Lecture 7

Kobayashi-Hitchin correspondence

(existence of pluri-harmonic metrics = stability + top. cond.)

Recall: categorical correspondence:

$$\mathcal{C}_{\mathrm{Dol}}(X, n) \simeq \mathcal{C}_{\mathrm{dR}}(X, n)$$

harmonic Higgs $\qquad\qquad$ harmonic flat

Q: When does a Higgs bundle (resp. flat bundle) admit a pluri-harmonic metric?

Today: solve **Q**!

Setting: (X, ω) cpt Kähler mfld. $\dim_{\mathbb{C}} X = m$

Main thm.

(1) A Higgs bundle over X admits a pluri-harmonic metric iff it is polystable with $c_1 = 0 = \mathrm{ch}_2$.

Moreover, such metric, if exists, is unique up to scalar multiplication. (on each stable component)

(2) A flat bundle over X admits a pluri-harmonic metric iff it is semisimple. Moreover, ...

Part I. Kobayashi-Hitchin correspondence for Higgs bundles

§1. Stability for Higgs bundles

Def 1.1 A Higgs bundle $(E, \bar{\partial}_E, \varphi)$ is called $\underbrace{\text{slope-stable}}_{(E, \bar{\partial}_E)}$ (resp. slope-semistable) if for $\forall$ proper torsion-free coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ s.t

(1) $0 < \mathrm{rk}\,\mathcal{F} < \mathrm{rk}\,\mathcal{E}$

(2) φ-invariant: $\varphi(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_X^1$ $\qquad \leftarrow \quad \varphi: \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1$

the following inequality holds

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\mathrm{rk}\,\mathcal{F}} \underset{(\leq)}{<} \frac{\deg(\mathcal{E})}{\mathrm{rk}\,\mathcal{E}} =: \mu(\mathcal{E})$$

It is slope-polystable if it decomposes as the direct sum of slope-stable Higgs bundle of the same slope.

Rmk (1) $m>1$. torsion-free sheaf $\mathcal{F}$ is locally free on $X\setminus \mathrm{Sing}(\mathcal{F})$

$\rightsquigarrow$ singular set. codim ≥ 2

(2) $m>1$. there are many stability conditions.

e.g. Gieseker stability defined by reduced Hilbert polynomials

$$p(\mathcal{F}) := \frac{\chi(\mathcal{F}(m))}{a_d} \quad m\geq 0 \qquad \dim \mathcal{F} = d, \quad \chi(\mathcal{F}(m)) = \sum_{i=0}^{d} a_i \cdot m^i$$

Bridgeland stability on $D^b(X)$ ...

From now on. we only use stable / semistable / polystable to denote slope stability.

(3) To check stability it's enough to check the inequality for saturated subsheaves.

i.e. $\mathcal{E}/\mathcal{F}$ torsion-free.

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{E}/\mathcal{F} \to 0$$

$$\mu(\mathcal{F}) < (\leq)\ \mu(\mathcal{E}) \iff \mu(\mathcal{E}) \underset{(\leq)}{<} \mu(\mathcal{E}/\mathcal{F})$$

$$\deg(\mathcal{E}) = \deg(\mathcal{F}) + \deg(\mathcal{E}/\mathcal{F})$$

$$\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{F}) + \mathrm{rk}(\mathcal{E}/\mathcal{F})$$

$T \subset \mathcal{E}/\mathcal{F}$ torsion part. i.e. $\left.\mathcal{E}/\mathcal{F}\right/ T$ torsion-free. $\Rightarrow \deg(T)\geq 0$

$$\Rightarrow (\mathcal{E}/\mathcal{F})_{tf} := \left(\left.\mathcal{E}/\mathcal{F}\right/T\right) \text{ has deg } \leq \deg(\mathcal{E}/\mathcal{F})$$

Ex $m=1 \quad g\geq 2$

$$\left(\mathcal{E} = K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}},\ \varphi = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix}\right) \qquad q_2 \in H^0(X, K_X^2)$$

$\otimes K_X$

Claim: $(\mathcal{E},\varphi)$ is stable.

Indeed. if $q_2 = 0$. the only φ-inv. proper subbundle is $K_X^{-\frac{1}{2}}$. $\deg = 1-g < 0 = \deg \mathcal{E}$

if $q_2 \neq 0$. there is no φ-inv. proper subbundle.

In general. $\left(\mathcal{E} = K_X^{\frac{n-1}{2}} \oplus K_X^{\frac{n-3}{2}} \oplus \cdots \oplus K_X^{-\frac{n-1}{2}},\ \varphi = \begin{pmatrix} 0 & q_2 & \cdots & q_n \\ 1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & q_2 \\ & & 1 & 0 \end{pmatrix}\right)$ is stable.

As harmonic + $ch_2 = 0 \iff$ pluri-harmonic

To show Thm (1), it suffices to show harmonic $\iff$ polystable + $c_1 = 0$

Thm 1.2 $(E, \bar\partial_E, \varphi)$ admits a pluri-harmonic metric

$$\Rightarrow (E, \bar\partial_E, \varphi) \text{ polystable} + c_1 = 0 = ch_2$$

Pf. By definition, if h is a pluri-harmonic metric on $(E, \bar\partial_E, \varphi)$.

Then $\nabla_h := D_h + \varphi + \varphi^{*h}$ is flat $\Rightarrow c_i = 0$

$\mathcal{F} \subset \mathcal{E}$ saturated subsheaf that is φ-inv. $(0 < \mathrm{rk}\,\mathcal{F} < \mathrm{rk}\,\mathcal{E})$

$Q := \mathcal{E}/\mathcal{F}$.

$$0 \to \mathcal{F} \underset{\pi}{\leftrightarrows} \mathcal{E} \to Q \to 0$$

given h

$\Rightarrow$ on $\underset{X'}{X \setminus S'}$, $\mathcal{E}$ splits

$$\mathcal{E} \overset{C^\infty}{\simeq} \mathcal{F} \oplus Q \quad \text{(not necessarily holo.)}$$

denote $\pi: \mathcal{E} \to \mathcal{F}$ projection.

$\quad Id - \pi: \mathcal{E} \to Q$

Aim write ∇_h in terms of π, D'', D_h' . . .

$\quad F_{\nabla_h}$ . . .

$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in A^0(X', \mathcal{E})$. $u_1 \in A^0(X', \mathcal{F})$. $u_2 \in A^0(X', Q)$

write

$$\nabla_h \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \pi \circ \nabla_h & \beta \\ \alpha & (Id - \pi) \circ \nabla_h \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\alpha \in A^1(X', \mathrm{Hom}(\mathcal{F}, Q))$

$\beta \in A^1(X', \mathrm{Hom}(Q, \mathcal{F}))$ $\quad 2^{nd}$ fundamental forms

- $\alpha(u_1) = (Id - \pi) \circ \nabla_h(u_1)$

$\quad = (Id - \pi) \circ (D_h' + D'')(u_1)$ $\quad D_h' = \partial_h + \varphi^{*h}$, $D'' = \bar\partial_E + \varphi$

$$= (Id - \pi) \circ D_h'(u_1)$$

$$= D_h' \circ \pi(u_1) - \pi \circ D_h'(u_1) \qquad D_h'(\pi) = D_h' \circ \pi - \pi \circ D_h'$$

$$= D_h'(\pi)(u_1) \qquad \text{i.e. } \alpha = D_h'(\pi)$$

- $\beta(u_2) = \nabla_h(u_2) - (Id - \pi) \circ \nabla_h(u_2)$

$$= \pi \circ \nabla_h(u_2)$$

$$= \pi \circ (D_h' + D'')(u_2)$$

claim $D_h'(u_2) = 0$. indeed.

$$0 = h(\bar\partial_E(u_1), u_2) = -h(u_1, \partial_h(u_2))$$

$$0 = h(\varphi(u_1), u_2) = h(u_1, \varphi^{*h}(u_2))$$

$$\Rightarrow \quad h(u_1, D_h'(u_2)) = 0$$

$$\Rightarrow \beta(u_2) = \pi \circ D''(u_2)$$

$$= (\pi \circ D'' - D'' \circ \pi)(u_2)$$

$$= -D''(\pi)(u_2) \qquad \text{i.e. } \beta = -D''(\pi)$$

$$\Rightarrow \quad \nabla_h \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \pi \circ \nabla_h & -D''(\pi) \\ D_h'(\pi) & (Id - \pi) \circ \nabla_h \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\Rightarrow \quad F_h \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \nabla_h^2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} F_{\pi \circ \nabla_h} - D''(\pi) \circ D_h'(\pi) & *_1 \\ *_2 & F_{(Id - \pi) \circ \nabla_h} - D_h'(\pi) \circ D''(\pi) \end{pmatrix}$$

In other words. let $\nabla_{\mathcal{F}} := \pi \circ \nabla_h$

$$\Rightarrow \quad \pi \circ F_h \circ \pi = F_{\nabla_{\mathcal{F}}} - D''(\pi) \circ D'(\pi)$$

$$\Rightarrow \quad \pi \circ \Lambda_\omega F_h \circ \pi = \Lambda_\omega F_{\nabla_{\mathcal{F}}} - \Lambda_\omega D''(\pi) D'(\pi)$$

$$\Rightarrow \quad \deg(\mathcal{F}) = \frac{\sqrt{-1}}{2\pi} \int_X \mathrm{Tr}(\Lambda_\omega F_{\nabla_{\mathcal{F}}}) \frac{\omega^m}{m!}$$

$$= \frac{\sqrt{-1}}{2\pi}\int_X \mathrm{Tr}\,(\pi \Lambda_\omega F_h)\,\frac{\omega^m}{m!} + \frac{\sqrt{-1}}{2\pi}\int_X \mathrm{Tr}\,(\Lambda_\omega D''(\pi)\, D'(\pi))\frac{\omega^m}{m!}$$

$$= -\,\|D''(\pi)\|_{L^2}^2 \le 0$$

i.e. $\deg(\mathcal{F}) \le \deg(\mathcal{E})$ "$=$" $\iff D''(\pi) = 0$

$$\iff (\underline{\bar\partial_E} + \underline{\varphi})(\pi) = 0$$

$\pi : \mathcal{E} \to \mathcal{F}$

i.e. $\mathcal{F} \subset \mathcal{E}$ φ-inv. holo. subbundle.

i.e. $\mathcal{E} \simeq \mathcal{F} \oplus Q$ holo. splitting & preserves Higgs fields

Continue. after finite steps. will stop.

$\rightsquigarrow (\mathcal{E}, \varphi) = \bigoplus_{i=1}^{\ell} (\mathcal{E}_i, \varphi_i)$ polystable.

$\boxtimes$

$$\bar\partial_E = \begin{pmatrix} \bar\partial_F & \alpha \\ 0 & \bar\partial_Q \end{pmatrix}$$

$$\varphi = \begin{pmatrix} \varphi_F & \beta \\ 0 & \varphi_Q \end{pmatrix}$$

$\nabla_h \quad F_h \quad \cdots$

$$\deg(\mathcal{E}) = \deg(\mathcal{F}) + \|\alpha\|_{L^2}^2 + \|\beta\|_{L^2}^2$$

§2. Existence of pluri-harmonic metrics.

Thm 2.1 (Uhlenbeck-Yau, Popovici (simplified proofs))

$(E, \bar\partial_E)$ holo. v.b. over X $\quad h$ hermitian metric.

$\pi \in W^{1,2}(X, \mathrm{End}(E))$ be a section s.t.

(1) $\pi^2 = \pi^{*h} = \pi$

(2) $(\mathrm{Id} - \pi) \circ \bar\partial_E(\pi) = 0$

holds almost everywhere.

$\Rightarrow \exists\ \mathcal{F} \subset \mathcal{E}$ coherent subsheaf & S codim ≥ 2 s.t.

(1) $\pi \in C^\infty(X \setminus S, \mathrm{End}(\mathcal{E}))$

(2) $\pi^2 = \pi^{*h} = \pi$ & $(Id - \pi) \circ \bar{\partial}_E(\pi) = 0$ holds on $X \setminus S$

(3) $\mathcal{F}|_{X \setminus S} = \pi_{X \setminus S}(\mathcal{E}|_{X \setminus S})$ holo. subbundle of $\mathcal{E}|_{X \setminus S}$

Rmk holds for $(E, \bar{\partial}_E, \varphi)$ & h

$$\bar{\partial}_E \rightsquigarrow D'' = \bar{\partial}_E + \varphi.$$

Thm 2.2 (Hitchin ($n=1$, $r=2$), Simpson (general))

Polystable + $c_i = 0$ $\Rightarrow$ there exists a harmonic metric.

Fix K. $H(t) = K \cdot h(t)$ $\quad h(t) \in \mathrm{Aut}(E)$ s.t. $h(t) = h(t)^{*K}$

$\Rightarrow h = K^{-1} \cdot H$

Lemma 2.3 (1) $D'_H = D'_K + h^{-1} D'_K(h)$

(2) $\sqrt{-1}\Lambda_\omega(F_H) = \sqrt{-1}\Lambda_\omega(F_K) + \sqrt{-1}\Lambda_\omega D''(h^{-1} D'_K(h))$

Introduce for $\forall$ h_0 hermitian metric

$$\Delta'_{h_0} := (D'_{h_0})^* D'_{h_0} + D'_{h_0}(D'_{h_0})^*$$

Cor 2.4

$$\Delta'_K(h) = h\sqrt{-1}\Lambda_\omega(F_H - F_K) + \sqrt{-1}\Lambda_\omega\left(D''(h) h^{-1} D'_K(h)\right)$$

pf. KI for Higgs.

$$\Delta'_K(h) = (D'_K)^* D'_K(h) + D'_K \underbrace{(D'_K)^*(h)}_{=0}$$

$$= \sqrt{-1}\Lambda_\omega D'' D'_K(h)$$

$$\overset{\text{Lem 2.3 (2)}}{=} \cdots$$

$\square$

Donaldson heat flow:

$$H(0) = K$$

$$H(t) = K\cdot h(t) \qquad \det(h) = 1$$

$$H^{-1}\frac{\partial H}{\partial t} = -\sqrt{-1}\Lambda_\omega F_H \qquad (*)$$

$$\Rightarrow \quad \frac{\partial h}{\partial t} = K^{-1}\frac{\partial H}{\partial t} = K^{-1}H\cdot H^{-1}\frac{\partial H}{\partial t}$$

$$= h\cdot H^{-1}\frac{\partial H}{\partial t}$$

$$\overset{(*)}{=} -h\sqrt{-1}\Lambda_\omega F_H.$$

$\Rightarrow$ Cor +

$$\left(\frac{\partial}{\partial t} + \Delta_K'\right)(h) = -h\sqrt{-1}\Lambda_\omega F_K + \sqrt{-1}\Lambda_\omega\left(D''(h)h^{-1}D_K'(h)\right) \qquad (**)$$

Donaldson functional:

$$M: \mathrm{Herm}(E)\times \mathrm{Herm}(E) \to \mathbb{R}$$

$$\frac{d}{dt}M(H,K) = \int_X \mathrm{Tr}\left(\sqrt{-1}\Lambda_\omega F_H\cdot H^{-1}\frac{\partial H}{\partial t}\right)\frac{\omega^m}{m!}$$

lem 2.5 $\quad M(H,K) = \int_0^1\int_X \mathrm{Tr}(\cdots)\frac{\omega^m}{m!}dt$ is path-independent.

$\forall\ H(t),\ H'(t)$. with

$$\begin{cases} H(0) = H'(0) = K \\ H(1) = H(1) \\ \det(H) = \det(K) = \det(H') \end{cases}$$

$$\Rightarrow M(H,K) = M(H',K)$$

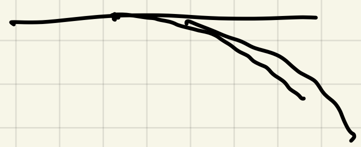
Hence choose a special path $H(t) = K\cdot e^{ts}$ $\quad \mathrm{Tr}(s) = 0$

$$H^{-1}\frac{\partial H}{\partial t} = s$$

$$\Rightarrow M(H,K) = \int_X \mathrm{Tr}(\sqrt{-1}\Lambda_\omega F_K\cdot s)\frac{\omega^m}{m!} + \int_X (\Psi(s)D''s), D's)\frac{\omega^m}{m!}$$

<u>idea:</u> along heat flow.

$$\frac{d}{dt} M(H(t), H(0)) = -\|\Lambda_\omega F_H\|^2_{L^2}$$
$$\le 0$$



Initial value
$$M(H(0), H(0)) = 0$$

<u>Thm 2.6</u> (Simpson's main estimate)

$$H = K e^{s} \qquad \mathrm{Tr}(s). \qquad \sup_X |s| < \infty$$

if $(E, \bar\partial_E, \varphi)$ stable with $\sup_X |\Lambda_\omega F_H| \le C$

$\Rightarrow$ $M(H, K)$ is bdd from below.

$$\sup_X |s| \le C_1 + C_2 M(H, K)$$

<u>Pf</u> need Uhlenbeck-Yau's construction to show. !

<u>Next</u>. $\sup_X |\Lambda_\omega (F_H)| \le C$

$$\frac{d}{dt} \Lambda_\omega F_H = - \sqrt{-1}\, \Lambda_\omega D'' D'_H (\Lambda_\omega F_H)$$

$$\Rightarrow \quad \frac{d}{dt} |\Lambda_\omega F_H|^2 = 2\sqrt{-1}\, \mathrm{Tr}(\Lambda_\omega F_H \cdot \Lambda_\omega D'' D'_H (\Lambda_\omega F_H))$$

$$\Delta' |\Lambda_\omega F_H|^2 = \cdots$$
$$= -2\sqrt{-1}\, \mathrm{Tr}\left(\Lambda_\omega F_H \cdot \Lambda_\omega D'' D'_H (\Lambda_\omega F_H)\right) - 2 |D''(\Lambda_\omega F_H)|^2$$

$\Rightarrow$
$$\left(\frac{d}{dt} + \Delta'\right) |\Lambda_\omega F_H|^2 = -2 |D'' \Lambda_\omega F_H|^2$$
$$\le 0$$

$\rightarrow$ parabolic PDE $\quad \sup_X |\Lambda_\omega F_H| \le C.$

Left: long existence of solutions to heat flow.

$$\frac{d^2}{dt^2} M(H(t), K) \geq 0$$

$$\Rightarrow \quad \frac{d}{dt} M(H, K) \to 0 \qquad t \to \infty$$

i.e. $\Lambda_\omega F_{H_t} \to 0$ in L^2, $H_t = K \cdot e^{s(t)}$

taking $s_t \to s_\infty$ in $W^{2,p}$ $\qquad$ $\underline{H_\infty} := K \cdot h_\infty = K \cdot e^{s_\infty}$

$\Rightarrow$

$$\Delta_K'(h_\infty) := -h_\infty \sqrt{-1} \Lambda_\omega F_K + \sqrt{-1} \Lambda_\omega D''(h_\infty) h_\infty^{-1} D_K'(h_\infty).$$

elliptic regularity to show h_∞ is $\underline{C^\infty}$

$\rightsquigarrow$ $H_\infty = K \cdot h_\infty$ satisfies

$$\underline{\Lambda_\omega F_{H_\infty} = 0} \qquad \text{i.e. } H_\infty \text{ harmonic.}$$

uniqueness:

suppose stable.

$$\Lambda_\omega F_{H_1} = \Lambda_\omega F_{H_2} = 0 \qquad h = H_1^{-1} \cdot H_2$$

$$\Rightarrow \quad \frac{1}{2} \Delta(\mathrm{Tr}\, h) = \Delta'(\mathrm{Tr}\, h)$$

$$= -|D''(h) h^{-\frac{1}{2}}|^2$$

$$\leq 0$$

$\Rightarrow$ $\mathrm{Tr}\, h$ subharmonic $\overset{MP}{\Rightarrow}$ $\mathrm{Tr}\, h = c$

$$D''(h) = 0 = D'(h)$$

i.e. h is a holo. morphism of stable ~~bundle~~ bundle.

stable $\Rightarrow$ simple i.e. $\mathrm{Aut} \simeq \mathbb{C} \cdot \mathrm{Id}$.

$\Rightarrow h = G'$.

$\Rightarrow$ uniqueness.

☒

Part II: Kobayashi-Hitchin correspondence for flat bundles

§3. Stability of flat bundles

Def 3.1 A holo. flat bundle $(\mathcal{E}, D)$ over X is

- irreducible/simple if it has no non-trivial proper $\underline{\text{holo. flat subbundle}}$.

$\rightarrow$ $(\mathcal{F}, D_{\mathcal{F}})$ s.t. $\mathcal{F} \subset \mathcal{E}$ holo subbundle

$D(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_X^1$

$D_{\mathcal{F}} := D|_{\mathcal{F}}$

- semisimple / completely reducible if it decomposes as the direct sum of irreducible ones

$$(\mathcal{E}, D) = \bigoplus_{i=1}^{\ell} (\mathcal{E}_i, D_i) \quad \text{for each } (\mathcal{E}_i, D_i) \text{ is irred.}$$

Rmk (1) similarly for (E, ∇)

(2) Not like Higgs bundles, we only consider subbundles for stability.

Lem 3.2 (Katz) Any coherent sheaf of $\mathcal{O}_X$-modules with integrable connection is locally free.

Lem 3.3 (André) Any coherent sheaf of $\mathcal{O}_X$-modules with a connection is locally free

Thm 3.4 (E, ∇) flat, harmonic metric $\Rightarrow$ semisimple.

Pf. h (E, ∇)

$\rightsquigarrow$ $(E, \bar\partial_h, \varphi_h)$ also admits h as a pluri-harmonic metric.

$\Rightarrow$ $(E, \bar\partial_h, \varphi_h)$ polystable $c_i = 0$

i.e. $(E, \bar\partial_h, \varphi_h) = \bigoplus_{i=1}^{\ell} (E_i, \bar\partial_{E_i}, \varphi_i)$

stable. $h_i = h|_{E_i}$ $c_i = 0$

$$\rightsquigarrow (E, \nabla) = \bigoplus_{i=1}^{\ell} (E_i, \nabla_i)$$

$$\nabla_i = \bar\partial_{E_i} + \partial_{h_i} + \varphi_i + \varphi_i^{*h_i}$$

direct sum of irreducible flat bundles. ∎

§4. Existence of pluri-harmonic metrics

Thm 4.1 (Donaldson ($m=1, n=2$). Corlette (general))

semisimple for (E, ∇) $\Rightarrow$ $\exists$ harmonic metric.

! up to scalar multiplication..

Pf ① Modify $H^{-1}\frac{\partial H}{\partial t} = -\sqrt{-1}\Lambda_\omega F_H$

to $H^{-1}\frac{\partial H}{\partial t} = -\sqrt{-1}\Lambda_\omega G_H$

$\rightsquigarrow$ . . .

Pf ② R-H.

$\rho: \pi_1(X, x_0) \to GL(n, \mathbb{C}) \longleftrightarrow (E, \nabla)$

irreducible / simple $\longleftrightarrow$ irreducible / simple

semisimple $\longleftrightarrow$ semisimple

$U(n) \subset GL(n, \mathbb{C})$ maximal cpt.

$\mathfrak{u}(n)$ $\mathfrak{gl}(n, \mathbb{C})$

$\rightsquigarrow \qquad \mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{p}$

$\rightarrow$ Killing form B on $\mathfrak{gl}(n,\mathbb{C})$ is positive-define on $\mathfrak{p}$

$\rightsquigarrow GL(n,\mathbb{C})/U(n)$ is a Riemann mfd.

$\left(T_{eU(n)} GL(n,\mathbb{C})/U(n) \simeq \mathfrak{p} \right)$

$\mathcal{M} := \{ M \in GL(n,\mathbb{C}) : M \text{ positive-define} . M^* = M \}$

space of hermitian matrices

$GL(n,\mathbb{C}) \curvearrowright \mathcal{M}$

$$(g.M) := (g^{-1})^* M g^{-1}$$

$\mathcal{M} \simeq GL(n,\mathbb{C})/U(n)$ via

$\searrow \; g \cdot U(n) \mapsto (g^{-1})^* g^{-1}$

regarding $GL(n,\mathbb{C}) \curvearrowright GL(n,\mathbb{C})/U(n)$

$$g \cdot M := (g^{-1})^* M g^{-1}$$

$\rho : \pi_1(X,x) \to GL(n,\mathbb{C}) \iff (E := \tilde{X} \times_\rho \mathbb{C}^n , \nabla = d)$

Prop 4.1 h on $(E,\nabla) \iff \rho$-equivariant map $h_\rho : \tilde{X} \to GL(n,\mathbb{C})/U(n)$

i.e. $h_\rho(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot h_\rho(\tilde{x})$

$= (\rho(\gamma)^{-1})^* \cdot h_\rho(\tilde{x}) \cdot (\rho(\gamma))^{-1}$

$\forall \gamma \in \pi_1(X,x), \; \tilde{x} \in \tilde{X}$

Pf:

"$\Leftarrow$" $h_\rho : \tilde{X} \to GL(n,\mathbb{C})/U(n)$

$$I(X,E) = I(X, \tilde{X} \times_\rho \mathbb{C}^n) \Leftrightarrow C^\infty(u : \tilde{X} \to \mathbb{C}^n, \rho\text{-equiv. map})$$

$$\text{i.e. } u(\gamma \cdot \tilde{x}) = \rho(\gamma) u(\tilde{x})$$

define h via

$$h(u,u')(\tilde{x}) := \langle u(\tilde{x}), h_\rho(\tilde{x}) u'(\tilde{x}) \rangle_{\mathbb{C}^n}$$
$$= u(\tilde{x})^T \overline{h_\rho(\tilde{x}) u'(\tilde{x})}$$

"$\Rightarrow$" h on (E,∇). define $h_\rho : \tilde{X} \to GL(n,\mathbb{C})/U(n)$

via

$$h(u,u')(\tilde{x}) = \langle u(\tilde{x}), h_\rho(\tilde{x}) u'(\tilde{x}) \rangle_{\mathbb{C}^n}$$

check h_ρ is ρ-equiv.

$\boxtimes$

<u>Recall</u>.

$$\nabla = \underbrace{D_h}_{\text{unitary conn. w.r.t. } h} + \overbrace{\Psi_h}^{\text{self-adjoint w.r.t. } h}$$

<u>Prop 4.3</u> $\Psi_h = -\frac{1}{2} h_\rho^{-1} \cdot dh_\rho$

pf.

$$dh(u,v) = h(D_h(u), v) + h(u, D_h(v))$$
$$\parallel$$
$$d(\langle u, h_\rho v \rangle_{\mathbb{C}^n}) = d(u^T \overline{h_\rho v})$$
$$= \cdots$$
$$= h(\underset{(D_h+\Psi_h)}{du}, v) + h(u, h_\rho^{-1} dh_\rho v) + h(u, \underset{D_h+\Psi_h}{dv})$$

$\boxtimes$

<u>Prop 4.4</u>. $h_\rho : \tilde{X} \to GL(n,\mathbb{C})/U(n)$ is harmonic $\Leftrightarrow$ h is <u>harmonic metric</u>.

$\Leftrightarrow \Lambda_\omega G_h = 0$

h_ρ is critical pt of

$$E(h_\rho) := \frac{1}{2} \int_X |dh_\rho|^2 \frac{\omega^m}{m!}$$

i.e. h_φ satisfies EL eqn.

$$d_\nabla^* d h_\varphi = 0$$

Pf. Prop 4.3 $\quad \Phi_h = -\frac{1}{2} h_\varphi^{-1} d h_\varphi$

$$\Rightarrow \quad E(h_\varphi) = 4\int_X |\Phi_h|^2 \frac{\omega^m}{m!}$$

find EL equation for $\int_X |\Phi_h|^2 \frac{\omega^m}{m!}$

$$\tilde{h} = h \cdot e^{s(t)} \qquad e^{s(t)} \in \mathrm{Aut}(E). \qquad s(0) = 0$$

$$\Rightarrow \quad \delta'_{\tilde{h}} = \delta'_h + \delta'_h(s)$$

$$\delta''_{\tilde{h}} = \delta''_h + \delta''_h(s)$$

$$\Rightarrow \quad \varphi_{\tilde{h}} = \frac{1}{2}(d' - \delta'_{\tilde{h}}) = \varphi_h - \frac{1}{2}\delta'_h(s)$$

$$\varphi_{\tilde{h}}^{*\tilde{h}} = \frac{1}{2}(d'' - \delta''_{\tilde{h}}) = \varphi_h^{*h} - \frac{1}{2}\delta''_h(s)$$

$$\Rightarrow \quad \int_X |\Phi_{\tilde{h}}|^2 \frac{\omega^m}{m!} = \int_X \langle \varphi_{\tilde{h}} + \varphi_{\tilde{h}}^{*\tilde{h}}, \varphi_{\tilde{h}} + \varphi_{\tilde{h}}^{*\tilde{h}} \rangle_{g,\tilde{h}} \frac{\omega^m}{m!}$$

$$= \int_X |\Phi_h|^2 \frac{\omega^m}{m!} - \int_X \langle (\delta'_h)^*(\varphi_h) + (\delta''_h)^*(\varphi_h^{*h}), s \rangle$$

$$+ \frac{1}{4}\int_X |\delta'_h(s) + \delta''_h(s)|^2 \frac{\omega^m}{m!}$$

$\Rightarrow$ EL for $\int_X |\Phi_h|^2 \frac{\omega^m}{m!}$ is

$$(\delta'_h)^*(\varphi_h) + (\delta''_h)^*(\varphi_h^{*h}) = 0 \qquad (***)$$

To show $(***) \Leftrightarrow \Lambda_\omega G_h = 0$

KI :

$$(d')^* = \sqrt{-1}[\Lambda_\omega, \delta''_h]$$

$$(d'')^* = -\sqrt{-1}[\Lambda_\omega, \delta'_h]$$

$$(\delta'_h)^* = \sqrt{-1}[\Lambda_\omega, d'']$$

$(\delta_h'')^* = -\sqrt{-1}[\Lambda_\omega, d']$

$$\overset{(***)}{\Rightarrow} 0 = \Lambda_\omega\left(d''(\varphi_h) - d'(\varphi_h^{*_h})\right)$$

$$= \Lambda_\omega\left(\bar\partial_h(\varphi_h) + \cancel{[\varphi_h^{*_h}, \varphi_h]} - \partial_h(\varphi_h^{*_h}) - \cancel{[\varphi_h, \varphi_h^{*_h}]}\right)$$

$$= \Lambda_\omega\left(\bar\partial_h(\varphi_h) - \partial_h(\varphi_h^{*_h})\right)$$

$$= 2\Lambda_\omega\left(\bar\partial_h(\varphi_h)\right)$$

$$= 2\Lambda_\omega G_h$$

$\cdot\ \bar\partial_h(\varphi_h) = -\partial_h(\varphi_h^{*_h})$

$$G_h = (D_h'')^2 = (\bar\partial_h + \varphi_h)^2 = \bar\partial_h(\varphi_h) + (2.0) + (0.2)$$

$\blacksquare$

Left: when is h_ρ harmonic map.

Thm 4.5 h_ρ harmonic map $\Leftrightarrow$ $\rho: \pi_1(X, x) \to GL(n, \mathbb{C})$ semisimple

$$\begin{cases} \dfrac{\partial h_{\rho,t}}{\partial t} = -d_\nabla^* d h_{\rho,t} \\ h_{\rho,0} = h_0 \end{cases}$$

$h_{\rho,\infty}$ is the solution. C^∞

§5. Conclusion

Algebraic description of $\mathcal{C}_{Dol}(X, n)$ & $\mathcal{C}_{dR}(X, n)$

By abuse of notations, use $\mathcal{C}_{Dol}(X, n)$: category of polystable Higgs bundle + $c_i = 0$ $rk = n$

$\mathcal{C}_{dR}(X, n)$: category of semisimple flat bundles $rk = n$.

Thm 5.1 (Nonabelian Hodge correspondence, categorical version)

$$\mathcal{C}_{Dol}(X,n) \simeq \mathcal{C}_{dR}(X,n) \quad \left(\simeq \mathcal{C}^{s.s}_{Loc}(X,n) \right)$$

Rmk. In fact, this can be generalized to semistable Higgs bundles.

Thm 5.2 (Simpson (projective), Nie-Zhang (Kähler mfds))

$$\mathcal{C}_{Higgs}(X,n) \simeq \mathcal{C}_{Flat}(X,n)$$

category of semistable Higgs bundles of rk n

category of flat bundles of rk n

Simpson idea: semistable + $c_i = 0$ are extension of stable + $c_i = 0$.

depends on Mehta-Ramanathan hyperplane restriction thm.