Introduction to K-stability @ Nankai by Yuchen Liu.

Lecture 1. Basic Concepts.

K-stability : introduced by Tian ('97)

algebraic reformulation by Donaldson ('02)

characterise $\exists$ of Kähler-Einstein metrics on Fano mfds.

(X, ω) Kähler mfd, KE metric : $Ric(\omega) = \lambda \cdot \omega$.

$$[Ric(\omega)] = c_1(X) \in H^2(X, \mathbb{R}).$$

			$\exists$KE?
$\lambda > 0 \Rightarrow$	$c_1(X) > 0$,	X is Fano.	Not always
$\lambda = 0 \Rightarrow$	$c_1(X) = 0$,	X is Calabi-Yau	✓ Yau
$\lambda < 0 \Rightarrow$	$c_1(X) < 0$,	X is of general type	✓ Aubin, Yau

K-moduli theory: a well-behaved moduli theory for Fano var.

§ Original definition.

- X is a Fano variety if X is a normal proj. var. $/\mathbb{C}$, and $-K_X$ is $\mathbb{Q}$-Cartier ample.

A smooth Fano variety is a Fano manifold.

Ex. $\mathbb{P}^n$, $Q^n = (\sum_{i=0}^{n+1} x_i^2 = 0) \subseteq \mathbb{P}^{n+1}$, $Gr(k,n)$.

$X_d = (f(x_0, \cdots, x_{n+1}) = 0) \subseteq \mathbb{P}^{n+1}$, $\deg f = d \leq n+1$.

$K_{X_d} = \mathcal{O}(-n-2+d)$ by adjunction.

del Pezzo surfaces (10 families), Fano 3-folds (105 families).

<u>Ex</u>. Weighted proj. spaces $a_0, \cdots, a_n \in \mathbb{N}$

$$\mathbb{P}(a_0, \cdots, a_n) = \mathbb{A}^{n+1} \setminus \{0\} \Big/ \mathbb{G}_m$$

$\mathbb{G}_m$-action on $\mathbb{A}^{n+1}$: $t \cdot (x_0, \cdots, x_n) \longmapsto (t^{a_0} x_0, t^{a_1} x_1, \cdots, t^{a_n} x_n)$.

$$-K_{\mathbb{P}(a_0, \cdots, a_n)} = \mathcal{O}(a_0 + \cdots + a_n) \quad \text{ample}.$$

weighted hypersurface, $\cdots$.

(test the theory on examples !)

$L = -rK_X$

Def. A test configuration (TC) of (X, L) is $(\mathcal{X}, \mathcal{L})$ together with $\pi : \mathcal{X} \to \mathbb{A}^1$ s.t.

(i) $\mathcal{X}$ is normal, π is a flat proj. morphism and $\mathcal{L}$ is a π-ample line bundle.

(ii) $\exists$ a $\mathbb{G}_m$-action on $\mathcal{X}$ s.t. π is $\mathbb{G}_m$-equivariant w.r.t. standard $\mathbb{G}_m$-action on $\mathbb{A}^1$

$$t \cdot x = tx.$$

(iii) $\exists\, r \in \mathbb{Q}_{>0}$ s.t. $-rK_X$ is an ample line bundle, and

$$\left(\mathcal{X} \setminus \mathcal{X}_0,\ \mathcal{L}\big|_{\mathcal{X} \setminus \mathcal{X}_0}\right) \overset{\mathbb{G}_m\text{-equiv}}{\cong} (X, -rK_X) \times (\mathbb{A}^1 \setminus \{0\})$$

where $\mathbb{G}_m$ acts trivially on $(X, -rK_X)$, standard on $\mathbb{A}^1 \setminus \{0\}$.

Ex. • Trivial TC: $(\mathcal{X}, \mathcal{L}) \overset{\mathbb{G}_m\text{-equiv}}{\cong} (X, -rK_X) \times \mathbb{A}^1$.

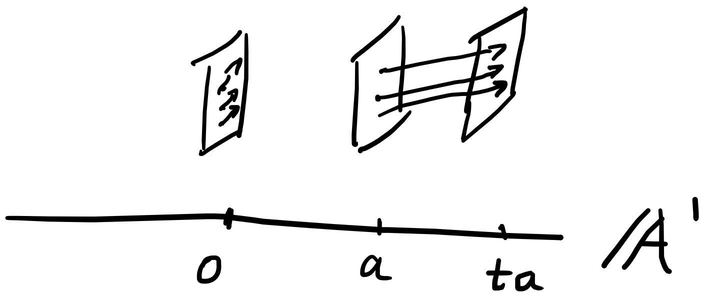
$\mathbb{G}_m$ acts trivially on X.

• Product TC: $\mathcal{X} \cong X \times \mathbb{A}^1$, but different $\mathbb{G}_m$-action.

λ : $\mathbb{G}_m$-action on $(X, -rK_X)$.

λ induces an action on $\mathcal{X}$:

$$t \cdot (x, a) := (\lambda(t)\, x, \; ta).$$



- Non-product TC with nice fiber.

$$\mathcal{X} = (x^2+y^2+z^2+a^2w^2=0) \subseteq \mathbb{P}^3_{[x,y,z,w]} \times \mathbb{A}^1_a .$$

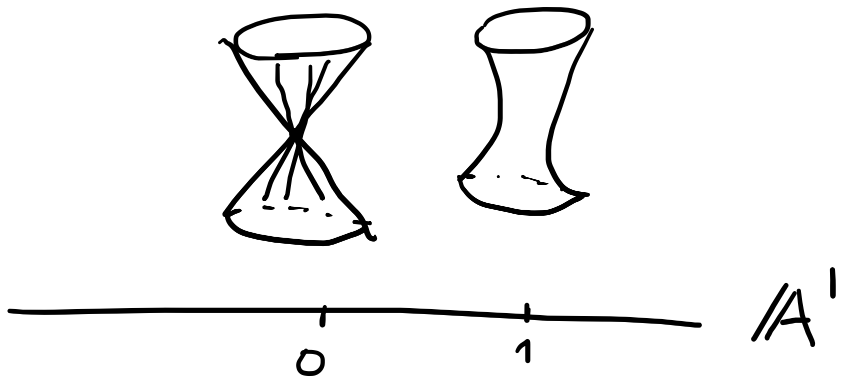
$$\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1), \quad r = \tfrac{1}{2}.$$

$a \neq 0$: $\mathcal{X}_a$ is a smooth quadric surface

$a = 0$: $\mathcal{X}_0$ is a cone over conic curve.

$\mathbb{G}_m$-action :

$$t \cdot ([x,y,z,w], a) = ([x, y, z, t^{-1}w], ta).$$



$$X = \mathbb{P}^1 \times \mathbb{P}^1 \xhookrightarrow{\mathcal{O}(1,1)} \mathbb{P}^3$$

$$\mathcal{X}_0 = \mathbb{P}(1,1,2) \xhookrightarrow{\mathcal{O}(2)} \mathbb{P}^3 .$$

$$\mathcal{X}_0 = (x^2+y^2+z^2=0) \subseteq \mathbb{P}^3.$$

$$\cong (xy - z^2 = 0).$$

$$\mathbb{P}(1,1,2) \xrightarrow{\mathcal{O}(2)} \mathbb{P}^3$$

$$[u,v,s] \longmapsto [u^2, v^2, uv, s] = [x, y, z, w]$$

$$\text{image} = (xy - z^2 = 0).$$

$[0,0,1] \in \mathbb{P}(1,1,2)$ locally $\mathbb{C}^2/\{\pm 1\}$ A_1-singularity.

Futaki invariant.

Def. X Fano var., $(\mathcal{X}, \mathcal{L})$ a test config. of (X, L). ($n = \dim X$, $L = -rK_X$)

Riemann-Roch: $N_m = h^0(X, L^{\otimes m}) = h^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes m})$

$$= a_0 m^n + a_1 m^{n-1} + O(m^{n-2}).$$

w_m = total $\mathbb{G}_m$-weight on $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes m})$

$$= b_0 m^{n+1} + b_1 m^n + O(m^{n-1}).$$

The generalized Futaki invariant of $(\mathcal{X}, \mathcal{L})$ is

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0^2}.$$

* Intersection formula (X. Wang, Odaka)

$$\mathrm{Fut}(\mathcal{X},\mathcal{L}) = \frac{1}{(-K_X)^n}\left(\frac{n}{n+1}\frac{(\bar{\mathcal{L}})^{n+1}}{r^{n+1}} + \frac{(\bar{\mathcal{L}}\cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1})}{r^n}\right).$$

$$\begin{array}{ccccc} (\bar{\mathcal{X}},\bar{\mathcal{L}}) & = & (\mathcal{X},\mathcal{L}) & \cup & (X,L)\times(\mathbb{P}^1\setminus\{0\}) \\ \downarrow & & & \uparrow & \\ \mathbb{P}^1. & & & & (\mathcal{X}\setminus\mathcal{X}_0,\ \mathcal{L}|_{\mathcal{X}\setminus\mathcal{X}_0}) \cong (X,L)\times(\mathbb{A}^1\setminus\{0\}) \end{array}$$

<u>Rmk</u>. $(\mathcal{X}_\lambda,\mathcal{L}_\lambda)$ product TC, $(\mathcal{X}_{\lambda^{-1}},\mathcal{L}_{\lambda^{-1}})$ product TC.

$$\mathrm{Fut}(\mathcal{X}_\lambda,\mathcal{L}_\lambda) + \mathrm{Fut}(\mathcal{X}_{\lambda^{-1}},\mathcal{L}_{\lambda^{-1}}) = 0.$$

Def. A Fano variety X is

- K-semistable if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ $\forall$ TC $(\mathcal{X}, \mathcal{L})$ $(\forall r)$.
- K-stable if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) > 0$ $\forall$ non-trivial TC $(\mathcal{X}, \mathcal{L})$.
- <u>K-polystable</u> if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ $\forall$ TC $(\mathcal{X}, \mathcal{L})$, and "=" iff $(\mathcal{X}, \mathcal{L})$ is product TC.

$\Updownarrow$ YTD

$\exists$ KE metric.

(Historically, Futaki showed that X admits KE $\Rightarrow$ Futaki-invariant for all holo. vector field $= 0$).

§ Fujita-Li's valuative criteria.

Def. X a Fano var.

$\mu : Y \to X$ proper birational morphism, Y normal

$E \subseteq Y$ is a prime divisor

We say E is prime divisor over X.

$$\begin{array}{c} \mathrm{ord}_E : K(Y)^\times \longrightarrow \mathbb{Z} \\ \cong \\ K(X)^\times \end{array}$$

discrete valuation

Def. E : prime div over X.

Log discrepancy : $A_X(E) := 1 + \mathrm{coeff}_E(K_Y - \mu^* K_X)$.

pseff threshold : $T_X(E) := \sup\{t \geq 0 \mid \mu^*(-K_X) - tE \text{ is big}\}$.
(T-invariant)

Expected vanishing order : $S_X(E) := \frac{1}{(-K_X)^n} \int_0^{T_X(E)} \mathrm{vol}_Y(\mu^*(-K_X) - tE)\, dt$.
(S-invariant)

$+\infty$

β-invariant : $\beta_X(E) := A_X(E) - S_X(E)$.

Recall. D divisor on Y. $m \longmapsto h^0(Y, mD)$ polynomial growth

$$\mathrm{vol}_Y(D) := \lim_{m \to \infty} \frac{h^0(Y, mD)}{m^n/n!}.$$

- If D is nef, then $\mathrm{vol}_Y(D) = (D^n)$.
- D is big iff $\mathrm{vol}(D) > 0$.

Thm (Fujita-Li's valuative criteria) X : a Fano var.

(1) X is K-semistable iff $\beta_X(E) \geq 0$ $\forall E$ prime div/X.

(2)[Blum-Xu] X is K-stable iff $\beta_X(E) > 0$ $\forall E$ prime div/X

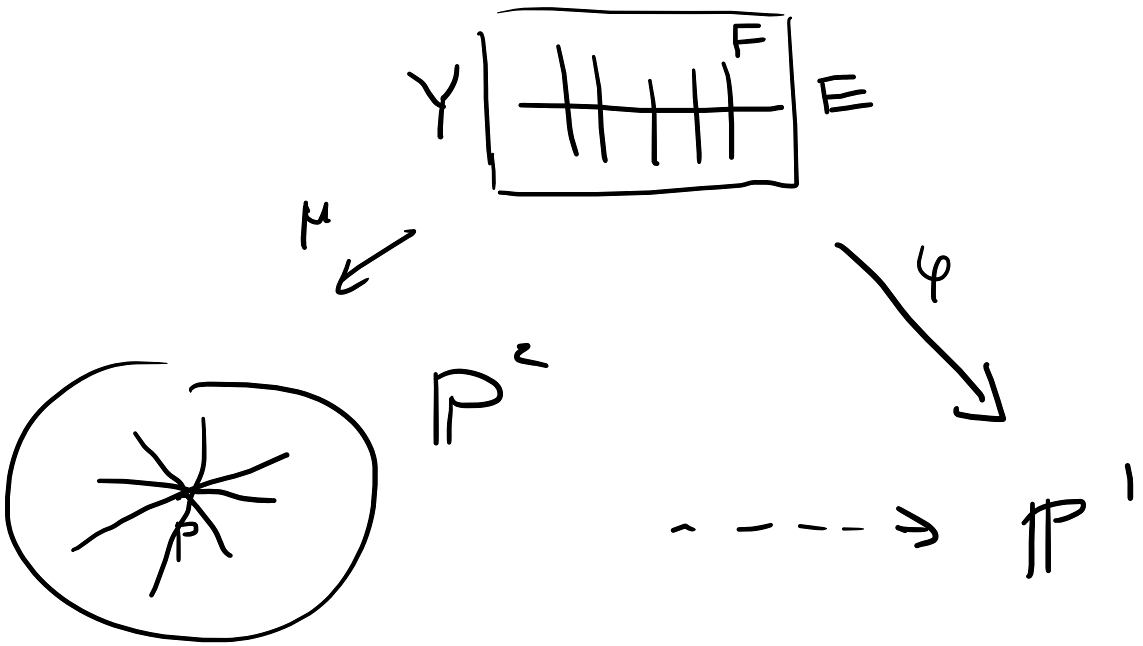
Ex 1 $X = \mathbb{P}^2$. $Y = Bl_p \mathbb{P}^2 \xrightarrow{\mu} X$, $E \subseteq Y$: μ-exc. curve.

$$A_X(E) = 1 + \mathrm{coeff}_E(K_Y - \mu^* K_X). \qquad K_Y = \mu^* K_X + E.$$

$$= 2.$$

$$-K_X = \mathcal{O}(3). \qquad (-K_X)^2 = \mathcal{O}(3)^2 = 9.$$

$$\mu^*(-K_X) - tE = \mu^* \mathcal{O}(3) - tE.$$



$$F = \varphi^* \mathcal{O}_{\mathbb{P}^1}(1)$$
$$= \mu^* \mathcal{O}(1) - E.$$

$$\mathrm{vol}(F) = 0$$

$\mu^* \mathcal{O}(3) - tE$ is nef when $0 \le t \le 3$.

(cone of curves on Y is generated by E & F).

$$\Rightarrow \quad \mathrm{vol}_Y(\mu^* \mathcal{O}(3) - tE) = (\mu^* \mathcal{O}(3) - tE)^2$$
$$= 9 + t^2 (E^2) = 9 - t^2.$$

$$S_X(E) = \frac{1}{(-K_X)^2}\int_0^{T_X(E)} \mathrm{vol}(\mu^*(-K_X) - tE)\,dt$$

$$= \frac{1}{9}\int_0^3 (9 - t^2)\,dt = \frac{1}{9}\left(27 - \frac{27}{3}\right) = 2.$$

$$\beta_X(E) = A_X(E) - S_X(E) = 2 - 2 = 0.$$

In the end, $\mathbb{P}^2$ is not K-stable.

(In fact, $\mathbb{P}^2$ is K-polystable).

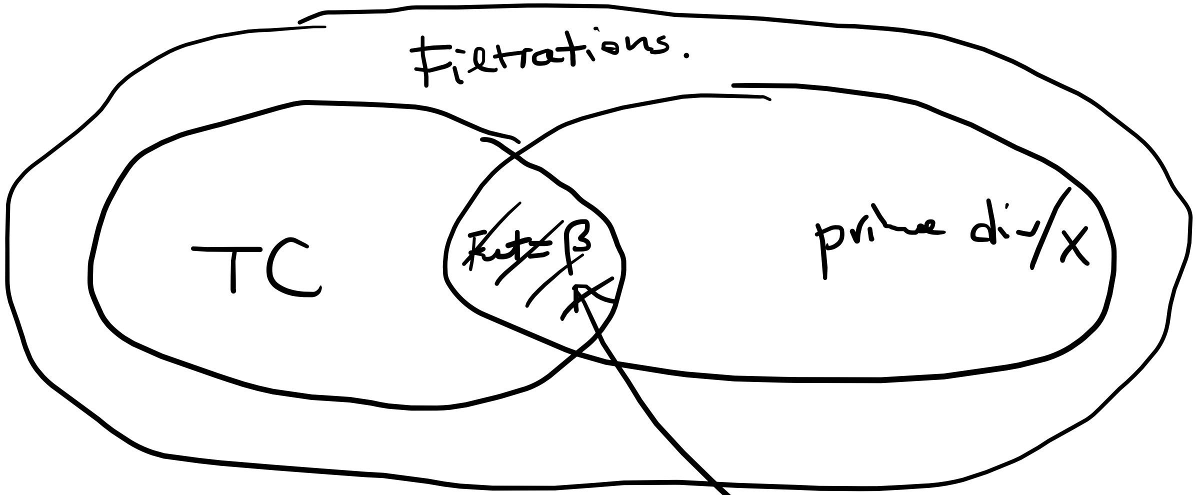
<u>Ex 2</u>. $X = \mathrm{Bl}_p\,\mathbb{P}^2$, $Y = X$, $E = (-1)$-curve.

$A_X(E) = 1$. $S_X(E) = \frac{7}{6}$. $\Rightarrow \beta_X(E) = -\frac{1}{6} < 0$

Conclusion: $\mathrm{Bl}_p\,\mathbb{P}^2$ is K-unstable, i.e. not K-semistable.

Homework. Check $Bl_p \mathbb{P}^n$ is K-unstable.

Understand original def & valuative criteria.



① TC $\longrightarrow$ divisors.

TC with $\mathcal{X}_0$ integral scheme

$(\mathcal{X}, \mathcal{L})$ TC with $\mathcal{X}_0$ integral.

$\mathcal{X} \setminus \mathcal{X}_0 \cong X \times (\mathbb{A}^1 \setminus \{0\})$

$\Downarrow$

$\mathcal{X} \overset{bir}{\dashrightarrow} X \times \mathbb{A}^1$

$ord_{\mathcal{X}_0} : K(\mathcal{X})^\times \longrightarrow \mathbb{Z}$ discrete valuation.

$$\mathrm{ord}_{\mathcal{X}_0} : K(\mathcal{X})^\times \to \mathbb{Z}.$$

$$K(\mathcal{X}) \cong K(X \times \mathbb{A}^1) = K(X)(t).$$

$\Rightarrow$ $\mathrm{ord}_{\mathcal{X}_0}\big|_{K(X)} : K(X)^\times \to \mathbb{Z}$ is also a discrete valuation.

Boucksom-Hisamoto-Jonsson: $\exists\, b \in \mathbb{Z}_{\geq 0}$, $\exists\, E$ prime div $/X$

s.t. $\mathrm{ord}_{\mathcal{X}_0}\big|_{K(X)} = b \cdot \mathrm{ord}_E$.

② From divisors to filtrations.

E prime div $/X$.

section ring $\quad R = \bigoplus_{m=0}^{\infty} H^0(X, -mK_X)$. $\quad \mathrm{Proj}\, R = X$.

$$\mathcal{F}^p R_m := \{ s \in H^0(-mK_X) \mid \mathrm{ord}_E(s) \geq p \}$$

$$\cong H^0(Y, \mu^*(-mK_X) - pE).$$

$$R_m = \mathcal{F}^0 R_m \supseteq \mathcal{F}^1 R_m \supseteq \mathcal{F}^2 R_m \supseteq \cdots \supseteq \mathcal{F}^{p_{max}} R_m \supseteq \mathcal{F}^{p_{max}+1} R_m = 0.$$

(3) Filtrations $\to$ TC.

$\mathcal{F}$: filtration on R.

Rees algebra:

$$\mathrm{Rees}(\mathcal{F}) = \bigoplus_{m=0}^{\infty} \bigoplus_{p=-\infty}^{+\infty} t^{-p} \mathcal{F}^p R_m \subseteq R[t, t^{-1}].$$

If $\mathcal{F}$ is finitely generated, then $\mathcal{X} = \mathrm{Proj}\, \mathrm{Rees}(\mathcal{F}) \to \mathbb{A}^1_t$

$(\mathcal{X}, \mathcal{L})$ is a TC. $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1)$.

Def. A prime divisor E/X is called <u>dreamy</u> if

$$\bigoplus_{m,p\geq 0} H^0(Y, \mu^*(-mK_X) - pE) \text{ is finitely generated.}$$

($\Leftrightarrow$ $\mathcal{F}$ induced by E is f.g.).

Thm. $\{\text{TC with integral } \mathcal{X}_0\} \overset{1-1}{\longleftrightarrow} \{\text{dreamy divisors}/X\}$ (up to rescaling).

Lem. $(\mathcal{X}, \mathcal{L})$ TC with $\mathcal{X}_0$ integral $\rightsquigarrow$ $\mathrm{ord}_{\mathcal{X}_0}|_{K(X)} = b \cdot \mathrm{ord}_E$.

Then $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = b \cdot \beta_X(E)$.