Lecture 4

Betti spaces

Recall.

$k = \bar{k}$, $\operatorname{char} k = 0$

G reductive alg. group $/k$

X: affine G-variety

$\rightsquigarrow$ affine GIT quotient $\varphi : X \to X/\!/G := \operatorname{Spec}(k[X]^G)$

Prop $\varphi : X \to X/\!/G$ is good quotient. in particular, φ is surjective

Lem: $\forall$ ideal $I \subset k[X]^G$

$$k[X]I \cap k[X]^G = I$$

Pf of lemma.

A $\forall$ k-algebra. $G \curvearrowright A$

$\Rightarrow \exists$ a linear map of k-algs $R : A \to A^G$ s.t.

$$R(ff') = R(f) f' \qquad \forall f \in A.\ f' \in A^G$$

In particular, $R(f) = f$ whenever $f \in A^G$.

This operator is called the Reynolds operator.

Applying R to $A := k[X]$, $A^G := k[X]^G$.

It suffices to show $k[X]I \cap k[X]^G \subset I$:

$\forall\ f = \sum_i f_i h_i \in k[X]^G \qquad f_i \in k[X],\ h_i \in I \subset k[X]^G$

$$\Rightarrow f = R(f) = \sum_i R(f_i h_i) = \sum_i \underbrace{R(f_i)}_{\in k[X]^G} \underbrace{h_i}_{\in I} \in k[X]^G I \subset I$$

$\blacksquare$

proof of proposition:

$\forall\ y \in X/\!/G \longleftrightarrow$ maximal ideal $I_y \subset k[X]^G$

Lemma $\Rightarrow$ $k[X]I_y \subset k[X]$ proper ideal

$\Rightarrow$ $k[X]I_y \subset I_x$, for some $I_x \subset k[X]$ maximal ideal

by definition $\varphi(x) = y$.

$\blacksquare$

$x \in X$ is

- polystable if $G \cdot x \subset X$ is closed
- stable if
  - $G \cdot x \subset X$ is closed
  - G_x finite i.e. $\dim(G_x) = 0$
- $x_1, x_2 \in X$ are S-equivalent if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi$

denote $x_1 \sim_S x_2$

$X^{ps} \subset X$ subset of polystable points

$X^s \subset X$ subset of stable points

Thm (1) $X^s \subset X$ open & G-invariant

(2) $\varphi(X^s) \subset X /\!/ G$ open & $\varphi^{-1}(\varphi(X^s)) = X^s$

(3) $\varphi|_{X^s} : X^s \to \varphi(X^s)$ geometric quotient

proof

(1) $X^s \subset X$ open: $\forall x \in X^s$, $\exists$ open nbhd of x in X^s

- define $X_+ := \{x \in X : \dim(G_x) > 0\}$

then $X_+ \subset X$ closed

- $G \cdot x \subset X$ closed

$\Rightarrow$ $X_+ \cap G \cdot x \neq \phi$. so they can be distinguished by some G-invariant function $f \in k[X]^G$.

i.e. $f(X_+) = 0$, $f(G \cdot x) = 1$

define $X_f := \{x \in X : f(x) \neq 0\} \subset X$ open.

Claim: X_f is the desired ngbh. i.e. $x \in X_f \subset X^s$

1st: $\forall x' \in X_f$. $f(x') \neq 0 \Rightarrow x' \notin X_+ \Rightarrow \dim(G_{x'}) = 0$

2nd: suppose $\exists x' \in X_f$ s.t. $G \cdot x'$ not closed.

$\Rightarrow \overline{G \cdot x'} \setminus G \cdot x'$ contains a closed orbit. say $G \cdot x''$ with

$\dim(G \cdot x'') < \dim(G \cdot x') \leq \dim(G) \Rightarrow \dim(G_{x''}) > 0$

but $x'' \in \overline{G \cdot x'} \setminus G \cdot x' \Rightarrow f(x'') \neq 0 \Rightarrow x'' \notin X_+ \Rightarrow \dim(G_{x''}) = 0$ ↯

$\Rightarrow X_f \subset X^s$

(2) $\varphi(X^s) \subset X /\!/ G$ open.

$f \in k[X]^G \simeq k[X/\!/G] \rightsquigarrow \varphi(X_f) = \{y \in X/\!/G : f(y) \neq 0\} \subset X/\!/G$ open

$\Rightarrow \varphi(X^s) \subset X/\!/G$ open

$X_f = \varphi^{-1}(\varphi(X_f))$:

if not. let $x \in \varphi^{-1}(\varphi(X_f)) \setminus X_f \Rightarrow f(x) = 0$

$f \in k[X/\!/G] \Rightarrow f(\varphi(x)) = 0$ ↯

$\Rightarrow X^s = \varphi^{-1}(\varphi(X^s))$

∎

§2. Local systems = fundamental group representations

X connected. locally simply connected top. space.

Def 2.1 A k-local system of rk n on X is a rk n locally constant sheaf of k-vector spaces. $\mathcal{F}$. i.e. $\forall x \in X$ $\exists$ open $x \in U \subset X$ st.

$$\mathcal{F}|_U \simeq \underline{k}^n_U \leftarrow \text{constant sheaf on } U$$

$\mathcal{C}_{Loc}(X,n)$: category of k-local systems of rk n on X

objects : ...

morphisms : morphism of sheaves.

$\phi : \mathcal{F} \to \mathcal{F}'$ is morphism of sheaves is collection of morphism of k-vector spaces. $\phi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, "r^{\mathcal{F}}_{UV}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}'(U) \arrow[d, "r^{\mathcal{F}'}_{UV}"] \\
\mathcal{F}(V) \arrow[r, "\phi_V"'] & \mathcal{F}'(U)
\end{tikzcd}$$

$\mathcal{C}_{Rep}(X,n)$: category of fundamental group representations $\rho : \pi_1(X,x) \to GL(n,k)$

fix $x \in X$ base point.

objects : ...

morphisms : a morphism of 2 representations $\rho_1, \rho_2 : \pi_1(X,x) \to GL(n,k)$

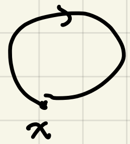
is a linear map $g : k^n \to k^n$ s.t.

$$\rho_1(\gamma) \circ g = g \circ \rho_2(\gamma) \quad \forall \gamma \in \pi_1(X,x)$$

ie $\rho_1(\gamma) = g \circ \rho_2(\gamma) \circ g^{-1}$

Thm 2.2 The above 2-categories are equivalent:

$$\mathcal{C}_{Loc}(X,n) \simeq \mathcal{C}_{Rep}(X,n)$$



pf (sketch)

"$\Rightarrow$" $\mathcal{F} \in \mathcal{C}_{Loc}(X,n)$. $\forall$ loop $\gamma : [0,1] \to X$ based at $x \in X$

$$\begin{array}{c} \mathcal{F} \\ \downarrow \\ X \end{array}$$

$\gamma^*\mathcal{F}$ is a k-local system on $[0,1]$

$[0,1]$ simply-connected $\Rightarrow$ $\gamma^*\mathcal{F}$ constant sheaf

$$\Rightarrow (\gamma^*\mathcal{F})_0 \xleftarrow[\cong]{r_{A0}} (\gamma^*\mathcal{F})_{[0,1]} \xrightarrow[\cong]{r_{A1}} (\gamma^*\mathcal{F})_1$$

$$\Rightarrow \begin{tikzcd} \gamma_*: \mathcal{F}_{\gamma(0)} \arrow[r, "\simeq"] \arrow[d, phantom, "\wr"] & \mathcal{F}_{\gamma(1)} \arrow[d, phantom, "\wr"] \\ k^n & k^n \end{tikzcd}$$

it suffices to show:

(check)! • γ_* is k-linear: $\gamma_*(v+\lambda w) = \gamma_*(v) + \lambda \gamma_*(w)$ $\quad \forall\, v,w \in \mathcal{F}_{\gamma(0)}$, $\lambda \in k$

• homotopy invariant: $\gamma \sim \gamma' \Rightarrow \gamma_* = \gamma'_*$.

homotopy inv.:

$\gamma \sim \gamma' \Rightarrow H: [0,1] \times [0,1] \to X$ s.t. $H(0,t) = \gamma(t)$

$H(1,t) = \gamma'(t)$

$$\Rightarrow \begin{tikzcd} \mathcal{F}_{H(0,0)} \arrow[r, "\gamma_*", "\simeq"'] \arrow[d, equal] & \mathcal{F}_{H(0,1)} \arrow[d, equal] \\ \mathcal{F}_{H(1,0)} \arrow[r, "\gamma'_*"] & \mathcal{F}_{H(1,1)} \end{tikzcd}$$

$\Rightarrow \gamma_* = \gamma'_*$

$\rightsquigarrow$ well-defined map $\rho: \pi_1(X,x) \to GL(n,k)$. called the
$[\gamma] \mapsto \gamma_*$

monodromy representation of the k-local system $\mathcal{F}$.

"$\Leftarrow$": $\rho: \pi_1(X,x) \to GL(n,k)$. let $\pi: \tilde{X} \to X$ universal covering.

$E_\rho := \tilde{X} \times_\rho k^n := \tilde{X} \times k^n / \pi_1$

$$\gamma \cdot (\tilde{x}, v) := (\gamma \tilde{x}, \rho(\gamma) v)$$

$\mathcal{E}_\rho$ the associated sheaf of sections of E_ρ.

$\forall\, U \subset X$ open

$$\mathcal{E}_\rho(U) := \Gamma(U, E_\rho)$$

$$\simeq \{ s : \pi^{-1}(U) \to k^n : s(\gamma \tilde{x}) = \rho(\gamma)\, s(\tilde{x}) \quad \forall \tilde{x} \in \tilde{X},\ \gamma \in \pi_1(X,x) \}$$

check $\mathcal{E}_\rho$ locally constant sheaf.. k-local system of rk n

Finally, show the functor $\mathcal{F} \mapsto (\gamma \mapsto \gamma_x)$ is an equivalence of categories.

$$\Rightarrow \quad \mathcal{C}_{Loc}(X,n) \simeq \mathcal{C}_{Rep}(X,n)$$

$\blacksquare$

§3. Betti spaces (abstract theory)

X smooth irreducible projective variety $/k$. fix $x \in X$

$\Rightarrow$ $\pi_1(X,x)$ is finitely generated:

$$\pi_1(X,x) = \langle \gamma_1, \dots, \gamma_\ell : r_1(\gamma_1, \dots, \gamma_\ell) = \dots = r_m(\gamma_1, \dots, \gamma_\ell) = Id \rangle,$$

where
- $\gamma_1, \dots, \gamma_\ell$ generators
- $r_1, \dots, r_m$ relations on generators.

Ex $\dim_k X = 1 \qquad g \geq 2$

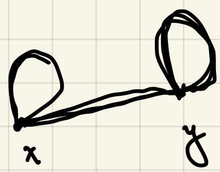
$$\pi_1(X,x) = \langle \gamma_1, \dots \gamma_g, \gamma_1', \dots, \gamma_g' : \prod_{i=1}^{g} \underbrace{[\gamma_i, \gamma_i']}_{\gamma_i \gamma_i' \gamma_i^{-1} \gamma_i'^{-1}} = Id \rangle$$

Define

$$\mathcal{U}(X,x,n) := \mathrm{Hom}(\pi_1(X,x), GL(n,k)) = \{ \rho : \pi_1(X,x) \to GL(n,k) \}$$

space of fundamental group representations.

Choose $y \in X$.



$\Rightarrow$ • $\pi_1(X,x) \simeq \pi_1(X,y)$

$\Rightarrow \mathcal{U}(X, x, n) \simeq \mathcal{U}(X, y, n)$

we omit the base point, and denote it as $\mathcal{U}(X, n)$

- $\rho \in \mathcal{U}(X, n)$ $\quad \rho : \pi_1(X, x) \to GL(n, k)$ is fully determined by $\rho(\gamma_1), \dots, \rho(\gamma_\ell)$ $\in GL(n, k)$

$\Rightarrow$ embedding
$$i : \mathcal{U}(X, n) \hookrightarrow GL(n, k)^\ell$$
$$\rho \mapsto (\rho(\gamma_1), \dots, \rho(\gamma_\ell))$$

with image
$$\mathcal{U}(X, n) \simeq i(\mathcal{U}(X, n)) = Rel^{-1}(Id, \dots, Id)$$
$$Rel : GL(n, k)^\ell \to GL(n, k)^m$$
$$(A_1, \dots, A_\ell) \mapsto (r_1(A_1, \dots, A_\ell), \dots, r_m(A_1, \dots, A_\ell))$$

$\Rightarrow \mathcal{U}(X, n)$ can be thought as a closed subvariety of $GL(n, k)^\ell$.

so it is an affine variety.

Ex $\dim_k X = 1$.
$$\mathcal{U}(X, n) \simeq \left\{ (A_1, \dots, A_g, A_1', \dots, A_g') \in GL(n, k)^{2g} : \prod_{i=1}^{g} [A_i, A_i'] = Id \right\}$$
$$\subset GL(n, k)^{2g}$$

$GL(n, k) \circlearrowright \mathcal{U}(X, n)$ by conjugation:
$$\sigma : GL(n, k) \times \mathcal{U}(X, n) \to \mathcal{U}(X, n)$$
$$(g, \rho) \mapsto g \cdot \rho \cdot g^{-1}$$
$$(g\rho g^{-1})(\gamma) := g \cdot \rho(\gamma) \cdot g^{-1}$$

this is compatible with the identification
$$\mathcal{U}(X, x, n) \simeq \mathcal{U}(X, y, n)$$

$\Rightarrow$ well-defined.!

$\Rightarrow$ Affine GIT quotient

$$\varphi : \mathcal{U}(X,n) \longrightarrow \mathcal{U}(X,n) /\!/ GL(n,\mathbb{k}) := \operatorname{Spec}\left(\mathbb{k}[\mathcal{U}(X,n)]^{GL(n,\mathbb{k})} \right)$$
$$=: \mathcal{M}_B(X,n)$$

called the moduli space of fundamental group representations.

"Betti moduli space".

Eg. $\dim_{\mathbb{C}} X = 1$

(1) $g=0$ $\quad \mathcal{M}_B(X,n) = \begin{cases} \phi & n \geq 2 \\ \{pt\} & n=1 \end{cases}$

(2) $g \geq 2$ $\quad \mathcal{M}_B(X,n) \simeq (\mathbb{C}^*)^{2g}$

For $\rho \in \mathcal{U}(X,n)$, recall

Def 3.1 (1) ρ is called irreducible/simple if $\nexists$ non-trivial ρ-invariant subspace $W < \mathbb{k}^n$ $\neq \{0\}, \mathbb{k}^n$. $\rho(\gamma)(W) \subseteq W$ $\forall \gamma \in \pi_1(X,x)$

(2) ρ is called completely reducible/semisimple if $\forall$ ρ-invariant subspace of $\mathbb{k}^n$ has a ρ-invariant complement.

or. direct sum of irreducible reps.

Also recall:

Def 3.2

(1) ρ is polystable if the orbit $GL(n,\mathbb{k}) \cdot \rho \subset \mathcal{U}(X,n)$ is closed.

(2) ρ is stable if
- $GL(n,\mathbb{k}) \cdot \rho \subset \mathcal{U}(X,n)$ closed
- $\dim(PGL(n,\mathbb{k})_\rho) = 0$

(3) $\rho_1 \sim_S \rho_2$ if $\overline{GL(n,\mathbb{k})\cdot\rho_1} \cap \overline{GL(n,\mathbb{k})\cdot\rho_2} \neq \phi$

Lem 3.3 : (1) ρ is stable $\Leftrightarrow$ ρ is irreducible

(2) ρ is polystable $\Leftrightarrow$ ρ is completely reducible.

pf

Recall Hilbert-Mumford criterion of stability:

For $\rho \in \mathcal{U}(X,n)$. $\exists$ 1-ps $\lambda : \mathbb{G}_m \to GL(n,\mathbb{k})$ s.t.

$$\lim_{t\to 0} \lambda(t)\cdot\rho := \lim_{t\to 0} \lambda(t)\rho\lambda(t)^{-1} \text{ exists \& } \in \mathcal{O}_\rho$$

the unique closed orbit inside $\overline{GL(n,\mathbb{k})\cdot\rho}$

In particular, ρ stable $\Leftrightarrow$ $\lim_{t\to 0}\lambda(t)\cdot\rho$ doesn't exist for $\forall$ non-trivial 1-ps $\lambda : \mathbb{G}_m \to GL(n,\mathbb{k})$

(1) $\rho : \pi_1(X,x) \to GL(n,\mathbb{k})$ stable $\Leftrightarrow$ $\lim_{t\to 0}\lambda(t)\cdot\rho$ does not exist. $\forall$ nontrivial 1-ps λ

By Borel thm. up to conjugation.

$$\lambda(t) = \begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_n} \end{pmatrix} \in GL(n,\mathbb{k})$$

$a_1 \geq a_2 \geq \cdots \geq a_n \in \mathbb{Z}$

w.r.t. $\{e_1,\ldots,e_n\}$

write $\underbrace{a_1 = a_2 = \cdots = a_{n_1}}_{n_1} > \underbrace{a_{n_1+1} = \cdots = a_{n_1+n_2}}_{n_2} > \cdots \underbrace{= a_{n_1+\cdots n_m} = a_n}_{n_m}$

write

$$\rho(\gamma) = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}$$

$\Rightarrow$ $\lambda(t)\cdot\rho(\gamma) := \lambda(t)\,\rho(\gamma)\,\lambda(t)^{-1}$

$$= \begin{pmatrix} \rho_{11} & t^{a_1-a_2}\rho_{12} & \cdots & t^{a_1-a_n}\rho_{1n} \\ t^{a_2-a_1}\rho_{21} & \rho_{22} & \cdots & t^{a_2-a_n}\rho_{2n} \\ \vdots & \vdots & & \vdots \\ t^{a_n-a_1}\rho_{n1} & t^{a_n-a_2}\rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}$$

$\Rightarrow$ $\lim_{t\to 0} \lambda(t)\cdot\rho$ exists $\Leftrightarrow$ $\rho_{ij} = 0$ whenever $a_i < a_j$

$$\Leftrightarrow \rho(\gamma) = \begin{pmatrix} \rho_{11} & \rho_{12} \cdots & \rho_{1n} \\ \vdots & \vdots & \vdots \\ \rho_{n_1 1} & \vdots & \vdots \\ 0 & \rho_{n_1+n_2\, 1} & \vdots \\ \vdots & 0 & \\ 0 & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{} & & & \\ 0 & \boxed{} & & \\ 0 & 0 & \ddots & \\ & & & \boxed{} \end{pmatrix}$$

$\Leftrightarrow$ ρ preserves the flag

$$0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_m = k^n$$

for $W_i = \mathrm{span}\langle e_1, \ldots, e_{n_1+\cdots+n_i} \rangle$

$\Rightarrow$

ρ stable $\Leftrightarrow$ ρ irreducible

(2) ρ polystable i.e. $GL(n,k)\cdot\rho \subset \mathcal{U}(X,n)$ closed $\Rightarrow$ ρ completely reducible

if not. $\Rightarrow \exists\ 0 \subsetneq W \subsetneq k^n$ s.t. $\rho(\gamma)(W) \subset W \quad \forall \gamma \in \pi_1(X,x)$

but W has no ρ-invariant complement.

Take 1-ps $\lambda : \mathbb{G}_m \to GL(n,k)$ s.t. $\lambda(t)w = tw \quad \forall w \in W$
& $\exists\, W'$ complement W' s.t. $\lambda(t)w' = w' \quad \forall w' \in W'$

define $\rho' := \lim_{t\to 0} \lambda(t)\cdot\rho = \lim_{t\to 0} \lambda(t)\,\rho\,\lambda(t)^{-1}$

$\Rightarrow \forall w \in W,\ \gamma \in \pi_1(X,x) \qquad \rho'(\gamma)w = \rho(\gamma)w \in W \qquad \forall w \in W$

$\forall w' \in W' \quad \gamma \in \pi_1(X,x) \qquad \rho'(\gamma)w' = 0 \ \text{ or } \in W'$

$\Rightarrow$ both W & W' are ρ'-invariant

$\Rightarrow \rho' \not\sim \rho$

$\Rightarrow GL(n,\mathbb{R})\cdot\rho$ not closed !

- ρ completely reducible $\Rightarrow$ ρ polystable, i.e. $GL(n,\mathbb{R})\cdot\rho \subset \mathcal{U}$ closed:

ρ c.r. $\Rightarrow \exists$ flag $0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_m = \mathbb{R}^n$ preserved by ρ

$$\rho = \begin{pmatrix} \rho_1 & & & * \\ & \rho_2 & & \\ & & \rho_3 & \\ 0 & & & \ddots \ \rho_m \end{pmatrix}$$

$$\rho' = \lim_{t\to 0} \lambda(t)\cdot\rho = \begin{pmatrix} \rho_1 & & & 0 \\ & \rho_2 & & \\ & & \rho_3 & \\ 0 & & & \ddots \ \rho_m \end{pmatrix}$$

For simplicity, assume $m=1$

$\Rightarrow \exists$ complement $W_1^{\perp}$ of W_1 in $\mathbb{R}^n$ which is ρ-inv.

$$\mathbb{R}^n = W_1 \oplus W_1^{\perp}$$

ρ & ρ' act on W_1 & $W_1^{\perp}$ in ~~the~~ same way.

$\Rightarrow \rho \sim \rho'$

$\square$

Cor 3.4 $\mathcal{U}^{ss}(X,n) \subset \mathcal{U}(X,n)$ semisimple reps.

$\mathcal{U}^{irr}(X,n) \subset \mathcal{U}(X,n)$ irreducible reps.

$$\Rightarrow \quad \varphi : \mathcal{U}^{irr}(X,n) \to \varphi(\mathcal{U}^{irr}(X,n)) =: \mathcal{M}_B^s(X,n)$$

geometric quotient.

In particular, $\mathcal{M}_B^s(X,n)$ is non-singular.

need deformation

Cor 3.5 Bijections of sets:

$$\mathcal{M}_B(X,n)(k) \simeq \mathcal{U}^{ss}(X,n)(k) / GL(n,k) \simeq \mathcal{U}(X,n)(k) / \sim_S$$

known. each $\overline{GL(n,k)\cdot\rho} \supset \underbrace{GL(n,k)\cdot\rho'}$

- closed $\Rightarrow$ ρ' polystable
- unique (up to iso.)

ρ' is the semisimple representative of ρ. called the semisimplification

Lem 3.6 $\forall \rho \in \mathcal{U}(X,n)$ admits a Jordan-Hölder filtration:

$$0 \subsetneq \rho_1 \subsetneq \cdots \subsetneq \rho_n = \rho$$

s.t. each quotient $gr_i = \rho_i / \rho_{i-1}$ is irreducible.

such filtration is unique up to iso. of $gr(\rho) := \bigoplus_{i=1}^n gr_i$

- subrep: $\rho' \subset \rho : \pi_1(X,x) \to GL(n,k)$ if $\exists\, 0 \subsetneq W \subsetneq k^n$ ρ-inv. s.t. $\rho' = \rho|_W : \pi_1(X,x) \to GL(W)$.

- quotient rep.: ρ/ρ' for ρ' subrep $: \pi_1(X,x) \to GL(k^n/W)$

$\rho/\rho'(\gamma)(w+v) := \rho(\gamma)(w) + v.$

$\gamma \in \pi_1$
$w \in W$
$v \in k^n.$

Lem3.7 $\rho_1 \sim_s \rho_2 \iff gr(\rho_1) \simeq gr(\rho_2)$

$\rightsquigarrow \quad \rho' \simeq gr(\rho)$

In conclusion. we can write $\varphi: \mathcal{U}(X.n) \to \mathcal{M}_B(X.n)$
$\rho \longmapsto [gr(\rho)]$

§4. Twisted version

In this part. $\dim_{\mathbb{R}} X = 1$

Though $\mathcal{M}_B(X.n)$ contains an open smooth subvar. $\mathcal{M}_B^s(X.n)$

Problem: $\mathcal{M}_B(X.n)$ is not smooth $<$ non-closed orbits / infinite stabilizers.

How to solve this?

Taking a primitive n-th root of unity ξ. is $\xi = e^{\frac{2\pi i d}{n}}$
$\gcd(n.d) = 1.$

Defined the space of twisted representations.

$$\mathcal{U}(X.n,d) := \mathrm{Hom}\left(\pi_1^{tw}(X.x),\ GL(n,k)\right)$$

$$\uparrow$$

$$\left\langle \gamma_1,\dots,\gamma_g,\gamma_1',\dots,\gamma_g' : \prod_{i=1}^{g}[\gamma_i,\gamma_i'] = \xi\,\mathrm{Id} \right\rangle$$

$$\simeq \left\{ (A_1,\dots,A_g,A_1',\dots,A_g') \in GL(n,k)^{2g} : \prod_{i=1}^{g}[A_i,A_i'] = \xi\,\mathrm{Id} \right\}.$$

$\subset GL(n,\mathbb{k})^{2g}$ closed subvariety

$\rightsquigarrow \mathcal{U}(X,n,d)$ affine variety.

similarly, $GL(n,\mathbb{k}) \curvearrowright \mathcal{U}(X,n,d)$ by conjugation :

$\rightsquigarrow$ affine GIT quotient

$$\varphi : \mathcal{U}(X,n,d) \to \mathcal{U}(X,n,d) /\!\!/ GL(n,\mathbb{k}) =: \mathcal{M}_B(X,n,d)$$

"twisted Betti moduli space"

$$\mathcal{M}_B(X,n) \simeq \mathcal{M}_{Dol}(X,n) \simeq \mathcal{M}_{dR}(X,n)$$

$$\mathcal{M}_B(X,n,d) \simeq \mathcal{M}_{Dol}(X,n,d) \simeq \mathcal{M}_{dR}(X,n,d)$$

<u>Thm 4.1</u>

(1) $\mathcal{U}(X,n,d)$ and $\mathcal{M}_B(X,n,d)$ are connected.

(2) $\mathcal{M}_B(X,n,d)$ is non-singular

(3) $\dim \mathcal{U}(X,n,d) = n^2(2g-1)+1$

$\dim \mathcal{M}_B(X,n,d) = n^2(2g-2)+2$