Recall & Clarification :

The arguments in Lecture 8 only prove the Schauder estimate. They do not show any $C^{2,\alpha}$ regularity. Furthermore, we need to replace C^2 assumption by $C^{2,\alpha}$ assumption. (Actually, C^2 assumption is enough. This will be done in Lecture 9).

Now, we only prove the following in Lecture 8:

- Special case (small oscillation)

Proposition. $L=\sum_{i,j} a_{ij}\partial_{ij}+\sum_i b_i\partial_i+c$, $a_{ij}, b_i, c\in C^\alpha(\bar{B}_R)$, $\alpha\in(0,1)$

$\lambda I_n\le (a_{ij})\le\Lambda I_n$ $(\lambda,\Lambda>0)$, $\|a_{ij}\|^*_{C^\alpha(B_R)}, R\|b_i\|^*_{C^\alpha(B_R)}, R^2\|c\|^*_{C^\alpha(B_R)}\le\Lambda$

Suppose that $u\in C^{2,\alpha}(\bar{B}_R)$ satisfies $Lu=f$ in B_R for some $f\in C^\alpha(\bar{B}_R)$. Then $\exists\, \varepsilon(\lambda,\Lambda,\alpha,n)>0$, s.t. if $\sup_{x',x''\in B_R}|a_{ij}(x')-a_{ij}(x'')|\le\varepsilon$, then

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})}\le C(\lambda,\Lambda,\alpha,n)\left(\|u\|_{L^\infty(B_R)}+R^2\|f\|^*_{C^\alpha(B_R)}\right).$$

Remark. The regularity assumption $u\in C^{2,\alpha}(\bar{B}_R)$ guarantees that $\tilde{f}\in C^{2,\alpha}(\bar{B}_R)$ and $K<+\infty$. But this assumption can be replaced by a weaker assumption as follows.

Proposition. $L=\sum_{i,j} a_{ij}\partial_{ij}+\sum_i b_i\partial_i+c$, $a_{ij}, b_i, c\in C^\alpha(\bar{B}_R)$, $\alpha\in(0,1)$

$\lambda I_n\le (a_{ij})\le\Lambda I_n$ $(\lambda,\Lambda>0)$, $\|a_{ij}\|^*_{C^\alpha(B_R)}, R\|b_i\|^*_{C^\alpha(B_R)}, R^2\|c\|^*_{C^\alpha(B_R)}\le\Lambda$

Suppose that $u\in L^\infty(B_R)\cap C^{2,\alpha}(B_R)$ satisfies $Lu=f$ in B_R for some $f\in C^\alpha(\bar{B}_R)$. Then $\exists\, \varepsilon(\lambda,\Lambda,\alpha,n)>0$, s.t. if $\sup_{x',x''\in B_R}|a_{ij}(x')-a_{ij}(x'')|\le\varepsilon$, then

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})}\le C(\lambda,\Lambda,\alpha,n)\left(\|u\|_{L^\infty(B_R)}+R^2\|f\|^*_{C^\alpha(B_R)}\right).$$

Proof. $\forall\, r\in(0,R)$, $u\in C^{2,\alpha}(\bar{B}_r)$, the previous proposition $\Rightarrow$

$$\|u\|^*_{C^{2,\alpha}(B_{r/2})}\le C(\lambda,\Lambda,\alpha,n)\left(\|u\|_{L^\infty(B_r)}+r^2\|f\|^*_{C^\alpha(B_R)}\right)$$

Let $r\to R$ $\Rightarrow$ Proposition. #

- General case

Theorem. $L=\sum_{i,j} a_{ij}\partial_{ij}+\sum_i b_i\partial_i+c$, $a_{ij}, b_i, c\in C^\alpha(\bar{B}_R)$, $\alpha\in(0,1)$

$\lambda I_n\le(a_{ij})\le\Lambda I_n$ $(\lambda,\Lambda>0)$, $\|a_{ij}\|^*_{C^\alpha(B_R)}, R\|b_i\|^*_{C^\alpha(B_R)}, R^2\|c\|^*_{C^\alpha(B_R)}\le\Lambda$

Suppose that $u\in L^\infty(B_R)\cap C^{2,\alpha}(B_R)$ satisfies $Lu=f$ in B_R

for some $f\in C^\alpha(\bar{B}_R)$, then

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})}\le C(\lambda,\Lambda,\alpha,n)\left(\|u\|_{L^\infty(B_R)}+R^2\|f\|^*_{C^\alpha(B_R)}\right).$$

- Schauder estimate in domains

Theorem. Ω: bounded domain in $\mathbb{R}^n$.

$L=\sum_{i,j} a_{ij}\partial_{ij}+\sum_i b_i\partial_i+c$, $a_{ij}, b_i, c\in C^\alpha(\bar{\Omega})$, $\alpha\in(0,1)$.

$\lambda I_n\le(a_{ij})\le\Lambda I_n$ $(\lambda,\Lambda>0)$. $\|a_{ij}\|_{C^\alpha(\Omega)}, \|b_i\|_{C^\alpha(\Omega)}, \|c\|_{C^\alpha(\Omega)}\le\Lambda$.

If $u\in L^\infty(\Omega)\cap C^{2,\alpha}(\Omega)$ satisfies $Lu=f$ in Ω

for some $f\in C^\alpha(\bar{\Omega})$. then $\forall\ \Omega'\subset\subset\Omega$,

$$\|u\|_{C^{2,\alpha}(\Omega')}\le C\cdot\left(\|u\|_{L^\infty(\Omega)}+\|f\|_{C^\alpha(\Omega)}\right)$$

where $C=C(\Omega', \mathrm{dist}(\Omega',\partial\Omega), \lambda, \Lambda, \alpha, n)$.

- Schauder estimate revisited

Theorem. Ω: bounded domain in $\mathbb{R}^n$. $\alpha,\beta\in(0,1)$

$L=\sum_{i,j} a_{ij}\partial_{ij}+\sum_i b_i\partial_i+c$, $a_{ij}, b_i, c\in C^\alpha(\bar{\Omega})$.

$\lambda I_n\le(a_{ij})\le\Lambda I_n$ $(\lambda,\Lambda>0)$. $\|a_{ij}\|_{C^\alpha(\Omega)}, \|b_i\|_{C^\alpha(\Omega)}, \|c\|_{C^\alpha(\Omega)}\le\Lambda$.

If $u\in C^{2,\alpha}(\Omega)$ satisfies $Lu=f$ in Ω

for some $f\in C^\alpha(\Omega)$ with $\|u\|_{C_{-\beta}(\Omega)}, \|f\|_{C^\alpha_{2-\beta}(\Omega)}<+\infty$.

Then

$$\|u\|_{C^{2,\alpha}_{-\beta}(\Omega)}\le C(\lambda,\Lambda,\alpha,\beta,n)\left(\|u\|_{C_{-\beta}(\Omega)}+\|f\|_{C^\alpha_{2-\beta}(\Omega)}\right).$$

In particular, $u\in C^{2,\alpha}_{-\beta}(\Omega)$.

Corollary. $L=\sum_{i,j} a_{ij}\partial_{ij}+\sum_i b_i\partial_i+c$. $a_{ij}, b_i, c\in C^\alpha(\bar B_R)$, $\alpha\in(0,1)$

$\lambda I_n\le (a_{ij})\le \Lambda I_n$, $\|a_{ij}\|_{C^\alpha(B_R)}, \|b_i\|_{C^\alpha(B_R)}, \|c\|_{C^\alpha(B_R)}\le\Lambda$

$c\le 0$. If $u\in C(\bar B_R)\cap C^{2,\alpha}(B_R)$ satisfies

$$\begin{cases} Lu=f & \text{in } B_R \\ u=0 & \text{on } \partial B_R \end{cases}$$

for some $f\in C^\alpha(B_R)$ with $\|f\|_{C^\alpha_{2-\beta}(B_R)}<+\infty$.

Then $u\in C^{2,\alpha}_{-\beta}(B_R)$ and

$$\|u\|_{C^{2,\alpha}_{-\beta}(B_R)}\le C(R,\lambda,\Lambda,\alpha,n)\|f\|_{C^\alpha_{2-\beta}(B_R)}.$$

Lecture 9. Dirichlet Problem in balls

1. Zero boundary value

Theorem. $L=\sum_{i,j} a_{ij}\partial_{ij}+\sum_i b_i\partial_i+c$. $a_{ij}, b_i, c\in C^\alpha(\bar B_R)$. $\alpha,\beta\in(0,1)$

$\lambda I_n\le (a_{ij})\le\Lambda I_n$. $\|a_{ij}\|_{C^\alpha(B_R)}, \|b_i\|_{C^\alpha(B_R)}, \|c\|_{C^\alpha(B_R)}\le\Lambda$. $c\le 0$.

For any $f\in C^\alpha_{2-\beta}(B_R)$. $\exists$ unique solution $u\in C^{2,\alpha}_{-\beta}(B_R)$

of the Dirichlet problem

$$\begin{cases} Lu=f & \text{in } B_R \\ u=0 & \text{on } \partial B_R. \end{cases}$$

Remark. $C^\alpha_{2-\beta}(B_R)\subset C^\alpha(B_R)$. $C^{2,\alpha}_{-\beta}(B_R)\subset C(\bar B_R)\cap C^{2,\alpha}(B_R)$.

Proof. By scaling method, assume $R=1$.

Continuity method: define $L_t=tL+(1-t)\Delta$ $t\in[0,1]$.

$\Rightarrow L_0=\Delta$. $L_1=L$.

write $L_t=\sum_{i,j} a^t_{ij}\partial_{ij}+\sum_i b^t_i\partial_i+c^t$.

where $a^t_{ij}=ta_{ij}+(1-t)\delta_{ij}$. $b^t_i=tb_i$. $c^t=tc$.

Assumptions $\Rightarrow$ $\min(\lambda,1)I_n\le(a^t_{ij})\le\max(\Lambda,1)I_n$.

$\|a^t_{ij}\|_{C^\alpha(B_1)}, \|b^t_i\|_{C^\alpha(B_1)}, \|c^t\|_{C^\alpha(B_1)}\le\Lambda+1$.

$\forall\, v \in C^{2,\alpha}_{-\beta}(B_1)$. Compute

$$\|L_t v\|_{C^{\alpha}_{2-\beta}(B_1)} \le \sum_{i,j} \|a^t_{ij}\partial_{ij} v\|_{C^{\alpha}_{2-\beta}(B_1)} + \sum_i \|b^t_i \partial_i v\|_{C^{\alpha}_{2-\beta}(B_1)} + \|c^t v\|_{C^{\alpha}_{2-\beta}(B_1)}$$

[Recall when $\sigma \ge 0$. $\|wv\|_{C^{\alpha}_{\sigma}(B_1)} \le \|w\|_{C^{\alpha}_{0}(B_1)} \cdot \|v\|_{C^{\alpha}_{\sigma}(B_1)}$ (cf. Lecture 8)]

$$\le \sum_{i,j} \|a^t_{ij}\|_{C^{\alpha}_{0}(B_1)} \cdot \|\partial_{ij} v\|_{C^{\alpha}_{2-\beta}(B_1)}$$

$$+ \sum_i \|b^t_i\|_{C^{\alpha}_{0}(B_1)} \|\partial_i v\|_{C^{\alpha}_{2-\beta}(B_1)} + \|c^t\|_{C^{\alpha}_{0}(B_1)} \|v\|_{C^{\alpha}_{2-\beta}(B_1)}$$

[$\forall\, x, y \in B_1$. $d_x, d_y, d_{xy} \le 1 \Rightarrow$ · $\|w\|_{C^{\alpha}_{0}(B_1)} \le \|w\|_{C^{\alpha}(B_1)}$
· $\|w\|_{C^{\alpha}_{2-\beta}(B_1)} \le \|w\|_{C^{\alpha}_{1-\beta}(B_1)} \le \|w\|_{C^{\alpha}_{-\beta}(B_1)}$]

$$\le \sum_{i,j} \|a^t_{ij}\|_{C^{\alpha}(B_1)} \cdot \|\partial_{ij} v\|_{C^{\alpha}_{2-\beta}(B_1)}$$

$$+ \sum_i \|b^t_i\|_{C^{\alpha}(B_1)} \|\partial_i v\|_{C^{\alpha}_{1-\beta}(B_1)} + \|c^t\|_{C^{\alpha}(B_1)} \|v\|_{C^{\alpha}_{-\beta}(B_1)}$$

$$\le C(\Lambda, n)\, \|v\|_{C^{2,\alpha}_{-\beta}(B_1)}$$

$\Rightarrow \quad \|L_t v\|_{C^{\alpha}_{2-\beta}(B_1)} \le C \cdot \|v\|_{C^{2,\alpha}_{-\beta}(B_1)}$

$\Rightarrow \quad L_t : C^{2,\alpha}_{-\beta}(B_1) \longrightarrow C^{\alpha}_{2-\beta}(B_1)$ is a bounded linear operator.

Consider the Dirichlet Problem

$$(*_t) \quad \begin{cases} L_t u = h & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

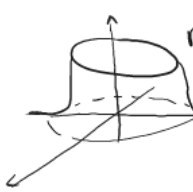
Define $I = \{ t \in [0,1] \mid (*_t)$ is solvable in $C^{2,\alpha}_{-\beta}(B_1)$ for any $h \in C^{\alpha}_{2-\beta}(B_1) \}$

Goal: $1 \in I$ ($\Leftrightarrow$ Theorem. uniqueness follows from maximum principle)

Step 1. $0 \in I$ ($L_0 = \Delta$)

$\forall\, h \in C^{\alpha}_{2-\beta}(B_1)$. h may not belong to $C^{\alpha}(\bar{B}_1)$. So we can not apply the result in Lecture 6. To avoid such case. We take $\eta_\varepsilon \in C^{\infty}(\bar{B}_1)$ s.t.

$$\begin{cases} 0 \le \eta_\varepsilon \le 1 & \text{in } \bar{B}_1 \\ \eta_\varepsilon \equiv 1 & \text{in } B_{1-2\varepsilon} \\ \eta_\varepsilon = 0 & \text{in } B_1 \setminus B_{1-\varepsilon} \end{cases}$$



η_ε

Set $h_\varepsilon = \eta_\varepsilon \cdot h \Rightarrow h_\varepsilon \in C^\alpha(\overline{B}_1)$

Lecture 6 $\Rightarrow \exists\, u_\varepsilon \in C(\overline{B}_1) \cap C^{2,\alpha}(B_1)$ s.t. $\begin{cases} \Delta u_\varepsilon = h_\varepsilon & \text{in } B_1 \\ u_\varepsilon = 0 & \text{on } \partial B_1 \end{cases}$

Lecture 4 $\Rightarrow$
$$\sup_{x\in B_1} d_x^{-\beta} |u_\varepsilon(x)| \le C \cdot \sup_{x\in B_1} d_x^{2-\beta} |h_\varepsilon(x)|$$
$$\le C \cdot \sup_{x\in B_1} d_x^{2-\beta} |h(x)| \le C \|h\|_{C_{2-\beta}(B_1)}$$
$$\Rightarrow |u_\varepsilon(x)| \le C \|h\|_{C_{2-\beta}(B_1)} d_x^{\beta} \quad \forall x \in B_1 \qquad (1)$$

$\beta \in (0,1) \Rightarrow \|u_\varepsilon\|_{L^\infty(B_1)} \le C \|h\|_{C_{2-\beta}(B_1)}$ (2).

$\forall r \in (0,1)$ set $\hat r = \frac{1}{2}(r+1)$. When $\varepsilon < \frac{1-\hat r}{100}$. $h_\varepsilon = h$ in $B_{\hat r}$

$h \in C^\alpha_{2-\beta}(B_1) \Rightarrow h \in C^\alpha(\overline{B}_{\hat r})$

For $u_\varepsilon \in C(\overline{B}_1) \cap C^{2,\alpha}(B_1)$. Schauder estimate $\Rightarrow$

$$\|u_\varepsilon\|_{C^{2,\alpha}(B_r)} \le C(r,\lambda,\Lambda,\alpha,\beta,n)\left(\|u_\varepsilon\|_{L^\infty(B_{\hat r})} + \|h\|_{C^\alpha(B_{\hat r})}\right)$$
$$(2) \Rightarrow \quad \le \underbrace{C(r,\lambda,\Lambda,\alpha,\beta,n)\left(\|h\|_{C_{2-\beta}(B_1)} + \|h\|_{C^\alpha(B_{\hat r})}\right)}_{\text{independent of } \varepsilon}$$

By Arzela-Ascoli theorem & diagnol argument $\Rightarrow$

$\exists\, \varepsilon_i \to 0$ s.t. $u_{\varepsilon_i} \to u$ in $C^2_{loc}(B_1)$ for some $u \in C^{2,\alpha}(B_1)$

$\Delta u_{\varepsilon_i} = h_{\varepsilon_i}$ in $B_1 \Rightarrow \Delta u = h$ in B_1

$(1) \Rightarrow \quad |u(x)| \le C \|h\|_{C_{2-\beta}(B_1)} \cdot d_x^{\beta}$

$\Rightarrow$ u can be extended to $\overline{B}_1$ continuously with $u=0$ on ∂B_1.

Thus. $u \in C(\overline{B}_1) \cap C^{2,\alpha}(B_1)$ satisfies

$$\begin{cases} \Delta u = h & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

$h \in C^\alpha_{2-\beta}(B_1) \Rightarrow u \in C^{2,\alpha}_{-\beta}(B_1)$ (cf. Lecture 8)

Step 2. $\exists\, \delta > 0$. if $t \in I$. $s \in [0,1]$ and $|s-t| < \delta$. then $s \in I$.

Step 2 $\Rightarrow$ Goal ($1 \in I$) : divide $[0,1]$ into some subintervals

$$[0,1] = \bigcup_{i=1}^{N} I_i . \quad |I_i| < \delta .$$

Step 1 $\Rightarrow$ $0 \in I$ $\Rightarrow$ $1 \in I$ $\Rightarrow$ Goal.

For $t \in I$. $(*_t)$ is soluable in $C^{2,\alpha}_{-\beta}(B_1)$ for any $h \in C^{\alpha}_{2-\beta}(B_1)$

Maximum principle $\Rightarrow$ solution is unique.

Denote the unique solution of $(*)_t$ by $L_t^{-1} h \in C^{2,\alpha}_{-\beta}(B_1)$.

i.e. $\begin{cases} L_t (L_t^{-1} h) = h & \text{in } B_1 \\ L_t^{-1} h = 0 & \text{on } \partial B_1 \end{cases}$

Lecture 8 $\Rightarrow$ $\| L_t^{-1} h \|_{C^{2,\alpha}_{-\beta}(B_1)} \le \underset{\substack{\uparrow \\ \text{independent of } t.}}{C} \| h \|_{C^{\alpha}_{2-\beta}(B_1)}$ (3)

$\Rightarrow$ $L_t^{-1} : C^{\alpha}_{2-\beta}(B_1) \to C^{2,\alpha}_{-\beta}(B)$ is a bounded linear operator.

Define $T : C^{2,\alpha}_{-\beta}(B_1) \to C^{2,\alpha}_{-\beta}(B_1)$ by

$$TV = L_t^{-1} (h + (t-s)(LV - \Delta V))$$

Motivation : $L_s V = h \Leftrightarrow L_t V = h + L_t V - L_s V \underset{\uparrow}{=} h + (t-s)(LV - \Delta V)$

$L_t = tL + (1-t)\Delta$.

$L_s = sL + (1-s)\Delta$

$(*_s)$ $\begin{cases} L_s V = h & \text{in } B_1 \\ V = 0 & \text{on } \partial B_1 \end{cases}$ $\Leftrightarrow$ $\begin{cases} L_t V = h + (t-s)(LV - \Delta V) & \text{in } B_1 \\ V = 0 & \text{on } \partial B_1 \end{cases}$

$\Leftrightarrow$ $V = L_t^{-1} (h + (t-s)(LV - \Delta V))$

$\Leftrightarrow$ $V = TV$

V solves $(*_s)$ $\Leftrightarrow$ V is a fixed point of T.

$\forall\ V, W \in C^{2,\alpha}_{-\beta}(B_1)$. compute

$$\|TV-TW\|_{C^{2,\alpha}_{-\beta}(B_1)} = \|(t-s)L_t^{-1}\left((L-\Delta)(V-W)\right)\|_{C^{2,\alpha}_{-\beta}(B_1)}$$

(3) $\Rightarrow$ $\qquad \leq C\cdot|t-s|\cdot\|(L-\Delta)(V-W)\|_{C^{\alpha}_{2-\beta}(B_1)}$

Similar calculation as before $\qquad \leq C\cdot|t-s|\cdot\|V-W\|_{C^{2,\alpha}_{-\beta}(B_1)}$

Choose $\delta = \frac{1}{2C}$ $\Rightarrow$ when $|t-s|<\delta$, T is a contraction map.

Contraction mapping theorem $\Rightarrow \exists\ V \in C^{2,\alpha}_{-\beta}(B_1)$ s.t. $TV=V$.

$\Leftrightarrow$ V solves $(*_s)$ $\Rightarrow s \in I$ #

2. Continuous boundary value

Theorem. $L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$. $a_{ij}, b_i, c \in C^{\alpha}(\bar{B}_R)$. $\alpha \in (0,1)$

$\lambda I_n \leq (a_{ij}) \leq \Lambda I_n$ $(\lambda, \Lambda > 0)$. $\|a_{ij}\|_{C^\alpha(B_R)}, \|b_i\|_{C^\alpha(B_R)}, \|c\|_{C^\alpha(B_R)} \leq \Lambda$.

$c \leq 0$. For any $f \in C^{\alpha}(\bar{B}_R)$ and $\varphi \in C(\partial B_R)$

$\exists$ unique solution $u \in C(\bar{B}_R) \cap C^{2,\alpha}(B_R)$ of the

Dirichlet problem $\begin{cases} Lu = f & \text{in } B_R \\ u = \varphi & \text{on } \partial B_R. \end{cases}$

Proof. By scaling method, assume $R=1$.

Let $\{\varphi_k\}$ be a consequence of functions in $C^{\infty}(\bar{B}_1)$ s.t.

$\|\varphi_k - \varphi\|_{C(\partial B_1)} \to 0$

Then $f - L\varphi_k \in C^{\alpha}_{2-\beta}(B_1)$ for $\beta \in (0,1)$

Previous theorem $\Rightarrow \exists\ V_k \in C^{2,\alpha}_{-\beta}(B_1) \subset C(\bar{B}_1) \cap C^{2,\alpha}(B_1)$ satisfies

$$\begin{cases} LV_k = f - L\varphi_k & \text{in } B_1 \\ V_k = 0 & \text{on } \partial B_1 \end{cases}$$

Set $u_k = V_k + \varphi_k \Rightarrow u_k \in C(\bar{B}_1) \cap C^{2,\alpha}(B_1)$ satisfies

$$\begin{cases} Lu_k = f & \text{in } B_1 \\ u_k = \varphi_k & \text{on } \partial B_1 \end{cases}$$

For any k, l we have

$$\begin{cases} L(u_k - u_l) = 0 & \text{in } B_1 \\ u_k - u_l = \varphi_k - \varphi_l & \text{on } \partial B_1 \end{cases}$$

maximum principle $\Rightarrow \|u_k - u_l\|_{C(B_1)} \le \|\varphi_k - \varphi_l\|_{C(\partial B_1)} \to 0$.

$\Rightarrow \exists\, u \in C(\bar{B})$ s.t. $u_k \to u$ in $C(\bar{B}_1) \Rightarrow u = \varphi$ on ∂B_1.

For $u_k \in C(\bar{B}_1) \cap C^{2,\alpha}(B_1)$. $\forall\, r \in (0,1)$. Schauder estimate $\Rightarrow$

$$\|u_k\|_{C^{2,\alpha}(B_r)} \le C(r, \lambda, \Lambda, \alpha, n)\left(\|u_k\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right)$$

$$u_k \to u \text{ in } C(\bar{B}_1) \Rightarrow \le \underbrace{C(r, \lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_1)} + 1 + \|f\|_{C^\alpha(B_1)}\right)}_{\text{independent of } k}$$

Arzela-Ascoli theorem + diagonal argument $\Rightarrow \exists$ subsequence k_i s.t.

$u_{k_i} \to u$ in $C^2_{loc}(B_1)$ and then $u \in C^{2,\alpha}(B_1)$

$Lu_{k_i} = f$ in $B_1 \Rightarrow Lu = f$ in B_1

$\Rightarrow u \in C(\bar{B}) \cap C^{2,\alpha}(B_1)$ satisfies $\begin{cases} Lu = f & \text{in } B_1 \\ u = \varphi & \text{on } \partial B_1 \end{cases}$

Uniqueness follows from maximum principle. #

3. $C^{2,\alpha}$ regularity

Corollary. Let Ω be a domain (not necessarily bounded) in $\mathbb{R}^n$.

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$. $a_{ij}, b_i, c \in C^\alpha(\Omega)$. $\alpha \in (0,1)$

$\lambda I_n \le (a_{ij}) \le \Lambda I_n \quad (\lambda, \Lambda > 0)$

If $u \in C^2(\Omega)$ satisfies $Lu = f$ in Ω

for some $f \in C^\alpha(\Omega)$. Then $u \in C^{2,\alpha}(\Omega)$.

Proof. Define $\tilde{L} = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i$, $\tilde{f} = f - cu$

$\forall\, B = B_R(x_0) \subset\subset \Omega$. $a_{ij}, b_i, c, \tilde{f} \in C^\alpha(\bar{B})$. $u \in C(\partial B)$

Previous theorem $\Rightarrow \exists$ unique $v \in C(\bar{B}) \cap C^{2,\alpha}(B)$ solves the Dirichlet problem (*) $\begin{cases} \tilde{L}v = \tilde{f} & \text{in } B \\ v = u & \text{on } \partial B \end{cases}$

Clearly. u is also a solution of (*).

Uniqueness $\Rightarrow$ $u = v \in C(\bar{B}) \cap C^{2,\alpha}(B)$ $\Rightarrow$ $u \in C^{2,\alpha}(B)$

B is arbitrary $\Rightarrow$ $u \in C^{2,\alpha}(\Omega)$ #

Remark. Using this corollary, in the previous Schauder estimate, the $C^{2,\alpha}$ assumption can be replaced by C^2 assumption.

Lecture 10. Dirichlet problem in domains.

Method: Perron's method

1. Subsolution & Supersolution

Definition: Ω: bounded domain in $\mathbb{R}^n$

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$ $a_{ij}, b_i, c \in C(\Omega)$

$(a_{ij}) > 0$ $c \le 0$. $f \in C(\Omega)$

A function $u \in C(\Omega)$ is called a subsolution of $Lv = f$ in Ω. Such that the following holds. For any ball $B = B_R(x) \subset\subset \Omega$ and any $C(\bar{B}) \cap C^2(B)$ - solution w of $Lw = f$ in B

if $u \le w$ on ∂B, then $u \le w$ in B (*)

Replace "$\le$" by "$\ge$" in (*) $\Rightarrow$ Definition of supersolution.

Remark. If $u \in C^2(\Omega)$ and $Lu \ge f$, then u is a subsolution as above.

Lemma. Ω: bounded domain in $\mathbb{R}^n$ $\alpha \in (0,1)$

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$, $a_{ij}, b_i, c \in C^\alpha(\Omega)$

$(a_{ij}) > 0$. $c \le 0$ $f \in C^\alpha(\Omega)$

$u \in C(\bar{\Omega})$: Subsolution in Ω

$v \in C(\bar{\Omega})$: supersolution in Ω.

If $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

Proof. Set $M = \max_{\bar{\Omega}} (u-v)$.

If $M \le 0$. $\Rightarrow u \le v$ in Ω ✓

If $M > 0$. then define $D = \{x \in \Omega \mid u(x) - v(x) = M\}$.

Claim: D is open and closed.

claim $\Rightarrow$ $D = \phi$ or $D = \Omega$

· If $D = \phi$ then $M = \max_{\partial\Omega} (u-v) \le 0$. Contradiction.

· If $D = \Omega$. then $u - v = M$ in Ω.

$\Rightarrow u - v = M$ on $\partial\Omega \Rightarrow M \le 0$ Contradiction.

Proof of claim: D is closed (trivial)

Goal: D is open.

$\forall x_0 \in D$ $\forall r < d = \mathrm{dist}(x_0, \partial\Omega) \Rightarrow B = B_r(x_0) \subset\subset \Omega$.

$\exists \bar{u}, \bar{v} \in C(\bar{B}) \cap C^{2,\alpha}(B)$ s.t.

$$(*_1) \begin{cases} L\bar{u} = f & \text{in } B \\ \bar{u} = u & \text{on } \partial B \end{cases} \qquad (*_2) \begin{cases} L\bar{v} = f & \text{in } B \\ \bar{v} = v & \text{on } \partial B \end{cases}$$

Definitions of subsolution & supersolution

$\Rightarrow$ $u \le \bar{u}$ in B. $v \ge \bar{v}$ in B.

$\Rightarrow$ $\bar{u} - \bar{v} \ge u - v$ in B. (1)

$$(*_1), (*_2) \Rightarrow (*_3) \begin{cases} L(\bar{u}-\bar{v}) = 0 & \text{in } B \\ \bar{u} - \bar{v} = u - v & \text{on } \partial B \end{cases}$$

maximum principle $\Rightarrow$ $\max_{\bar{B}} (\bar{u}-\bar{v}) \le \max_{\partial B} (u-v)^+ \overset{M>0}{\le} M$

$x_0 \in D \Rightarrow M = (u-v)(x_0) \overset{(1)}{\le} (\bar{u}-\bar{v})(x_0) \le M$

$\Rightarrow (\bar{u}-\bar{v})(x_0) = M$. $\quad \underline{B = B_r(x_0)}$

Strong maximum principle $\Rightarrow$ $\bar{u}-\bar{v} \equiv M$ in B

$\Rightarrow \bar{u} - \bar{v} \equiv M$ on ∂B

$(*_3) \Rightarrow u - v \equiv M$ on $\partial B = \partial B_r(x_0)$

$r \in (0,d)$ is arbitrary $\Rightarrow u - v = M$ in $B_d(x_0)$

$\Rightarrow B_d(x_0) \subset D \Rightarrow$ claim. #

Lemma Ω: bounded domain in $\mathbb{R}^n$. $\alpha \in (0,1)$

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$ $\quad a_{ij}, b_i, c \in C^{\alpha}(\Omega)$

$(a_{ij}) > 0$. $c \le 0$. $f \in C^{\alpha}(\Omega)$.

$u \in C(\Omega)$: subsolution in Ω. $\quad B = B_R(x_0) \subset\subset \Omega$.

Define $\hat{u} = \begin{cases} u & \text{in } \Omega \setminus B \\ v & \text{in } B. \end{cases}$

where v solves $\begin{cases} Lv = f & \text{in } B \\ v = u & \text{on } \partial B. \end{cases}$

Then $\hat{u}$ is a subsolution in Ω and $\hat{u} \ge u$ in Ω

Remark. $\hat{u}$ is called the lifting of u in B.

Proof. $u \le \hat{u}$ is trivial. $\forall$ ball $B' \subset\subset \Omega$.

$\forall\, w \in C(\bar{B}') \cap C^2(B')$. s.t. $\begin{cases} Lw = f & \text{in } B' \\ w \ge \hat{u} & \text{on } \partial B' \end{cases}$

Goal: $\hat{u} \le w$ in B'.

If $B \cap B' = \phi$. definition of subsolution $\Rightarrow$ Goal.

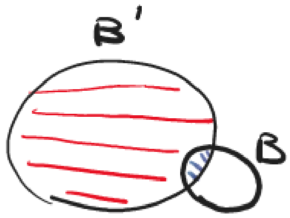
Assume $B \cap B' \ne \phi$.

Step 1. $\hat{u} \le w$ in $B' \setminus B$ (red region)

$u \le \hat{u} \le w$ on $\partial B'$

u is subsolution $\Rightarrow u \le w$ in B'

$\Rightarrow \hat{u} = u \le w$ in $B' \setminus B$.



Step 2. $\hat{u} \le w$ in $B' \cap B$ (blue region)

- Step 1 $\Rightarrow$ $\hat{u} \le w$ in $B' \setminus B$ $\Rightarrow$ $v = \hat{u} \le w$ on $\underline{\overline{B'} \cap \partial B}$
- $\hat{u} \le w$ on $\partial B'$ $\Rightarrow$ $v = \hat{u} \le w$ on $\underline{\partial B' \cap \overline{B}}$.

$$\Rightarrow \begin{cases} Lv = f = Lw & \text{in } B' \cap B \\ v \le w & \text{on } \partial(B' \cap B). \end{cases}$$

Comparison principle $\Rightarrow$ $v \le w$ in $B' \cap B$.

($\hat{u} = v$ in B $\Rightarrow$) $\Rightarrow$ $\hat{u} \le w$ in $B' \cap B$

Step 1 & 2 $\Rightarrow$ Goal. #.