Lecture 2 — A crash course on Hodge theory ②

§3. Differential forms on cplex mflds

X cplex mfld, $\dim_{\mathbb{C}} X = m$

$\forall\ 0 \le k \le 2m$, introduce

(1) k-th exterior cotangent bundle

$$\Lambda^k X := \Lambda^k (T^* X)$$

(2) k-th exterior complexified cotangent bundle

$$\Lambda^k_{\mathbb{C}} X := \Lambda^k (T^*_{\mathbb{C}} X)$$

$$= \Lambda^k (T^*_{1,0} X \oplus T^*_{0,1} X)$$

$$\Lambda^k (E \oplus F) = \bigoplus_{p+q=k} \Lambda^p E \otimes \Lambda^q F$$

$$= \bigoplus_{p+q=k} \left(\Lambda^p (T^*_{1,0} X) \otimes \Lambda^q (T^*_{0,1} X) \right)$$

$$=: \bigoplus_{p+q=k} \Lambda^{p,q} X$$

- $A^k(X, \mathbb{R}) := \Gamma(X, \Lambda^k X)$ space of C^∞ real k-forms on X
- $A^k(X, \mathbb{C}) := \Gamma(X, \Lambda^k_{\mathbb{C}} X)$ ... .. complex ..
- $A^{p,q}(X) := \Gamma(X, \Lambda^{p,q} X)$ . .. (p,q)-forms ..

$$\Rightarrow \Lambda^k_{\mathbb{C}} X = \Lambda^k X \otimes_{\mathbb{R}} \mathbb{C}$$

$$\Lambda^k_{\mathbb{C}} X = \Lambda^k (T^*_{\mathbb{C}} X) = \Lambda^k (T^* X \otimes_{\mathbb{R}} \mathbb{C}) = \Lambda^k (T^* X) \otimes_{\mathbb{R}} \mathbb{C}$$

$$A^k(X, \mathbb{C}) = A^k(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

$$A^k(X, \mathbb{C}) = \bigoplus_{p+q=k} A^{p,q}(X)$$

Taking local coordinate chart $(U, \varphi: z_1, \dots, z_m)$, then $\forall\ \alpha \in A^k(X, \mathbb{C})$, locally it looks like

$$\alpha = \sum_{p+q=k} \underbrace{\alpha_{i_1 \cdots i_p j_1 \cdots j_q}}_{\in C^\infty(U, \mathbb{C})} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \qquad (*)$$

Def 3.1 The exterior derivative $d: A^k(X,\mathbb{R}) \to A^{k+1}(X,\mathbb{R})$ extends $\mathbb{C}$-linearly to

$$d: A^k(X,\mathbb{C}) \to A^{k+1}(X,\mathbb{C})$$

define

$$\partial := \pi_{p+1,q} \circ d : A^{p,q}(X) \to A^{p+1,q}(X)$$

$$\bar\partial := \pi_{p,q+1} \circ d : A^{p,q}(X) \to A^{p,q+1}(X)$$

where $\pi_{p,q} : A^{p+q}(X,\mathbb{C}) \to A^{p,q}(X)$

Prop 3.2

X cplex mfld $\overset{(*)}{\Rightarrow}$ $d = \partial + \bar\partial$

i.e. $d : A^{p,q}(X) \to A^{p+1,q}(X) \oplus A^{p,q+1}(X)$

In particular, $d^2 = 0 \iff \partial^2 = 0 = \bar\partial^2$

$\partial\bar\partial = -\bar\partial\partial$

Rmk If X is merely an almost cplex mfld, then $d = \partial + \bar\partial$ not true!

$$d: A^{p,q}(X) \to A^{p+1,q}(X) + A^{p,q+1}(X) + \underline{A^{p-1,q+2}(X) + A^{p+2,q-1}(X)}$$

§4. Kähler mflds

(X, J) cplex mfld, $\dim_{\mathbb{C}} X = m$. g a Riemiannian metric

Def 4.1 g is Hermitian if J is an isometry w.r.t. g.

i.e.

$$g(J\cdot, J\cdot) = g(\cdot,\cdot)$$

$\rightsquigarrow$ (X, J, g) is called a <u>Hermitian mfld</u>.

Rmk Hermitian metrics always exist on cplex mflds!

g a Riemannian metric

$$\tilde g(\cdot,\cdot) := g(J\cdot, J\cdot) + g(\cdot,\cdot)$$

$\rightsquigarrow$ $\tilde g$ is Hermitian.

(X, J, g) Hermitian mfd. define the following 2-tensor

$$\omega(\cdot,\cdot) := g(J\cdot,\cdot) \qquad \left(\in \Gamma(X, T^*X \otimes T^*X) \right)$$

called the fundamental form.

Rmk (1) $g(JV, JW) = g(V,W) \Rightarrow \omega(V,W) = -\omega(W,V)$

$\rightarrow \omega$ is 2-form. $\left(\in A^2(X, \mathbb{R}) \right)$

(2) J is an isometry w.r.t. ω.

i.e. $\omega(JV, JW) = \omega(V,W)$

(3) $g \,\&\, J \Rightarrow \omega$; $\omega \,\&\, J \Rightarrow g$ $\qquad \left(g(\cdot,\cdot) = \omega(\cdot, J\cdot) \right)$

Def 4.2 (X, J, g) is called Kähler if ω is closed. i.e. $d\omega = 0$.

Hermitian metric

• $g : TX \times TX \to \mathbb{R}$ extends $\mathbb{C}$-linearly to $T_{\mathbb{C}}X$

$$g : T_{\mathbb{C}}X \times T_{\mathbb{C}}X \to \mathbb{C}$$

$$(V \otimes \lambda, W \otimes \mu) \mapsto g(V,W)\lambda\mu \qquad \lambda, \mu \in \mathbb{C}$$

$\Rightarrow$ (a) $g(\bar{V}, \bar{W}) = \overline{g(V,W)}$

(b) $g(V, \bar{V}) \geq 0$ & "$=0$" iff $V = 0$

(c) $g(V,W) = 0 \quad \forall V, W \in T_{1,0}X$

(d) $g(V,W) = 0 \quad \forall V, W \in T_{0,1}X$

Rmk (1) we say g Hermitian metric. because

$$g_{\mathbb{C}}(V,W) := g(V, \bar{W})$$

defines a Hermitian metric on the $\mathbb{C}$-v.b. $T_{\mathbb{C}}X$.

$T_{\mathbb{C}}X = T_{1,0}X \oplus T_{0,1}X$ is orthogonal w.r.t. $g_{\mathbb{C}}$

(2) $\omega \in A^2(X,\mathbb{R}) \cap A^{1,1}(X)$ from (c) & (d)

(3) local coord. char $(U, \varphi; z_1, \dots, z_m)$ $z_j = x_j + i y_j$

$$\mathrm{Vol}_g = \sqrt{\det(g)}\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_m \wedge dy_m$$

$$= \frac{\omega^m}{m!}$$

§ 5. Hodge theory.

(X, J, g) cplex mfld. for g a Riemannian metric. $\dim_{\mathbb{C}} X = m$

g inner product on TX

$\rightsquigarrow$ induces g on $T^*X = \mathrm{Hom}_{\mathbb{R}}(TX, \mathbb{R})$

$\rightsquigarrow$ induces g on $\Lambda^k X = \Lambda^k(T^*X)$ $1 \le k \le 2m$

Locally. choose orthogonal normal basis $\{e^1, \dots, e^{2m}\}$ for T^*X

then $\{e^{j_1} \wedge \dots \wedge e^{j_k}, 1 \le j_1 < j_2 < \dots < j_k \le 2m\}$ is an orthogonal normal basis for $\Lambda^k X$.

Def 5.1 $\forall\ 0 \le k \le 2m$, on $A^k(X, \mathbb{R})$ we define

(1) global L^2-norm:

$\forall\ \theta, \eta \in A^k(X, \mathbb{R})$

$$(\theta, \eta)_{L^2} := \int_X g(\theta, \eta)\, \mathrm{vol}_g$$

In particular, $\eta = \theta$, $(\theta, \theta)_{L^2} =: \|\theta\|^2_{L^2}$

(2) Hodge $*$:

$$* : A^k(X,\mathbb{R}) \longrightarrow A^{2m-k}(X,\mathbb{R})$$

via

$$\theta \wedge *\eta = g(\theta,\eta)\,\mathrm{Vol}_g.$$

well-defined since $\wedge : A^k(X,\mathbb{R}) \times A^{2m-k}(X,\mathbb{R}) \to A^{2m}(X,\mathbb{R})$
$(\theta\ ,\ \eta) \mapsto \theta\wedge\eta$

is non-degenerate.

(3) adjoint exterior derivative

$$d^* : A^k(X,\mathbb{R}) \to A^{k-1}(X,\mathbb{R})$$

via
$$d^* = -*d*$$

(4) Hodge Laplacian

$$\Delta := dd^* + d^*d : A^k(X,\mathbb{R}) \longrightarrow A^k(X,\mathbb{R})$$

(5) harmonic forms :

$$\mathcal{H}^k_g(X,\mathbb{R}) := \{\theta \in A^k(X,\mathbb{R}) : \Delta(\theta) = 0\}$$

space of "harmonic k-forms".

From now on, g is Hermitian metric. $\rightsquigarrow$ ω fundamental 2-form

(6) Lefschetz operator :

$$L_\omega : A^k(X,\mathbb{R}) \longrightarrow A^{k+2}(X,\mathbb{R})$$
$$\theta \longmapsto \theta\wedge\omega$$

(7) dual Lefschetz operator :

$$\Lambda_\omega : A^k(X,\mathbb{R}) \longrightarrow A^{k-2}(X,\mathbb{R})$$

via

$$g(\Lambda_\omega(\theta),\eta) = g(\theta, L_\omega(\eta)) \qquad \forall\ \theta\in A^k(X,\mathbb{R}),\ \eta\in A^{k-2}(X,\mathbb{R})$$

$\mathbb{R} \rightsquigarrow \mathbb{C}$

$*$, L_ω, Λ_ω extend $\mathbb{C}$-linearly to $A^k(X,\mathbb{C}) = A^k(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. $\rightsquigarrow d^*, \Delta$

real in the following sense:

$$\overline{*(\theta)} = *(\bar\theta), \qquad \overline{L_\omega(\theta)} = L_\omega(\bar\theta), \qquad \overline{\Lambda_\omega(\theta)} = \Lambda_\omega(\bar\theta)$$

g on TX $\rightsquigarrow$ $g_{\mathbb{C}}$ Hermitian metric on $T_{\mathbb{C}}X$

on $\Lambda^k X = \Lambda^k(T^*X)$ $\rightsquigarrow$ $g_{\mathbb{C}}$ Hermitian metric $\Lambda^k_{\mathbb{C}} X$

$$g_{\mathbb{C}}(\theta,\eta) := g(\theta,\bar\eta) \qquad \forall\, \theta,\eta \in A^k(X,\mathbb{C})$$

$\rightsquigarrow$ global L^2-norm on $A^k(X,\mathbb{C})$:

$$(\theta,\eta)_{L^2} := \int_X g_{\mathbb{C}}(\theta,\bar\eta)\, \mathrm{vol}_g$$

Rmk.:

(1) $* : A^{p,q}(X) \longrightarrow A^{m-q,m-p}(X)$

(2) $*^2 = (-1)^{k(2m-k)}\, \mathrm{Id}$ on $A^k(X,\mathbb{C})$

(3) $g(*\theta,*\eta) = g(\theta,\eta)$ $\rightsquigarrow$ $g_{\mathbb{C}}(*\theta,*\eta) = g(*\theta,\overline{*\eta})$

$$= g(*\theta,*\bar\eta)$$
$$= g(\theta,\bar\eta)$$
$$= g_{\mathbb{C}}(\theta,\eta)$$

(4) $A^k(X,\mathbb{C}) = \bigoplus_{p+q=k} A^{p,q}(X)$ orthogonal decom. w.r.t. $g_{\mathbb{C}}$

Def 5.2 $\partial^* := -*\bar\partial* : A^{p,q}(X) \longrightarrow A^{p-1,q}(X)$ adjoint

$\bar\partial^* := -*\partial* : A^{p,q}(X) \longrightarrow A^{p,q-1}(X)$ adjoint

$\Delta_\partial := \partial\partial^* + \partial^*\partial : A^{p,q}(X) \to A^{p,q}(X)$

$\Delta_{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : A^{p,q}(X) \longrightarrow A^{p,q}(X)$

Prop 5.3 (1) $\Lambda_\omega(\theta) = (-1)^k * L_\omega * \theta \quad \forall\, \theta \in A^k$

(2) X cpt (i.e. without boundary). then

$$(\partial\theta, \eta)_{L^2} = (\theta, \partial^*\eta)_{L^2} \qquad \theta \in A^{p,q} \quad \eta \in A^{p+1,q}$$

$$(\bar\partial\theta, \eta)_{L^2} = (\theta, \bar\partial^*\eta)_{L^2} \qquad \theta \in A^{p,q} \quad \eta \in A^{p,q+1}$$

$$(d\theta, \eta)_{L^2} = (\theta, d^*\eta)_{L^2} \qquad \theta \in A^k \quad \eta \in A^{k+1}$$

pf (sketch)

(1) $g(\Lambda_\omega(\theta), \eta) = g(\theta, L_\omega(\eta)) \qquad \forall\, \theta \in A^k,\ \eta \in A^{k-2}$

$$\Rightarrow \eta \wedge * \Lambda_\omega(\theta) = L_\omega(\eta) \wedge * \theta$$
$$= \eta \wedge \omega \wedge * \theta$$
$$= \eta \wedge * \theta \wedge \omega$$
$$= \eta \wedge L_\omega(*\theta)$$

$$\Rightarrow * \Lambda_\omega(\theta) = L_\omega * \theta$$

(2) show $(d\theta, \eta)_{L^2} = (\theta, d^*\eta)_{L^2} \quad \forall\, \theta \in A^k \quad \eta \in A^{k+1}$

$$g(d\theta, \eta)\,\mathrm{vol}_g = d\theta \wedge * \eta = d(\theta \wedge * \eta) - (-1)^k \theta \wedge d * \eta$$
$$= d(\theta \wedge * \eta) - (-1)^k (-1)^{k(2m-k)} \theta \wedge * * d * \eta$$
$$= d(\theta \wedge * \eta) - g(\theta, * d * \eta)\,\mathrm{vol}_g$$
$$= d(\theta \wedge * \eta) + g(\theta, d^*\eta)\,\mathrm{vol}_g$$

∎

Prop 5.4 (Kähler identities) (X, J, g, ω) Kähler mfld

(1) $[\bar\partial, L_\omega] = [\partial, L_\omega] = [\bar\partial^*, \Lambda_\omega] = [\partial^*, \Lambda_\omega] = 0$

(2) $[\bar\partial^*, L_\omega] = i\partial. \quad [\partial^*, L_\omega] = -i\bar\partial$

$[\Lambda_\omega, \bar\partial] = -i\partial^*$. $[\Lambda_\omega, \partial] = i\bar\partial^*$

(3) $\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}$

Δ commutes with $*$, ∂, $\bar\partial$, ∂^*, $\bar\partial^*$, L_ω, and Λ_ω.

<u>pf</u> (sketch)

(1) $[\bar\partial, L_\omega](\theta) = \bar\partial(L_\omega(\theta)) - L_\omega(\bar\partial\theta)$

$$= \bar\partial(\theta\wedge\omega) - \bar\partial\theta\wedge\omega$$

$$= \bar\partial\theta\wedge\omega + (-1)^k\,\theta\wedge\bar\partial\omega - \bar\partial\theta\wedge\omega$$

$$= 0$$

(2) define $H : A^k(X,\mathbb{C}) \longrightarrow A^k(X,\mathbb{C})$ $\quad H = (k-m)\,Id.$

$$\theta \longmapsto (k-m)\,\theta$$

check: $[L_\omega, \Lambda_\omega] = H$

$$-i\bar\partial = [\partial^*, L_\omega]$$

$[\Lambda_\omega, \bar\partial] = i[\Lambda_\omega, [\partial^*, L_\omega]]$

$$= -i\left([\partial^*, [L_\omega, \Lambda_\omega]] + [L_\omega, [\Lambda_\omega, \partial^*]]\right)$$

$$= -i\left([\partial^*, H]\right)$$

$$= -i\partial^*$$

(3) $\underline{\partial\bar\partial^* + \bar\partial^*\partial = 0}$ because

$$i(\partial\bar\partial^* + \bar\partial^*\partial) = \partial[\Lambda_\omega, \partial] + [\Lambda_\omega, \partial]\partial$$

$$= \partial\Lambda_\omega\partial - \partial^2\Lambda_\omega + \Lambda_\omega\partial^2 - \partial\Lambda_\omega\partial$$

$$= 0$$

$\underline{\bar\partial\partial^* + \partial^*\bar\partial = 0}$

$\Rightarrow \Delta_\partial = \Delta_{\bar\partial}$

- $\Delta = \Delta_\partial + \Delta_{\bar\partial}$

"spaces of harmonic forms": (X, J, g) Hermitian mfd. ☒

(1) $\mathcal{H}^k(X, g) := \{\theta \in A^k(X, \mathbb{C}) : \Delta(\theta) = 0\}$

(2) $\mathcal{H}^k_\partial(X, g) := \{\theta \in A^k(X, \mathbb{C}) : \Delta_\partial(\theta) = 0\}$

(3) $\mathcal{H}^k_{\bar\partial}(X, g) := \{\theta \in A^k(X, \mathbb{C}) : \Delta_{\bar\partial}(\theta) = 0\}$

(4) $\mathcal{H}^{p,q}(X, g) := \{\theta \in A^{p,q}(X) : \Delta(\theta) = 0\}$

(5) $\mathcal{H}^{p,q}_\partial(X, g) := \{\theta \in A^{p,q}(X) : \Delta_\partial(\theta) = 0\}$

(6) $\mathcal{H}^{p,q}_{\bar\partial}(X, g) := \{\theta \in A^{p,q}(X) : \Delta_{\bar\partial}(\theta) = 0\}$

$\rightsquigarrow$

$$\mathcal{H}^k(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, g)$$

orthogonal decomp. w.r.t. $g_{\mathbb{C}}$

$$\mathcal{H}^k_\partial(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_\partial(X, g)$$

$$\mathcal{H}^k_{\bar\partial}(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar\partial}(X, g)$$

In particular, X Kähler $\left(\Rightarrow \Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}\right)$

$\Rightarrow$

$$\mathcal{H}^k(X, g) = \mathcal{H}^k_\partial(X, g) = \mathcal{H}^k_{\bar\partial}(X, g)$$

$$\mathcal{H}^{p,q}(X, g) = \mathcal{H}^{p,q}_\partial(X, g) = \mathcal{H}^{p,q}_{\bar\partial}(X, g)$$

Prop 5.5

(1) (Poincaré duality)

$$* : \mathcal{H}^k(X, g) \xrightarrow{\cong} \mathcal{H}^{2m-k}(X, g)$$
$$\theta \longmapsto *\theta$$

$\Big| [\Delta, *] = 0$

(2) (Hodge duality)

$$* : \mathcal{H}^{p,q}_{\partial}(X, g) \xrightarrow{\cong} \mathcal{H}^{m-q, m-p}_{\partial}(X, g)$$
$$\theta \longmapsto *\theta$$

$\Big| [\Delta_\partial, *] = 0$

(3) (Serre duality)

$$\bar{*} : \mathcal{H}^{p,q}_{\bar\partial}(X, g) \xrightarrow{\cong} \mathcal{H}^{m-p, m-q}_{\bar\partial}(X, g)$$
$$\theta \longmapsto *\bar\theta$$

Def 5.6 (X, J) cplex mfld. $\dim_{\mathbb{C}} X = m$. $\forall\, 0 \le k \le 2m$

(1) Betti cohomology:

$$H^k_B(X, \mathbb{Z}) := H^k(C^\bullet_{sing}, d)$$

where $(C^\bullet_{sing}, d)$: singular cochain complex. Betti complex

(2) De Rham cohomology:

$$H^k_{dR}(X, \mathbb{C}) := H^k(C^\bullet_{dR}, d)$$
$$= \frac{\ker(d : A^k(X, \mathbb{C}) \to A^{k+1}(X, \mathbb{C}))}{\mathrm{Im}(d : A^{k-1}(X, \mathbb{C}) \to A^k(X, \mathbb{C}))}$$

$$(C^\bullet_{dR}, d) : \quad 0 \to A^0(X, \mathbb{C}) \xrightarrow{d} A^1(X, \mathbb{C}) \xrightarrow{d} A^2(X, \mathbb{C}) \to \cdots$$

de Rham complex

(3) Dolbeault cohomology:

$$H^k_{Dol}(X,\mathbb{C}) := \bigoplus_{p+q=k} H^{p,q}(X)$$

$$:= \bigoplus_{p+q=k} H^q(C^{p,\bullet}_{Dol}, \bar\partial)$$

$$= \bigoplus_{p+q=k} \frac{\operatorname{Ker}(\bar\partial : A^{p,q}(X) \to A^{p,q+1}(X))}{\operatorname{Im}(\bar\partial : A^{p,q-1}(X) \to A^{p,q}(X))}$$

$(C^{p,\bullet}_{Dol}, \bar\partial)$:

$$0 \to A^{p,0}(X) \xrightarrow{\bar\partial} A^{p,1}(X) \xrightarrow{\bar\partial} \cdots \to A^{p,n}(X) \to 0$$

Dolbeault complex

Rmk (1) Betti coho is enough for X top space

(2) De Rham coho. is enough for X diff. mfld. $(d^2=0)$

(3) Dolbeault coho. need X cplex mfld $(d^2=0 \overset{d=\partial+\bar\partial}{\Rightarrow} \bar\partial^2=0)$

Thm 5.7 (Main thms in classical Hodge theory)

(1) De Rham thm:

$$H^k_B(X,\mathbb{C}) := H^k_B(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^k_{dR}(X,\mathbb{C})$$

(2) Dolbeault thm:

$$H^{p,q}(X) \simeq H^q(X, \Omega^p_X)$$

where Ω^p_X : sheaf of holo. p-forms.

$\forall\, U \subset X$ open $\Omega^p_X(U) := \{ \alpha \in A^{p,0}(U) : \bar\partial\alpha = 0 \}$

(3) Hodge decomposition thm:

(X, J, g, ω) compact Kähler mfld.

$$H^k_{dR}(X,\mathbb{C}) \simeq H^k_{Dol}(X,\mathbb{C})$$
$$= \bigoplus_{p+q=k} H^{p,q}(X)$$

$$\overline{H^{q,p}(X)} = H^{p,q}(X) \quad \leftarrow \text{ conjugation.}$$

$$H^{p,q}(X) \simeq H^{m-p,m-q}(X) \quad \sim \text{ Serre duality}$$

pf (sketch)

(1) $H^k_B(X, \mathbb{C}) \simeq H^k(X, \underline{\mathbb{C}})$

local Poincaré lemma implies

$\mathcal{A}^k_{\mathbb{C}}$: sheaf of C^∞ $\mathbb{C}$-valued k-forms

In particular, $\mathcal{A}^k_{\mathbb{C}}(X) = A^k(X, \mathbb{C})$

$$0 \to \underline{\mathbb{C}} \hookrightarrow \mathcal{A}^0_{\mathbb{C}} \to \mathcal{A}^1_{\mathbb{C}} \to \cdots \quad \text{exact}$$

$$\Rightarrow H^k(X, \underline{\mathbb{C}}) \simeq H^k(\Gamma(X, \mathcal{A}^\bullet_{\mathbb{C}})) = H^k_{dR}(X, \mathbb{C})$$

$$0 \to A^0(X, \mathbb{C}) \xrightarrow{d} A^1(X, \mathbb{C}) \to \cdots$$

(2) $\mathcal{A}^{p,q}_X$: sheaf of C^∞ (p,q)-forms

In part. $\mathcal{A}^{p,q}_X(X) = A^{p,q}(X)$

$$0 \to \Omega^p_X \hookrightarrow \mathcal{A}^{p,0}_X \xrightarrow{\bar\partial} \mathcal{A}^{p,1}_X \xrightarrow{\bar\partial} \cdots \quad \text{exact}$$

$$\rightsquigarrow H^q(X, \Omega^p_X) \simeq H^{p,q}(X)$$

(3) to show $H^k_{dR}(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X) \ni [\theta^{p,q}]$

$[\theta]_k \in H^k_{dR}$

$$\mathcal{H}^k(X, g) = \bigoplus_{p+q=k} \mathcal{H}^k(X, g) = \bigoplus_{p+q=k} \mathcal{H}^k_{\bar\partial}(X, g)$$

$\theta \in \mathcal{H}^k(X,g)$, $\theta^{p,q}$

Kähler

check independ of g

$\boxtimes$

Rmk $b_k := \dim_{\mathbb{C}} H^k_{dR}(X, \mathbb{C})$ Betti number

$h_{p,q} := \dim_{\mathbb{C}} H^{p,q}(X)$ Hodge number

(X, J, g, ω) compact Kähler $\dim_{\mathbb{C}} X = m$

$\Rightarrow$

- $b_k = b_{2m-k}$ (Poincaré duality)
- $h_{p,q} = h_{m-q,m-p}$ (Hodge duality)
- $h_{p,q} = h_{q,p}$ (conjugation)

$\Big\} \Rightarrow h_{p,q} = h_{m-p,m-q}$ (Serre duality)

- $b_k = \bigoplus_{p+q=k} h_{p,q}$ (Hodge decomposition)

$\rightsquigarrow b_k$ is even if k odd.

$\rightsquigarrow$ Hodge diamond

$$
\begin{array}{ccccccccc|c}
 & & & & h_{0,0} & & & & & b_0 \\
 & & & h_{1,0} & & h_{0,1} & & & & b_1 \\
 & & h_{2,0} & & h_{1,1} & & h_{0,2} & & & b_2 \\
 & \cdots & & & & & & \cdots & & \vdots \\
h_{m,0} & & \cdots & & & \cdots & & & h_{0,m} & b_m \\
 & \ddots & & & & & & \cdots & & \\
 & & h_{m,m-2} & & h_{m-1,m-1} & & h_{m-2,m} & & & b_{2m-2} \\
 & & & h_{m,m-1} & & h_{m-1,m} & & & & b_{2m-1} \\
 & & & & h_{m,m} & & & & & b_{2m}
\end{array}
$$

Hodge *

Serre

Poincaré

conjugation