}$$
$$(e,f) \longleftrightarrow \rho$$

$\mathcal{H}^1(X,\mathbb{C}) \longrightarrow$

$\shortparallel$

$\mathcal{H}^{0,1}(X)\oplus\mathcal{H}^{1,0}(X)$ $\underline{H^1_{dR}(X,\mathbb{C})}$

"space of harmonic 1-forms"

nonabelian analogue:

$\mathbb{C} \rightsquigarrow G$ general reductive gp. ($G = GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$)

$$\to \quad H^1_B(X, G) \simeq H^1_{dR}(X, G) \simeq H^1_{Dol}(X, G)$$

$H^1_B(X, G) = \mathrm{Hom}(\pi_1(X), G)$

$H^1_{Dol}(X, G) = \mathbb{H}^1(X, Gl(\mathcal{O})) \oplus H^0(X, ad(E) \otimes \Omega^1_X)$

$H^1_{dR}(X, G) \leftarrow$ "space of diff. object" $\leftarrow$ flat G-connections

$H^1_{Dol}(X, G) \leftarrow$ G-Higgs bundles

Riemann-Hilbert correspondence ; .. "harmonic objects"

§3. 2nd motivation : generalization of Narasimhan-Seshadri correspondence

<u>Thm 1</u> (Narasimhan-Seshadri 1965)

X cpt. R. S. $g \geq 2$

holomorphic vector bundle over X is <u>stable</u> $\iff$ comes from irreducible proj. unitary representation of $\pi_1(X)$

In particular holo. v.b. of deg 0 is stable $\iff$ comes from irreducible unitary representation of $\pi_1(X)$

- $\mathcal{C}^s_{Vect}(X, n)$ (resp $\mathcal{C}^s_{Vect}(X, n, 0)$) : category of stable v.b. of rk n (resp. rk n + deg 0)
- $\mathcal{C}^{irr}_{Rep}(X, PU(n))$ (resp. $\mathcal{C}^{irr}_{Rep}(X, U(n))$) : category of irreducible reps $\pi_1(X) \to PU(n)$ (resp. $\pi_1(X) \to U(n)$)

$$\mathcal{C}^s_{Vect}(X, n) \simeq \mathcal{C}^{irr}_{Rep}(X, PU(n))$$

$$\mathcal{C}^s_{Vect}(X, n, 0) \simeq \mathcal{C}^{irr}_{Rep}(X, U(n))$$

higher dim. generalization:

Thm2 (Donaldson, Uhlenbeck-Yau)

(X, ω) cpt. Kähler mfd.

$$\mathcal{C}^{s}_{\mathrm{Vect}}(X, n) \simeq \mathcal{C}^{\mathrm{irr}}_{\mathrm{Rep}}(X, PU(n))$$

$$\mathcal{C}^{s}_{\mathrm{Vect}}(X, n, c_1 = \cdots = 0) \simeq \mathcal{C}^{\mathrm{irr}}_{\mathrm{Rep}}(X, U(n))$$

Q: What objects correspond to $\mathcal{C}^{\mathrm{irr}}_{\mathrm{Rep}}(X, GL(n, \mathbb{C}))$?

Thm3 (Donaldson-Corlette-Hitchin-Simpson)

$$\begin{array}{ccc} \mathcal{C}^{s}_{\mathrm{Higgs}}(X, n, c_1 = \cdots = 0) & \simeq & \mathcal{C}^{\mathrm{irr}}_{\mathrm{Rep}}(X, GL(n, \mathbb{C})) \\ \shortparallel & & \shortparallel \\ \mathcal{C}^{s}_{\mathrm{Dol}}(X, n) & & \mathcal{C}^{\mathrm{irr}}_{B}(X, n) \end{array}$$

§4. key point: Donaldson's diff-geom reproof of NS correspondence.

Thm1' (Donaldson 1983) X cpt R.S $g \geq 2$ $\mathcal{E}$ indecomposable v.b.

$\mathcal{E}$ is stable $\Leftrightarrow$ $\mathcal{E}$ is projectively unitary flat bundle

In particular, $\mathcal{E}$ is stable of deg 0 $\Leftrightarrow$ $\mathcal{E}$ is unitary flat bundle.

where • proj. unitary flat bundle := $\exists$ unitary connection ∇ s.t.

$$\Lambda_\omega \underbrace{F_\nabla}_{\substack{\uparrow \\ \text{curvature.}}} = -2\pi\sqrt{-1}\, \underbrace{\mu(\mathcal{E})}_{\substack{\to \\ \mu(\mathcal{E}) := \frac{\int_X c_1(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}}} \mathrm{Id}$$

$\iff \exists$ hermitian metric h s.t.

$$\Lambda_\omega F_{\nabla_h} = -2\pi\sqrt{-1}\,\mu(E)\,\mathrm{Id}$$

∇_h : Chern connection.

E is unitary flat $\triangleq$ $\sqrt{-1}\Lambda_\omega F_{\nabla} = 0$

:

$$\boxed{\begin{aligned} \mathcal{C}^{s}_{\mathrm{Vect}}(X,n) &\simeq \mathcal{C}^{\mathrm{irr}}_{\mathrm{PUF}}(X,n) \\ \mathcal{C}^{s}_{\mathrm{Vect}}(X,n,0) &\simeq \mathcal{C}^{\mathrm{irr}}_{\mathrm{UF}}(X,n) \end{aligned}}$$

"stability" ~ "special metric" $\leftarrow$ called Kobayashi-Hitchin correspondence

Goal of this ~~short~~ course:

explain explicitly the idea on show Thm 3:

$$\mathcal{C}^{s}_{\mathrm{Bd}}(X,n) \simeq \mathcal{C}^{\mathrm{irr}}_{B}(X,n)$$

KHC $\to$ $\wr\wr$ $\quad$ $\wr\wr$ $\leftarrow$ Riemann-Hilbert corresp.

key!

$$\mathcal{C}^{\mathrm{irr}}_{\mathrm{dR}}(X,n)$$

$\uparrow$ category of irreducible flat bundle of rk n

PART II A crash course course on Hodge theory (1)

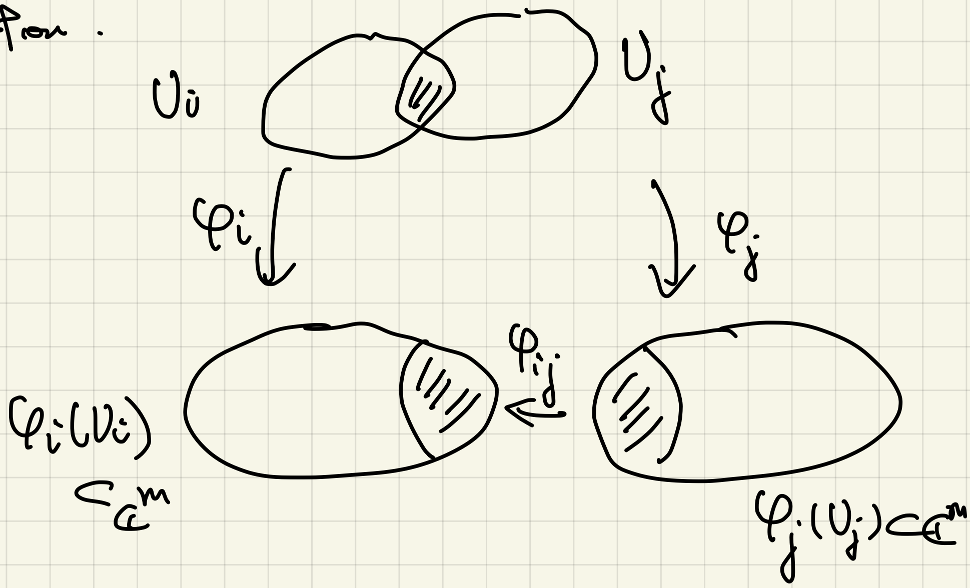
§1. Complex mflds (2-equiv. descriptions)

<u>Def 1.1</u> A cplex mfld of dim m is top. space X together with a complex structure (complex atlas) $\{(U_i, \varphi_i)\}_{i\in I}$ where:

- $\mathcal{U} = \{U_i\}_{i\in I}$ open covering of X
- $\forall i \qquad \varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^m$ homeomorphism
- whenever $U_i \cap U_j \neq \emptyset$, the transition function

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \underset{\subset \mathbb{C}^m}{\varphi_j(U_i \cap U_j)} \longrightarrow \underset{\subset \mathbb{C}^m}{\varphi_i(U_i \cap U_j)}$$

is holo. function.



- (U_i, φ_i) holo. chart
- $\varphi_i : U_i \longrightarrow \varphi_i(U_i) \subset \mathbb{C}^m$ coordinate function

$\qquad x \longmapsto (z_1, \dots, z_m)$

<u>Ex</u> (complex proj. space)

$$\mathbb{CP}^m := \mathbb{C}^{m+1} - \{0\} / \sim$$

$(z_0, \dots, z_m) \sim (z_0', \dots, z_m') \Leftrightarrow (z_0, \dots, z_m) = \lambda(z_0', \dots, z_m')$

for some $\lambda \in \mathbb{C}^*$.

$[z_0 : \cdots : z_m]$

$\{(U_i, \varphi_i)\}_{0 \le i \le m}$

- $U_i := \{[z_0 : \cdots : z_m] : z_i \neq 0\}$
- $\varphi_i : U_i \longrightarrow \varphi_i(U_i) \subset \mathbb{C}^m$

$$[z_0 : \cdots : z_m] \longmapsto \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_m}{z_i}\right)$$

- $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j) \quad (i < j)$

$$(w_1, \dots, w_m) \mapsto \left(\frac{w_1}{w_i}, \dots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \dots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, \dots, \frac{w_m}{w_i}\right)$$

Def 1.2 An almost cplex mfld is a diff. mfld X together with an almost cplex str. J, i.e. $J : TX \to TX$ s.t. $J^2 = -\mathrm{Id}$.

Prop 1.3 (1) Any almost cplex mfld has even real dim.

(2) complex mfld $\Rightarrow$ almost cplex mfld.

Pf

(1) $\forall x \in X \quad \dim_{\mathbb{R}} T_xX = \dim_{\mathbb{R}} X$

$\forall v \in T_xX$. v & Jv $\mathbb{R}$-linear independent

$V := \mathrm{Span}_{\mathbb{R}}\{v, Jv\}$ J-inv. $V^\perp \subset T_xX$ is J-inv.

induction on dim gives $2 \mid \dim_{\mathbb{R}} T_xX$

(2) X cplex ~~of~~ $\dim_{\mathbb{C}} X = m \Rightarrow$ real mfld. of dim $2m$.

$\forall x \in X \quad (U, \varphi; z_1, \dots, z_m)$

$z_j = x_j + \sqrt{-1}\, y_j. \quad 1 \le j \le m.$

$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ local basis of $T_x U$

define $J_x : T_x U \to T_x U$

$$\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial y_j} \quad \rightsquigarrow \quad J_x^2 = -\mathrm{Id}$$

$$\frac{\partial}{\partial y_j} \mapsto -\frac{\partial}{\partial x_j}$$

transition functs are holo.

$\rightsquigarrow$ globally defined $J : TX \to TX$ s.t. $J^2 = -\mathrm{Id}$. $\square$

Rmk. W. $\mathbb{R}$-v.s. $J : W \to W$ $J^2 = -\mathrm{Id}$

$\Rightarrow W$ is $\mathbb{C}$-v.s. via

$$(a + \sqrt{-1}b) \cdot w := a \cdot w + b J(w)$$

But. almost cplex mflds are not necessarily cplex mflds.

(X, J) almost cplex mfld. $\dim_{\mathbb{R}} X = 2n$

$J : TX \to TX$ $J^2 = -\mathrm{Id}$ $\rightsquigarrow$ $J : T^*X \longrightarrow T^*X$ via

$T^*X = \mathrm{Hom}_{\mathbb{R}}(TX, \mathbb{R})$

$(J\theta)(v) := \theta(J(v))$

$\forall \theta \in T^*X$.

$v \in TX$

let $T_{\mathbb{C}}X := TX \otimes_{\mathbb{R}} \mathbb{C}$ complexified bundle.

$T^*_{\mathbb{C}}X := T^*X \otimes_{\mathbb{R}} \mathbb{C}$

$J : TX \to TX$ & $J : T^*X \to T^*X$ extend $\mathbb{C}$-linearly

- $J : T_{\mathbb{C}}X \longrightarrow T_{\mathbb{C}}X$ $v \in TX$. $\lambda \in \mathbb{C}$

$v \otimes \lambda \longmapsto J(v) \otimes \lambda$

$\bullet\ J : T^*_{\mathbb{C}}X \longrightarrow T^*_{\mathbb{C}}X \qquad J^2 = -Id$

$\theta\otimes\lambda \mapsto J(\theta)\otimes\lambda$

$\rightsquigarrow \pm i$-eigenbundles

$\bullet\ T_{\mathbb{C}}X = T_{1,0}X \oplus T_{0,1}X$

$$\begin{cases} T_{1,0}X = \{v - iJv : v\in TX\} \\ T_{0,1}X = \{v + iJv : v\in TX\} \end{cases}$$

$\bullet\ T^*_{\mathbb{C}}X = T^*_{1,0}X \oplus T^*_{0,1}X$

$$\begin{cases} T^*_{1,0}X = \{\theta - iJ\theta : \theta\in T^*X\} \\ T^*_{0,1}X = \{\theta + iJ\theta : \theta\in T^*X\} \end{cases}$$

define conjugation:

$\bar{\ } : T_{\mathbb{C}}X \longrightarrow T_{\mathbb{C}}X$

$v\otimes\lambda \mapsto v\otimes\bar{\lambda}$

$\bar{\ } : T^*_{\mathbb{C}}X \longrightarrow T^*_{\mathbb{C}}X$

$\theta\otimes\lambda \mapsto \theta\otimes\bar{\lambda}$

$\rightsquigarrow \overline{T_{1,0}X} = T_{0,1}X$

$\overline{T^*_{1,0}X} = T^*_{0,1}X$

<u>Thm 1.4</u> (Newlander-Nirenberg) (X, J) almost cplex str. TFAE:

(1) X cplex mfld

(2) The Nijenhuis tensor $N^J \equiv 0$

where $N^J(v,w) := J\big([Jv,w] + [v,Jw]\big) + [v,w] - [Jv,Jw]$

(3) $[\cdot,\cdot]$ is closed under $T_{0,1}X$ i.e. $[v,w] \in T_{0,1}X$ for $\forall v, w \in T_{0,1}X$

§2. Vector bundles and sheaves

$\mathbb{k} = \mathbb{C}$ or. $\mathbb{R}$

<u>Def 2.1</u> A diff. surj map of diff. mflds $\pi : E \to X$ is <u>$\mathbb{k}$-vector bundle of rk n</u> if $\forall x \in X$. $\exists$ open nbhd $x \in U \subseteq X$ & homeomorphism $\psi_U : \pi^{-1}(U) \to U \times \mathbb{k}^n$ ← local trivialization

s.t.

(1) $pr_1 \circ \psi_U = \pi$ $\longleftrightarrow$

$$\begin{tikzcd} & U \times \mathbb{k}^n \arrow[d, "pr_1"] \\ \pi^{-1}(U) \arrow[ur, "\psi_U"] \arrow[r, "\pi"'] \arrow[ur, phantom, "\circlearrowright", shift left=4] & U \end{tikzcd}$$

(2) each fiber $E_x := \pi^{-1}(x)$ has a str. of $\mathbb{k}$-v.s. s.t.

$E_x \to \mathbb{k}^n$ $\mathbb{k}$-linear iso.

A <u>section</u> of $\pi : E \to X$ is a C^∞ map $s : X \to E$ s.t. $\pi \circ s = id_X$

- $A^0(X, E)$: space of sections of E

$\Updownarrow$ transition functions:

$\forall$ 2 local trivializations $\psi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{k}^n$

$\psi_j : \pi^{-1}(U_j) \xrightarrow{\sim} U_j \times \mathbb{k}^n$

with $U_i \cap U_j \neq \phi$

$$\rightsquigarrow \psi_{ij} := \psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{k}^n \longrightarrow (U_i \cap U_j) \times \mathbb{k}^n$$
$$(x, v) \longmapsto (x, g_{ij}(x) v)$$

determines $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{k})$ ← transition functions

$$x \mapsto g_{ij}(x)$$

s.t. • $g_{ii} = Id$ on U_i

• $g_{ij}\, g_{jk}\, g_{ki} = Id$ on $U_i \cap U_j \cap U_k \neq \emptyset$

__Def 2.2__ X cplex mfld. A __holomorphic vector bundle__ over X is a $\mathbb{C}$-v.b. over X s.t. the transitions functions $\{g_{ij}\}_{i,j}$ are holomorphic.

__Def 2.3__ X top. space. a __presheaf__ $\mathcal{F}$ of abelian gps (resp. k-v.s., rings. modules, ...) over X is

(1) $\forall\, U \subset X$ open. $\mathcal{F}(U)$ is abelian gp (resp. k-v.s., ...) ← space of sections on U

(2) $\forall\, U, V \subseteq X$ open. with $U \subset V$. the __restriction map__

$$r_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$
$$s \longmapsto s|_U$$

is gp. homorphism. (resp. linear map of k-v.s. ..)

s.t.

(a) $r_{U,U} = id_{\mathcal{F}(U)}$

(b) $\forall\, U, V, W \subset X$ open. $U \subset V \subset W$

$$r_{V,U} \circ r_{W,V} = r_{W,U}$$

$s \in \mathcal{F}(W)$

$$(s|_V)|_U = s|_U$$

It is a __sheaf__ if moreover,

(c) $\forall\, U = \bigcup_i U_i \subset X$ open. $\forall\, s_1, s_2 \in \mathcal{F}(U)$ with $s_1|_{U_i} = s_2|_{U_i}$ $\forall i$

$\Rightarrow s_1 = s_2$ (uniqueness)

(d) $s_i \in \mathcal{F}(U_i)$ s.t. whenever $U_i \cap U_j \neq \emptyset$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \forall\, i, j$$

$\Rightarrow \exists_! s \in \mathcal{F}(U)$ s.t.

$s|_{U_i} = s_i \quad \forall i$ gluing

<u>Ex</u> X cplex mfd.

(1) C^∞_X : sheaf of C^∞ $\mathbb{C}$-valued functions on X

(2) $\mathcal{O}_X$: sheaf of holo. functions on X

(3) $E \to X$ $\mathbb{C}$-v.b. $\rightsquigarrow$ $\mathcal{E}$ sheaf of sections

i.e. $\forall U \subset X$ open.

$\mathcal{E}(U) := \{ s : U \to E \text{ section} \}$

(4) Ω^p_X : sheaf of holo. p-forms on X

$\Omega^p_X(U) := \{ \alpha \in \underset{\sim}{A^{p,0}(U)} : \bar{\partial}\alpha = 0 \}$

• $\mathcal{F}$: sheaf of $\mathcal{O}_X$-modules over X

$\mathcal{O}_X(U)$: ring

(1) $\mathcal{F}$ is <u>coherent</u> if $\forall x \in X$ $\exists$ $x \in U \subseteq X$ open s.t.

$\mathcal{O}^q_U \to \mathcal{O}^p_U \to \mathcal{F}_U \to 0$ exact for some $p, q \in \mathbb{Z}_{>0}$

$\Updownarrow$ $\mathcal{F}$ locally of finite type

& $\ker(\mathcal{O}^p_U \to \mathcal{F}_U)$ finite.

(2) $\mathcal{F}$ is <u>torsion free</u> if $\forall x \in X$ $\mathcal{F}_x$ is torsion free $\mathcal{O}_{X,x}$-module

$\Uparrow$

(3) $\mathcal{F}$ is <u>locally free</u> if $\forall x \in X$. $\exists_{x \in} U \subset X$ s.t.

$\mathcal{F}_U \underset{\uparrow}{\simeq} \mathcal{O}^p_U$ for some $p \in \mathbb{Z}_{>0}$ $\nwarrow$ 秩 of $\mathcal{F}$

iso. of sheaves of $\mathcal{O}_U$-modules.

for us. X cplex mfld

$\mathcal{O}_X$: sheaf of holo. functions on X

$(X, \mathcal{O}_X)$

<u>Prop 2.5</u> We have the following equiv. of categories (analytic version)

$$\left\{ \begin{array}{c} \text{holo. v.b. of rk } n \\ \text{over } X \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{Locally free sheaves of } \mathcal{O}_X\text{-modules} \\ \text{of rk } n \end{array} \right\}$$

$$E \longmapsto \mathcal{E}$$

associated sheaf of holo sections

<u>pf</u>

- fully faithful: $\mathrm{Hom}(E, F) \qquad \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$

$$\exists\, \tilde{f} \in \mathrm{Hom}(E,F) \longleftrightarrow f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$$

- essentially surj. $\mathcal{E}$ on right side $\longrightarrow$ find $E = \coprod_{x \in X} \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$

$\blacksquare$

<u>Prop 2.6</u>

(1) (X, J) almost cplex $\Rightarrow$ TX (and T^*X) is $\mathbb{C}$-v.b.

(2) X cplex mfld. $\Rightarrow$ $T_{1,0}X$ (and $T^*_{1,0}X$) is a holo. v.b.

$T_{1,0}X$: holo. tangent; $T^*_{1,0}X$: holo. cotangent.

2. Hodge theory (left)

3. affine GIT

4. Betti spaces

5. de Rham spaces

6. Dolbeault spaces

7. Proof of KHC

8. Other topics

X cplex mfld X analytic space (allow singularities)

$\mathcal{O}_X$ sheaf of $\mathcal{O}_X$-modules

E v.b. $F \subseteq E$ torsion-free sheaf

E/F may have torsion

$0 \to F \to E \to E/F \to 0$