Recall. · Dirichlet Problem in balls (Continuity method)

· Dirichlet problem in domains (Perron's method)

· Definitions of subsolution & supersolution

· u: subsolution. v: supersolution

$u \le v$ on $\partial\Omega \Rightarrow u \le v$ in Ω

· u: subsolution. $B \subset\subset \Omega$

$$\hat{u} = \begin{cases} u & \Omega \setminus B \\ v & B \end{cases} \quad \text{where} \quad \begin{cases} Lv = f & \text{in } B \\ v = u & \text{on } \partial B \end{cases}$$

$\hat{u}$: the lifting of u in B.

$\hat{u}$ is still a subsolution and $\hat{u} \ge u$.

Lecture 10. Dirichlet problem in domains (Continued)

2. Perron's method

Proposition. Ω: bounded domain in $\mathbb{R}^n$.

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$. $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C^\alpha(\Omega)$, $\alpha \in (0,1)$

$(a_{ij}) \ge \lambda I_n$ $(\lambda > 0)$. $c \le 0$. $f \in L^\infty(\Omega) \cap C^\alpha(\Omega)$. $\varphi \in C(\partial\Omega)$

Define

$$S_\varphi = \{ v \mid v \in C(\bar\Omega) \text{ is a subsolution in } \Omega \text{ and } v \le \varphi \text{ on } \partial\Omega \}$$

and

$$u_\varphi(x) = \sup\{ v(x) \mid v \in S_\varphi \} \quad \forall x \in \Omega.$$

Then $u_\varphi \in C^{2,\alpha}(\Omega)$ and $Lu_\varphi = f$ in Ω.

Proof. Step 1. u_φ is well-defined ($S_\varphi \ne \emptyset$. $|u_\varphi(x)| < +\infty$ $\forall x \in \Omega$)

After translation, assume $\Omega \subset \{0 < x_1 < d\}$

Set $F = \|f\|_{L^\infty(\Omega)}$, $\Phi = \|\varphi\|_{L^\infty(\partial\Omega)}$.

Define $V_+(x) = \Phi + (e^{Ad} - e^{Ax_1})F$. $V_-(x) = -V_+(x)$.

where A is a constant to be determined.

Compute $LV_+ = -(a_{11}A^2 + b_1 A)F e^{Ax_1} + cV_+$.

$a_{11} \ge \lambda$. $b_1 \in L^\infty(\Omega) \Rightarrow$ choose $A \gg 1$ s.t. $a_{11}A^2 + b_1 A \ge 1$.

$\Rightarrow \quad LV_+ \leq -Fe^{Ax_1} + cV_+$

$V_+ \geq 0 \quad c \leq 0 \quad e^{Ax_1} \geq 1 \Rightarrow LV_+ \leq -F$.

$$\Rightarrow \begin{cases} LV_+ \leq -F \leq f & \text{in } \Omega \\ V_+ \geq \Phi \geq \varphi & \text{on } \partial\Omega \end{cases} \Rightarrow V_+ \text{ is a supersolution.}$$

Recall $V_- = -V_+ \quad \Rightarrow \begin{cases} LV_- \geq F \geq f & \text{in } \Omega \\ V_- \leq -\Phi \leq \varphi & \text{on } \partial\Omega \end{cases}$

$\Rightarrow V_-$ is a Subsolution.

Furthermore. $V_- \in S_\varphi \Rightarrow S_\varphi \neq \phi$.

$\forall\, v \in S_\varphi \quad \Rightarrow \quad v \leq \varphi \leq V_+ \text{ on } \partial\Omega \Rightarrow v \leq V_+ \text{ in } \Omega$.

$u_\varphi(x) = \sup\limits_{v \in S_\varphi} v(x) \Rightarrow \quad V_- \leq u_\varphi \leq V_+ \quad \text{in } \Omega$.

$\Rightarrow \quad |u_\varphi(x)| < +\infty \;\; \forall x \in \Omega$

$\Rightarrow \quad u_\varphi$ is well-defined in Ω.

Step 2. $\forall\, v_1, v_2 \in S_\varphi \Rightarrow v = \max(v_1, v_2) \in S_\varphi$

$v_i \leq \varphi$ on $\partial\Omega \Rightarrow \quad v \leq \varphi$ on $\partial\Omega \quad (1)$

To prove v is a subsolution, for any ball $B \subset\subset \Omega$ and any $w \in C(\bar{B}) \cap C^2(B)$ s.t. $\begin{cases} Lw = f & \text{in } B \\ w \geq v & \text{on } \partial B \end{cases}$

$\Rightarrow \begin{cases} Lw = f & \text{in } B \\ w \geq v_i & \text{on } \partial B \end{cases} \quad v_i$: subsolution $\Rightarrow w \geq v_i$ in B

$\Rightarrow w \geq v$ in $B \Rightarrow v$ is a subsolution (2)

$(1), (2) \Rightarrow v \in S_\varphi$.

Step 3. Prove $u_\varphi \in C^{2,\alpha}(\Omega)$ & $Lu_\varphi = f$

Goal: $\forall\, B_r(x_0) \subset\subset \Omega \quad u_\varphi \in C^{2,\alpha}(B_r(x_0))$ & $Lu_\varphi = f$ in $B_r(x_0)$.

$u_\varphi(x_0) = \sup\limits_{v \in S_\varphi} v(x_0) \Rightarrow \exists\, v_i \in S_\varphi$ s.t. $\lim\limits_{i \to +\infty} v_i(x_0) = u_\varphi(x_0)$.

Assume $V_- \leq v_i \leq u_\varphi$ in Ω

Otherwise, replace v_i by $\max(v_i, V_-)$. (step 2)

Let w_i be the lifting of v_i in $B_r(x_0) \Rightarrow w_i \in S_\varphi$

$\Rightarrow \quad V_- \le V_i \le W_i \le u_\varphi \le V_+ \quad \text{in } \Omega$

$L W_i = f \quad \text{in } B_r(x_0) \qquad \lim\limits_{i\to\infty} W_i(x_0) = u_\varphi(x_0)$

$\Rightarrow \quad \|W_i\|_{L^\infty(B_r(x_0))} \le C \quad$ (independent of i)

Schauder estimate : $\forall\, r' \in (0, r)$

$\|W_i\|_{C^{2,\alpha}(B_{r'}(x_0))} \le C_{r'} \quad$ (independent of i).

Arzela-Ascoli theorem + diagonal argument

$\Rightarrow$ $\exists$ subsequence of $\{W_i\}$ (still denote by $\{W_i\}$) s.t.

$W_i \to W$ in $C^2_{loc}(B_r(x_0))$ for some $W \in C^{2,\alpha}(B_r(x_0))$.

$\Rightarrow \quad W \le u_\varphi. \quad LW = f \quad \text{in } B_r(x_0). \quad \& \quad W(x_0) = u_\varphi(x_0).$

Claim : $W = u_\varphi$ in $B_r(x_0)$. $\qquad$ claim $\Rightarrow$ Step 3

$\forall\, \bar{x} \in B_r(x_0) \qquad u_\varphi(\bar{x}) = \sup\limits_{v \in S_\varphi} v(\bar{x})$

$\Rightarrow \quad \exists\, \bar{v}_i \in S_\varphi. \quad$ s.t. $\lim\limits_{i\to+\infty} \bar{v}_i(\bar{x}) = u_\varphi(\bar{x})$.

Assume $W_i \le \bar{v}_i \le u_\varphi$ in Ω

Otherwise. replace $\bar{v}_i$ by $\max(\bar{v}_i, W_i)$ (Step 2)

Let $\bar{w}_i$ be the lifting of $\bar{v}_i$ in $B_r(x_0)$

$\Rightarrow \quad V_- \le V_i \le W_i \le \bar{v}_i \le \bar{w}_i \le u_\varphi \le V_+ \quad \text{in } \Omega.$

$\& \quad \lim\limits_{i\to+\infty} \bar{w}_i(\bar{x}) = u_\varphi(\bar{x})$

Similarly. $\exists$ subsequence $\bar{w}_i \to \bar{w}$ in $C^2_{loc}(B_r(x_0))$

for some $\bar{w} \in C^{2,\alpha}(B_r(x_0))$.

$\Rightarrow \quad W \le \bar{w} \le u_\varphi. \qquad LW = L\bar{w} = f \quad \text{in } B_r(x_0).$

$W(x_0) = \bar{w}(x_0) = u_\varphi(x_0) \quad . \qquad \bar{w}(\bar{x}) = u_\varphi(\bar{x})$

$$\Rightarrow \quad \begin{cases} L(W - \bar{w}) = 0 & \text{in } B_r(x_0) \\ W - \bar{w} \le 0 & \text{on } \partial B_r(x_0). \end{cases}$$

$(w-\bar{w})(x_0) = 0$, strong maximum principle

$\Rightarrow$ $w-\bar{w} \equiv$ constant $\Rightarrow$ $w-\bar{w} \equiv 0$

$\Rightarrow$ $w(\bar{x}) = \bar{w}(\bar{x}) = u_\varphi(\bar{x})$.

$\bar{x} \in B_r(x_0)$ is arbitrary $\Rightarrow$ $w = u_\varphi$ in $B_r(x_0)$

#.

Proposition. Ω: bounded domain in $\mathbb{R}^n$

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$ $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C^\alpha(\Omega)$. $\alpha \in (0,1)$

$(a_{ij}) \geq \lambda I_n$ $(\lambda > 0)$ $c \leq 0$. $f \in L^\infty(\Omega) \cap C^\alpha(\Omega)$ $\varphi \in C(\partial\Omega)$

For $x_0 \in \partial\Omega$. if $\exists\, w_{x_0} \in C(\bar{\Omega}) \cap C^2(\Omega)$. s.t.

$Lw_{x_0} \leq -1$ in Ω, $w_{x_0}(x_0) = 0$. $w_{x_0} > 0$ on $\partial\Omega \setminus \{x_0\}$.

then $\lim_{x \to x_0} u_\varphi(x) = \varphi(x_0)$

Remark. The function w_{x_0} is called a barrier function at x_0.

Proof. Write $w = w_{x_0}$. $\Phi = \|\varphi\|_{L^\infty(\partial\Omega)}$.

$\varphi \in C(\partial\Omega)$ $\Rightarrow$ $\forall\, \varepsilon > 0$. $\exists\, \delta > 0$. s.t. $|\varphi - \varphi(x_0)| < \varepsilon$ in $B_\delta(x_0)$.

$w > 0$ on $\partial\Omega \setminus \{x_0\}$ $\Rightarrow$ $\min_{\partial\Omega \setminus B_\delta(x_0)} w > 0$

$\Rightarrow$ $\exists\, k_\varepsilon$ s.t. $|\varphi - \varphi(x_0)| \leq 2\Phi \leq k_\varepsilon w$ on $\partial\Omega \setminus B_\delta(x_0)$.

$\Rightarrow$ $|\varphi - \varphi(x_0)| \leq \varepsilon + k_\varepsilon w$ on $\partial\Omega$ (1)

Set $A_\varepsilon = \max(k_\varepsilon, \|f - c\varphi(x_0)\|_{L^\infty(\Omega)})$

$L(\varphi(x_0) - \varepsilon - A_\varepsilon w) = c\varphi(x_0) - c\varepsilon - A_\varepsilon Lw$

$(c \leq 0.\ Lw \leq -1 \Rightarrow)$ $\geq c\varphi(x_0) + A_\varepsilon \geq f$ in Ω.

Definition of A_ε.

$\Rightarrow$ $\varphi(x_0) - \varepsilon - A_\varepsilon w$ is a subsolution.

(1). Definition of $A_\varepsilon \Rightarrow |\varphi - \varphi(x_0)| \le \varepsilon + A_\varepsilon W$ on $\partial\Omega$

$\Rightarrow \varphi(x_0) - \varepsilon - A_\varepsilon W \le \varphi$ on $\partial\Omega$.

$\Rightarrow \varphi(x_0) - \varepsilon - A_\varepsilon W \in S_\varphi$.

Recall : $u_\varphi = \sup\limits_{v \in S_\varphi} v$

$\Rightarrow \varphi(x_0) - \varepsilon - A_\varepsilon W \le u_\varphi$ in Ω (2)

Similarly $\begin{cases} L(\varphi(x_0) + \varepsilon + A_\varepsilon W) \le f & \text{in } \Omega \\ \varphi(x_0) + \varepsilon + A_\varepsilon W \ge \varphi & \text{on } \partial\Omega. \end{cases}$

$\Rightarrow \varphi(x_0) + \varepsilon + A_\varepsilon W$ is a supersolution with

$\varphi(x_0) + \varepsilon + A_\varepsilon W \ge \varphi$ on $\partial\Omega$.

$\Rightarrow \varphi(x_0) + \varepsilon + A_\varepsilon W \ge u_\varphi$ in Ω (3)

(2). (3) $\Rightarrow |u_\varphi - \varphi(x_0)| \le \varepsilon + A_\varepsilon W$ in Ω.

$$\Rightarrow \limsup_{x \to x_0} |u_\varphi(x) - \varphi(x_0)| \le \varepsilon + A_\varepsilon \limsup_{x \to x_0} W(x)$$

$$= \varepsilon + A_\varepsilon W(x_0) = \varepsilon.$$

Let $\varepsilon \to 0 \Rightarrow \limsup\limits_{x \to x_0} |u_\varphi(x) - \varphi(x_0)| = 0$

$$\Rightarrow \lim_{x \to x_0} |u_\varphi(x) - \varphi(x_0)| = 0$$

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Theorem. Ω : bounded domain in $\mathbb{R}^n$.

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c$. $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C^\alpha(\Omega)$. $\alpha \in (0,1)$

$(a_{ij}) \geq \lambda I_n$ $(\lambda > 0)$ $c \leq 0$. $f \in L^\infty(\Omega) \cap C^\alpha(\Omega)$. $\varphi \in C(\partial\Omega)$

For each $x_0 \in \partial\Omega$. $\exists$ barrier function.

$\exists$ unique solution $u \in C(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$ of the

Dirichlet Problem $\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$

Remark. Review Perron's method.

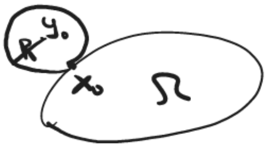
- Interior : maximum principle & Solvability of Dirichlet problem in balls.
- Boundary : barrier function (local geometry of domain)

Question : How to guarantee the existence of barrier function?

3. Exterior sphere condition.

Definition. Ω : bounded domain in $\mathbb{R}^n$. $x_0 \in \partial\Omega$

If $\exists B_R(y_0)$ s.t. $\Omega \cap B_R(y_0) = \phi$. $\bar{\Omega} \cap \bar{B}_R(y_0) = \{x_0\}$



then we say Ω satisfies the exterior sphere condition at $x_0 \in \partial\Omega$.

If Ω satisfies the exterior sphere condition at each boundary point. then we say Ω satisfies the exterior sphere condition.

Lemma. Ω : bounded domain in $\mathbb{R}^n$

$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$ $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$

$(a_{ij}) \ge \lambda I_n \quad (\lambda > 0) \qquad c \le 0.$

If Ω satisfies the exterior sphere condition at $x_0 \in \partial\Omega$, then $\exists$ barrier function at x_0.

Proof. $\exists B_R(y_0)$ s.t. $\Omega \cap B_R(y_0) = \phi$. $\bar{\Omega} \cap \bar{B}_R(y_0) = \{x_0\}$

Write $d(x) = |x - y_0|$, $y_0 = (y_1^0, \cdots, y_n^0)$

$\Rightarrow R \le d(x) \le R + \operatorname{diam}(\Omega)$ in $\bar{\Omega}$.

Define $W(x) = K(R^{-A} - d^{-A})$ in $\bar{\Omega}$

where K and A are constants to be determined.

$\Rightarrow W(x_0) = 0 \qquad W > 0$ on $\partial\Omega \setminus \{x_0\}$ ($\Leftarrow d > R$ on $\partial\Omega \setminus \{x_0\}$)

Compute

$$L(R^{-A} - d^{-A}) = A d^{-A-4}\Big[-(A+2)\underline{\sum_{i,j} a_{ij}(x_i - y_i^0)(x_j - y_j^0)} + d^2 \sum_i a_{ii} + d^2 \sum_i b_i (x_i - y_i^0)\Big] + c(R^{-A} - d^{-A})$$

$(a_{ij}) \ge \lambda I_n \Rightarrow$ " $\sim$ " $\ge \lambda |x - y_0|^2 = \lambda d^2$. $\qquad c \le 0 \Rightarrow$

$$\Rightarrow L(R^{-A} - d^{-A}) \le A d^{-A-4}\Big[-(A+2)\lambda d^2 + d^2 \sum_i a_{ii} + d^2 \sum_i (x_i - y_i^0)\Big]$$

$$= A d^{-A-2}\Big[-(A+2)\lambda + \sum_i a_{ii} + \sum_i \underline{(x_i - y_i^0)}\Big]$$

choose $A \gg 1$ s.t. $[\cdots] \le -1$

$$\Rightarrow L(R^{-A} - d^{-A}) \le -A d^{-A-2}$$

$R \le d \le R + \operatorname{diam}(\Omega) \Rightarrow \quad \le -A(R + \operatorname{diam}(\Omega))^{-A-2}$

Choose $K = A^{-1}(R + \operatorname{diam}(\Omega))^{A+2} \Rightarrow$

$$LW = K L(R^{-A} - d^{-A}) \le -1$$

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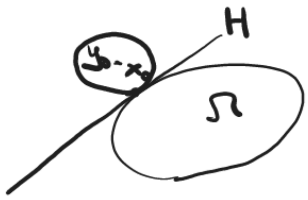
Example. Ω : bounded domain in $\mathbb{R}^n$

(1) If Ω is convex, then Ω satisfies the exterior sphere condition.

(2) If $\partial\Omega$ is C^2, then Ω satisfies the exterior sphere condition.

(1). is trivial.



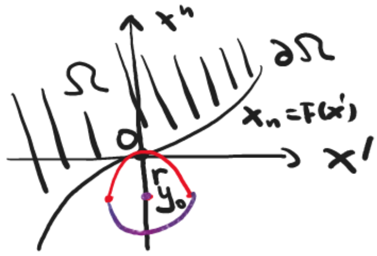
(2) $\forall x_0 \in \partial\Omega$. $\partial\Omega$ is $C^2 \Rightarrow$ Near x_0, $\partial\Omega$ can be expressed as a graph of some C^2 function.

Choose coordinate system s.t. $x_0 = 0$.

$\partial\Omega$: $x_n = F(x_1, \cdots, x_{n-1})$. $F(0) = 0$ $\nabla F(0) = \vec{0}$.

for some C^2 function F near 0.

Write $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$.



Taylor expansion $\Rightarrow$

$$F(x') = \frac{1}{2}\sum_{i,j=1}^{n-1} \partial_{ij}F(0)x_i x_j + o(|x'|^2)$$

$$\geq -C|x'|^2.$$

Set $S_r(x') = \sqrt{r^2 - |x'|^2} - r$ $\qquad S_r(x') \geq \tilde{S}_r(x')$.

$\tilde{S}_r(x') = -\sqrt{r^2 - |x'|^2} - r$

$$\Rightarrow S_r(x') = \frac{1}{2}\sum_{i,j} \partial_{ij}S_r(0)x_i x_j + o(|x'|^2)$$

$$= -\frac{|x'|^2}{2r} + o(|x'|^2)$$

Choose $r << 1$ $(r > 0)$ $\Rightarrow S_r(x') \leq -2C|x'|^2$

$\Rightarrow F(x') > S_r(x')$.

Set $y_0 = (0, \cdots, 0, -r) \Rightarrow B_r(y_0)$ is the desired exterior ball.

Theorem. Ω: bounded domain in $\mathbb{R}^n$. Ω is convex or $\partial\Omega$ is C^2.

$L = \sum_{ij} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c$. $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C^\alpha(\Omega)$. $\alpha \in (0,1)$

$(a_{ij}) \geq \lambda I_n$. $(\lambda > 0)$ $\quad c \leq 0$. $f \in L^\infty(\Omega) \cap C^\alpha(\Omega)$. $\varphi \in C(\partial\Omega)$.

$\exists$ unique solution $u \in C(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$ of the

the Dirichlet problem $\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$