Introduction to K-stability @ Nankai by Yuchen Liu.

Lecture 6. Explicit K-stability II.

Recall. A log Fano pair is (X, D)

X : normal projective var.

D : effective $\mathbb{Q}$-divisor

s.t. $-K_X - D$ is $\mathbb{Q}$-Cartier ample.

Today : $D \sim_{\mathbb{Q}} -rK_X$, X $\mathbb{Q}$-Fano var.

(X, cD) is log Fano , i.e. $0 \leq c < \frac{1}{r}$.

Thm (L.-Zhu, Zhuang).

$\pi : Y \to X = Y/G$

$K_Y = \pi^*(K_X + D)$.

Let Y be a $\mathbb{Q}$-Fano variety.

Assume $G \curvearrowright Y$ (effective), G finite group. $(G \leq \mathrm{Aut}(Y))$.

$Y/G = (X, D)$ log Fano pair.

$$D = \sum \left(1 - \frac{1}{d_i}\right) D_i,$$

$D_i \subseteq X$ prime divisor

d_i = ramification index of D_i

= size of stab_x for $x \in D_i$ general.

Then Y is K-polystable/semistable

iff (X, D) is K-polystable/semistable.

<u>Ex</u>. $\mathbb{Z}/d\mathbb{Z} \curvearrowright \mathbb{P}^1 = Y$.

$$z \longmapsto e^{\frac{2\pi i}{d}} \cdot z.$$

$$\pi : Y \longrightarrow X$$
$$z \mapsto z^d.$$

$$X = \mathbb{P}^1/(\mathbb{Z}/d\mathbb{Z}) \cong \mathbb{P}^1.$$

$$D = \left(1-\tfrac{1}{d}\right)[0] + \left(1-\tfrac{1}{d}\right)[\infty].$$

$\mathbb{P}^1$ K-polystable $\Rightarrow$ $\left(\mathbb{P}^1, \left(1-\frac{1}{d}\right)([0]+[\infty])\right)$ is K-polystable.

Thm (Tian) Every smooth del Pezzo surface of deg 2 is K-stable.

Idea. Suppose X is smooth del Pezzo surface deg 2.

By classical results,

quatic curve

$$\pi = |-K_X| : X \xrightarrow{2:1} (\mathbb{P}^2, \tfrac{1}{2}C_4) \quad \text{log Fano}$$

(In general, if $(-K_X)^2 = d \geq 2$, then $|-K_X|$ is bpf

and $|-K_X| : X \longrightarrow \mathbb{P}^d$ $\begin{cases} \text{embedding if } d \geq 3 \\ \text{covering if } d = 2 \end{cases}$).

$X = Bl_{9-d}\mathbb{P}^2$.

$$K_X = \pi^*(K_{\mathbb{P}^2} + \tfrac{1}{2}C) \Rightarrow 2 = (K_X^2) = 2\cdot(K_{\mathbb{P}^2} + \tfrac{1}{2}C)^2 \Rightarrow \deg C = 4.$$

It suffices to show $(\mathbb{P}^2, \frac{1}{2}C_4)$ is K-polystable

where C_4 is a smooth quartic curve.

<u>Observation</u> Assume $D \sim_{\mathbb{Q}} r(-K_X)$.

$$\beta_{(X,cD)}(E) \text{ is linear in } c.$$

Moreover, if $c = r^{-1}$, i.e. $(X, r^{-1}D)$ is log CY

then $\beta_{(X,r^{-1}D)}(E) = A_{(X,r^{-1}D)}(E)$.

<u>Reason</u>. $\beta_{(X,cD)}(E) = A_{(X,cD)}(E) - S_{(X,cD)}(E)$.

linear in c.

$$A_{(X,cD)}(E) = A_X(E) - \mathrm{ord}_E(cD)$$
$$= A_X(E) - c \cdot \mathrm{ord}_E(D) \qquad \text{linear } \checkmark.$$

$$S_{(X,cD)}(E) = \frac{1}{(-K_X-cD)^n}\int_0^{+\infty} \mathrm{vol}(-K_X-cD-tE)\,dt$$

$$\left(-K_X - cD = (1-cr)(-K_X) \right)$$

$$= (1-cr)^{-n}\frac{1}{(-K_X)^n}\int_0^{+\infty} \mathrm{vol}\left((1-cr)(-K_X) - tE\right)dt$$

$$= \frac{1-cr}{(-K_X)^n}\int_0^{+\infty} \mathrm{vol}\left(-K_X - \frac{t}{1-cr}E\right) d\frac{t}{1-cr}$$

$$= (1-cr)\cdot S_X(E). \qquad \text{linear } \checkmark.$$

Prop. (interpolation). X $\mathbb{Q}$-Fano, $D \sim_{\mathbb{Q}} -rK_X$.

Assume X is K-semistable.

and $(X, \frac{1}{r}D)$ is klt. (lc)

Then $\forall\ c \in (0, \frac{1}{r})$, (X, cD) is K-stable.

(K-semistable).

Pf. Use Fujita-Li repeatedly.

X K-ss $\Longleftrightarrow$ $\beta_X(E) \geq 0$ $\forall E$ div$/X$.

$(X, \frac{1}{r}D)$ klt $\Longleftrightarrow$ $\beta_{(X, \frac{1}{r}D)}(E) = A_{(X, \frac{1}{r}D)}(E) > 0$

$\overset{\beta \text{ linear}}{\Longrightarrow}$ $\beta_{(X, cD)}(E) > 0$ $\forall c \in (0, \frac{1}{r})$ $\forall E$ div$/X$. #

$\Rightarrow (X, cD)$ K-stable

Back to the proof

$(\mathbb{P}^2, \frac{3}{4} C_4)$ is klt as C_4 is smooth.

$\mathbb{P}^2$ is K-polystable.

$\overset{\text{Prop}}{\Longrightarrow}$ $(\mathbb{P}^2, c \cdot C_4)$ is K-stable $\forall\ c \in (0, \frac{3}{4})$.

so $(\mathbb{P}^2, \frac{1}{2} C_4)$ is K-stable

$\overset{\text{Covering}}{\Longrightarrow}$ X is K-polystable #.

Similar method $\Rightarrow$ sm. quartic double solid is K-stable [Dervan].

Next, we'll describe the K-moduli space of del Pezzo surface of deg 2.

$$\left\{\begin{array}{c}\text{smooth del Pezzo surf } X \\ \text{of deg } 2\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\text{smooth } C_4 \subseteq \mathbb{P}^2\right\}.$$

$X \rightsquigarrow (\mathbb{P}^2, \tfrac{1}{2}C_4)$

Q. Is K-moduli M^K of X isomorphic to M^{GIT} of C_4?

A. False! they're birational, but not isomorphic.

○ Odaka-Spotti-Sun gives description of M^K.

Ascher-DeVleming-L. '19: Wall crossing for K-moduli spaces.

Consider K-moduli space of $(\mathbb{P}^2, c\cdot C_4)$, denoted by M_c^K.

Thm 1 (ADL). $M_\varepsilon^K \cong M^{GIT}$.

(holds for $(\mathbb{P}^n$, hypersurface$)$, more generally (X, D) C. Zhou)

K-ps Fano ↓ X

Thm 2 (ADL) $\exists$ finitely many walls $0 < c_1 < c_2 < \cdots < c_\ell < \frac{3}{4}$.
$(\frac{n}{r-1})$

s.t. K-moduli is the same for $c \in (c_i, c_{i+1})$.

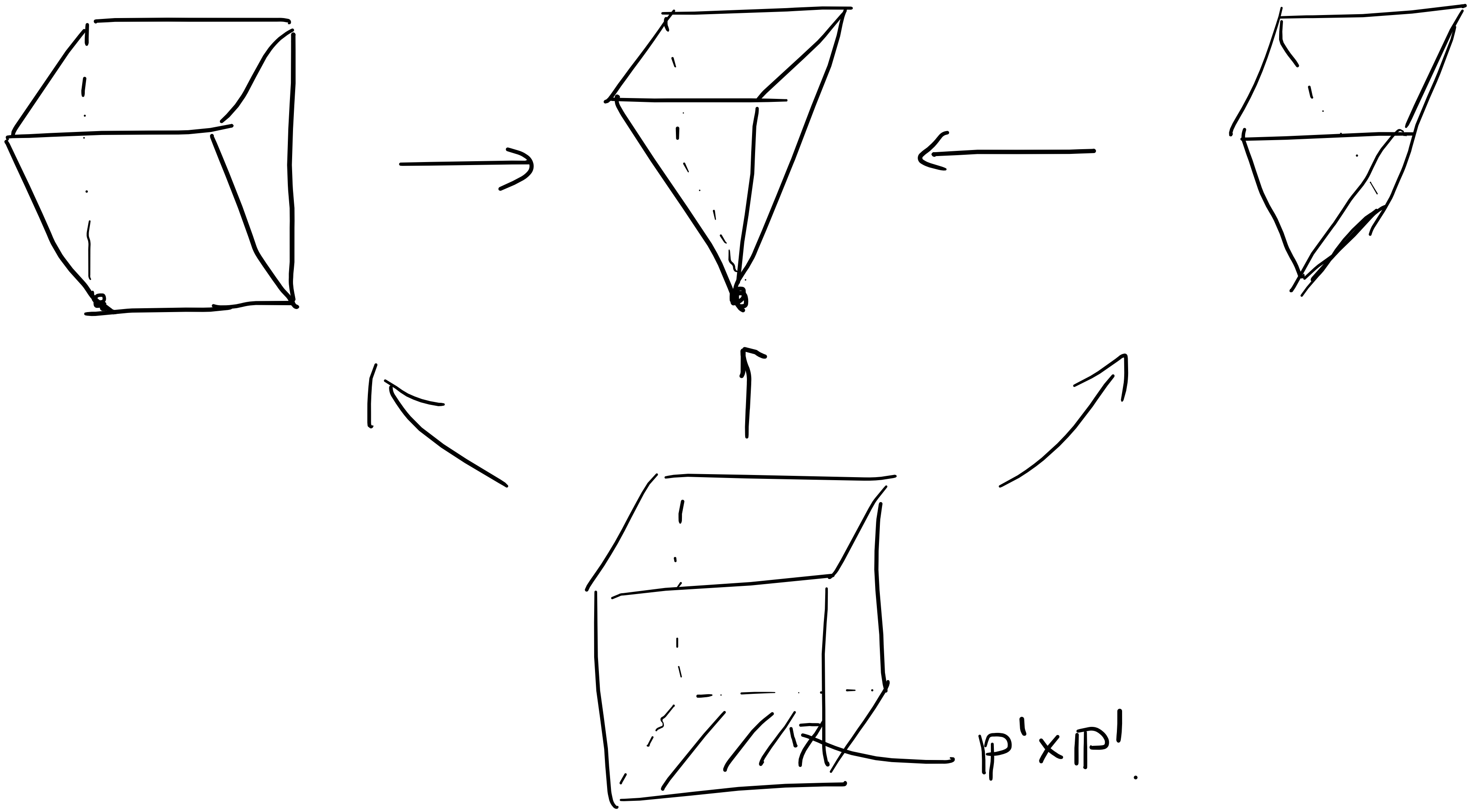
Moreover, $\exists$ wall crossing morphisms $M^K_{c_i-\varepsilon} \to M^K_{c_i} \leftarrow M^K_{c_i+\varepsilon}$

(holds for general (X, D), $D \sim_{\mathbb{Q}} -rK_X$).

flip / flop

$C(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$

$(xy = zw) \subseteq \mathbb{A}^4$

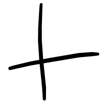


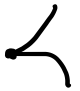
M^K = K-moduli of deg 2 del Pezzo surfaces.

By conj, $M^K \cong M^K_{\frac{1}{2}}$: K-moduli of $(\mathbb{P}^2, \frac{1}{2} C_4)$.

Thm 1 $\Rightarrow$ $M^{GIT} \cong M^K_{\varepsilon}$: K-moduli of $(\mathbb{P}^2, \varepsilon \cdot C_4)$.

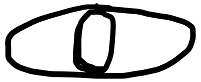
First, understand M^{GIT}.

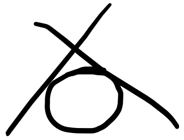

$(xy=0)$


$(x^3=y^2)$

Mumford: C_4 is GIT-stable iff it has only nodes or ordinary cusps

strictly GIT-polystable ($\cong \mathbb{P}^1$) :

 cat-eye 1-dim'l

 ox 1 pt

 double conic. 1 pt

$\mathrm{lct}(X; D) = \sup\{c \mid (X, cD) \text{ is log canonical}\}$.

GIT stable: C_4 has node or cusp.

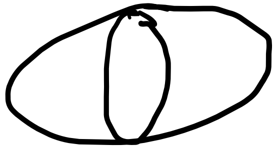
$\mathrm{lct}(\mathbb{P}^2; \text{node}) = 1$

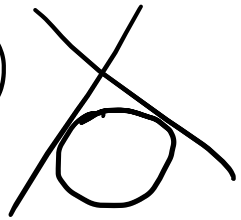
$\mathrm{lct}(\mathbb{P}^2, \text{cusp}) = \frac{5}{6} > \frac{3}{4}$

$\Rightarrow$ GIT stable curves C_4 satisfy $(\mathbb{P}^2, \frac{3}{4}C_4)$ is klt.

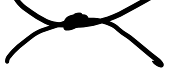
By interpolation, $(\mathbb{P}^2, cC_4)$ is K-stable $\forall c \in (0, \frac{3}{4})$.

strictly GIT polystable.

(1) 

(2) 

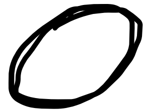
singularities are tacnodal

locally: $x^2 = y^4$. 

$$\mathrm{lct}(\mathbb{P}^2; \text{tacnode}) = \frac{3}{4}. \Rightarrow \left(\mathbb{P}^2, \frac{3}{4} C_4\right) \text{ is lc.}$$

$$\left(\mathrm{lct}(\mathbb{A}^2; \underset{p,q \geq 2}{x^p = y^q}) = \frac{1}{p} + \frac{1}{q} \right).$$

By interpolation, $(\mathbb{P}^2, c \cdot C_4)$ is K-polystable $\forall c \in (0, \frac{3}{4})$.
if C_4 is (1) or (2).

③ C_4 = double conic $= 2Q$  Q = smooth conic

$$\mathrm{lct}(\mathbb{P}^2; 2Q) = \frac{1}{2} < \frac{3}{4}.$$

<u>Prop</u>[Li-Sun]. $(\mathbb{P}^2, c\cdot 2Q)$ is K-semistable $\Longleftrightarrow$ $c \le \frac{3}{8}$.

K-polystable $\Longleftrightarrow$ $c < \frac{3}{8}$.

<u>Pf</u>. We compute β-invariant for $E = Q$.

$$A_{(\mathbb{P}^2,\, c\cdot 2Q)}(Q) = 1 - 2c.$$

$$S_{(\mathbb{P}^2,\, 2cQ)}(Q) = \left(1 - c \cdot \frac{4}{3}\right) S_{\mathbb{P}^2}(Q) = \frac{1}{2} - \frac{2}{3}c.$$

$$S_{\mathbb{P}^2}(Q) = \frac{1}{(-K_{\mathbb{P}^2})^2} \int_0^\infty \mathrm{vol}(-K_{\mathbb{P}^2} - tQ)\, dt$$

$$= \frac{1}{9} \int_0^{\frac{3}{2}} (3-2t)^2\, dt = \frac{1}{2}.$$

K-ss $\Rightarrow$ $A \geq S$, i.e. $1 - 2c \geq \frac{1}{2} - \frac{2}{3}c$.

$$6 - 12c \geq 3 - 4c$$

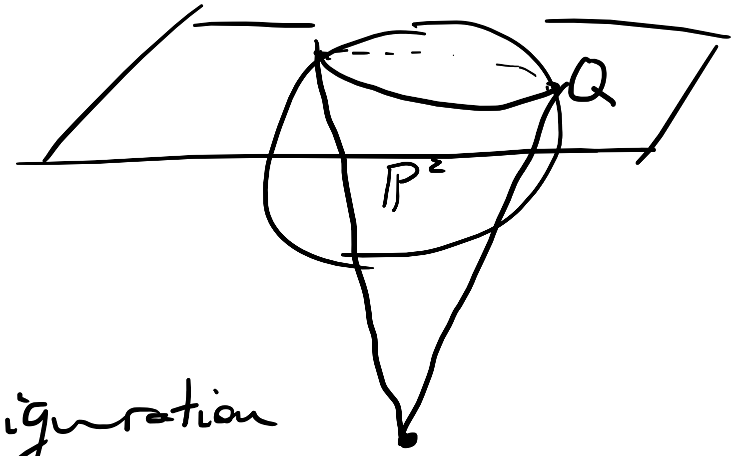
$$8c \leq 3$$

$$c \leq 3/8.$$

Next, we show $(\mathbb{P}^2, \frac{3}{8}\cdot 2Q) = (\mathbb{P}^2, \frac{3}{4}Q)$ is K-ss.

but not K-polystable

$$\mathbb{P}^2 \xrightarrow{\mathcal{O}(2)} \mathbb{P}^5$$



induces a test configuration

toric pair

K-polystable.

$$(\mathbb{P}^2, \tfrac{3}{4}Q) \rightsquigarrow \Big(\mathbb{P}(\overset{x}{1}, \overset{y}{1}, \overset{z}{4}), \tfrac{3}{4}(z=0)\Big).$$

$\mathbb{P}(1,1,4)$ $\|$ cone over rational normal quartic

$G = \mathbb{Z}/4\mathbb{Z}$ act on $\mathbb{P}^2$ $\qquad i = \sqrt{-1}$.

$$[u, v, w] \longmapsto [iu, iv, w]$$

$$\mathbb{P}^2/G \cong \left(\mathbb{P}(1,1,4), \tfrac{3}{4}(z=0)\right) \Longrightarrow \text{K-polystable}$$

#.

At $c_1 = \frac{3}{8}$, $\leftarrow$ first wall.

(weighted blow-up on M^{GIT})

$$\left(\mathbb{P}^2, \left(\tfrac{3}{8}-\varepsilon\right)\cdot 2Q\right) \qquad \left(\mathbb{P}(1,1,4), \left(\tfrac{3}{8}\right)\left(z^2 = f(x,y)\right)\right)$$

$$\left(\overset{x\ y\ z}{\mathbb{P}(1,1,4)}, \tfrac{3}{8}(z^2=0)\right)$$

$\deg f = 8$

f : binary form.

Prop. If f is GIT semistable (stable) as a binary form, then $\left(\mathbb{P}(1,1,4),\ c\cdot\left(\underset{C'}{\underset{\|}{z^2=f(x,y)}}\right)\right)$ is K-ss (K-stable).

pf. f is GIT-ss $\iff$ $\mathrm{ord}_p f \leq \frac{1}{2}\deg f = 4$ for $\forall\ p \in \mathbb{P}^1_{[x,y]}$.

Then $z^2 = f(x,y)$ is a (smooth or singular) hyperelliptic curve

$$\downarrow 2:1$$
$$\mathbb{P}^1$$

branch divisor $=$ $\mathrm{div}(f)$.

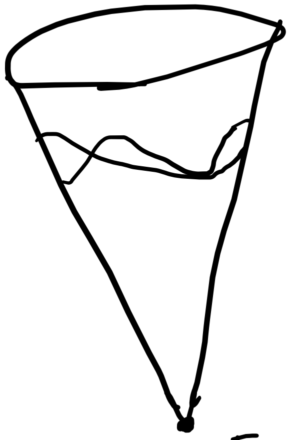
Thus C' has at worst tacnodal singularity.

$$\mathrm{lct}(\mathbb{P}(1,1,4), C') \geq \frac{3}{4}.$$

Hence $\left(\mathbb{P}(1,1,4), \frac{3}{4}C'\right)$ is lc log CY pair.

$$\xRightarrow{\text{interpolation}} \left(\mathbb{P}(1,1,4), cC'\right) \text{ is K-ss } \forall c \in \left(\frac{3}{8}, \frac{3}{4}\right).$$

#



$\mathbb{P}(1,1,4)$

$[0,0,1] \notin C'$.

Look at two affine charts:

$(x=1)$ & $(y=1)$

Thm (ADL) $^{\text{OSS } c=\frac{1}{2}}$ $c_1 = \frac{3}{8}$ unique wall.

If $0 < c < \frac{3}{8}$, then $M_c^K \cong M^{GIT}$.

If $\frac{3}{8} < c < \frac{3}{4}$, then $M_c^K \cong Bl_{[2Q]} M^{GIT}$.

Exceptional divisor parametrize $(\mathbb{P}(1,1,4), (z^2 = f(x,y)))$

as a compactification for moduli of genus 3

hyperelliptic curve.

Cor. K-moduli of deg 2 del Pezzo surface $\cong Bl_{[2Q]} M^{GIT}$.

Future problems.

Some Fano 3folds are double covers.

Ex. quartic double solids $X \xrightarrow{2:1} (\mathbb{P}^3, \frac{1}{2}S_4)$,

ADL'21 : K-moduli for quartic double solids ✓.
using wall crossing.

Q : Family 2.8. double cover of $Bl_{pt}\mathbb{P}^3$.

L. : smooth members are K-stable.

K-moduli ?

- del Pezzo 3-fold of deg 1 : $X \xrightarrow{2:1} \mathbb{P}(1,1,1,2)$.

Explicit Fano, hypersurface, how to compute/estimate δ-invariant?

Abban-Zhuang: adjunction/induction estimate.

Sm. hypersurf. Fano index $\leq \sqrt[3]{\dim} \Rightarrow$ K-stable.

cubic surface: Abban-Zhuang.

some hypersurface with singularity: Zhang-Zhou, L.-Zhuang.