Lecture 3

Algebraic groups and affine GIT

Setting: $k = \bar{k}$, $\operatorname{char} k = 0$. e.g. $k = \mathbb{C}$

algebraic varieties: defined as $Z(I)$ + Zariski top.

- Z: zero locus
- I: (homogeneous) ideal generated by (homog.) polynomials in $k[x_1, \dots, x_n]$

e.g. affine varieties

proj. varieties

quasi-proj. varieties

$\updownarrow$

integral separated scheme of finite type.

Q: Classification problems in algebraic geometry:

$(A, \sim)$ for

- A: collection of objects e.g. varieties, top. spaces, bundles, ...
- $\sim$: equivalence relation e.g. homeomorphisms, iso., ...

find an algebraic variety M to describe $A/\sim$ so that
$\downarrow$
$/k$

$$M(k) \sim A/\sim$$
$$x \longleftrightarrow [a]$$

(where $x \in M(k)$, $[a] \in A/\sim$)

via GIT (geometric invariant theory).

§1. Linear algebraic groups

Def 1.1 An algebraic group G/k is a group that admits a structure of algebraic variety $/k$ s.t.

(1) multiplication $m: G \times G \to G$

$(g, h) \mapsto gh$

(2) inverse $i: G \to G$

$g \mapsto g^{-1}$

are morphisms of alg. varieties.

Def 1.2 A <u>homomorphism</u> of alg. gps $f: G \to G'$ is both a morphism of alg. varieties and a homomorphism of gps.

Prop 1.3 Every algebraic group is non-singular as an alg. variety.

- non-singular at some $g \in G$.
- $\forall h = hg^{-1} \cdot g \rightarrow$ non-singular at h.

E.g. $GL(n, k) := \{ X \in M_{n\times n}(k) : \det X \neq 0 \}$

$$\simeq \{ (g, \lambda) \in M_{n\times n}(k) \times k : \det(g) \cdot \lambda = 1 \} \subseteq \mathbb{A}_k^{n^2+1}$$

closed subvariety. so affine

Def 1.4 A <u>linear algebraic group</u> is an algebraic group that can be identified with a closed subgroup of some $GL(n, k)$.

An <u>affine algebraic group</u> is an algebraic group that is also an affine variety.

Rmk over $k = \bar{k}$, char $k = 0$

linear alg. gps = affine alg. groups.

Ex. (1) $G_m := (k^*, \times) \simeq GL(1, k)$

(2) $G_a := (k, +)$

as $G_a \simeq \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in k \right\} \subset GL(2, k)$

(3) $SL(n, k) \cong GL(n, k) \cap Z(\det - 1)$

(4) $O(n, k)$

$SO(n, k)$

$PGL(n, k)$

...

(5) (Non-linear alg. group) · elliptic curves.

· abelian varieties

<u>Now</u>. G is linear alg. group. / k.

V. finite dimensional vector space

$\rho : G \to GL(V)$ representation

Def 1.5 (1) ρ is irreducible if there is no <u>non-trivial</u> ρ-invariant subspace of V.

· non-trivial: except $\{0\}$. and V.

· ρ-invariant subspace: $W \subseteq V$ subspace s.t. $\rho(g)(W) \subseteq W$ $\forall g \in G$

(2) ρ is semisimple / completely reducible if $\forall$ non-trivial ρ-invariant subspace has ρ-invariant complement.

or equivalently. $\rho = \bigoplus_{i=1}^{l} \rho_i$ for each ρ_i irreducible

Check!

i.e., · $V = \bigoplus_{i=1}^{l} V_i$ $V_i \subset V$ subspace, ρ-inv.

· $\rho_i := \rho|_{V_i} : G \to GL(V_i)$

(3) G is <u>reductive</u> if $\forall$. finite dimensional representation of G

is completely reducible.

$\Updownarrow$ Weyl, Nagata, Mumford

G is reductive if unipotent radical is trivial. i.e. the maximal connected unipotent closed subgroup is trivial.

e.g. $GL(n, \mathbb{k})$. the maximal normal connected closed subgp is $T \subset GL(n, \mathbb{k})$ maximal torus. i.e.

$$\mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

unipotent : $\mathrm{diag}(1, \dots, 1)$

$SL(n, \mathbb{k})$. $PGL(n, \mathbb{k})$

$O(n, \mathbb{k})$. $SO(n, \mathbb{k})$

__But__. G_a is __not__ reductive!

$$G_a \simeq \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{k} \right\} \subset GL(2, \mathbb{k})$$

but the unipotent radical is itself. non-trivial.

or equiv. $\rho : G_a \to GL(\mathbb{k}^2)$

$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle < \mathbb{k}^2$ G_a-inv.

but it has no G_a-inv. complement!

§2. Algebraic group action.

__Def 2.1__ __G-variety__ is an alg. variety $X/\mathbb{k}$ s.t. $G \curvearrowright X$

i.e. $\exists$ a group action $G \times X \to X$ also morphism of varieties

$(g, x) \mapsto gx$

- $g.(h.x) = (gh).x \quad \forall g, h \in G$
- $e.x = x \quad \forall x \in X$

Prop 2.2 X is a G-variety. then for $\forall x \in X$

(1) $G\cdot x \subset X$ is smooth locally closed. i.e. $G\cdot x \subset \overline{G\cdot x}$ open

(2) $G\cdot x$ is equi-dimensional. of $\dim(G\cdot x) = \dim(G) - \dim(G_x)$

for $G_x := \{ g \in G : gx = x \}$

(3) $\overline{G\cdot x} \setminus G\cdot x$ is the union of some closed orbits of $< \dim(G\cdot x)$

$\Rightarrow$ any orbit of minimal dimension is closed.

$\forall$ G-inv. $Y \subseteq X$ contains a closed orbit. (In particular. $\overline{G\cdot x}$ contains a closed orbit.)

Pf. **Lem 2.3 (Chevalley)** $\varphi : X \to Y$ regular map of varieties $/k$.

$\Rightarrow \forall\, U \subset X$ constructible, its image $\varphi(U)$ is constructible.

↑ finite union of locally closed subsets.

In particular, $\varphi(X)$ is constructible

(1) $\sigma_x : G \to X$, $g \mapsto g\cdot x$ $\Rightarrow$ $G\cdot x = \mathrm{Im}(\sigma_x)$ $\Rightarrow$ constructible

$\Rightarrow U \subset G\cdot x \subset \overline{G\cdot x}$ for $U \subset \overline{G\cdot x}$ open.

$G \curvearrowright \overline{G\cdot x}$ transitively $\forall\, g \in G\cdot x$ can be covered by moving U.

$\Rightarrow$ $G\cdot x$ locally closed

(2) $\sigma_x : G \to G\cdot x$ flat with the fiber at x is $\sigma_x^{-1}(x) = G_x$

$\Rightarrow \dim G = \dim(G\cdot x) + \dim(G_x)$

(3) $\overline{G\cdot x} \setminus G\cdot x \subset \overline{G\cdot x}$ closed by (1)

$G \curvearrowright \overline{G\cdot x} \setminus G\cdot x \quad \Rightarrow \quad \overline{G\cdot x}\setminus G\cdot x = \bigcup_{y \in U} G\cdot y$ some $U \subset G$

∎

§3. Quotients and affine GIT.

X G-variety. G linear alg. group. $/ \mathbb{k}$

affine

$$G \times X \to X$$
$$(g, x) \mapsto g\cdot x$$

⇝ want "quotient".

(1) "stronger sense": $X/G := \{ G\cdot x : x \in X\}$ has the str. of alg. var.

at least $\forall\, G\cdot x \subset G$ closed

(2) "weaker sense": allow $\exists$ non-closed orbits.. ⇒ we already know closed orbits alway exist! Need a method to identify non-closed orbits with non-empty intersection of their closures.

Def 3.1 A <u>categorical quotient</u> of $G \curvearrowright X$ is (Y, φ) s.t.

$\varphi : X \to Y$ <u>G-invariant</u> morphism of varieties, which is universal

i.e. $\forall$ G-invariant morphism $f: X \to Z$ uniquely factors through φ.

$G \curvearrowright X$
$\varphi(g\cdot x) = \varphi(x) \quad \forall g \in G$

$$\begin{array}{ccc} & & Y \\ & \overset{\varphi}{\nearrow} & \circlearrowright \ \downarrow \exists! g \\ X & \xrightarrow{f} & Z \end{array}$$

Rmk 1) Not necessarily surjective (for general alg. var. see. arXiv: 9806049)

2) categorical quotient if exist. is unique up to isomorphism (check!)

3). $\overline{G\cdot x_1} \cap \overline{G\cdot x_2} \neq \phi \Rightarrow \varphi(G\cdot x_1) = \varphi(G\cdot x_2)$

$$\begin{matrix} \varphi(G\cdot x_1) & = & \varphi(G\cdot x_2) \\ \| & & \| \\ \varphi(x_1) & & \varphi(x_2) \\ \| & & \| \\ \varphi(\overline{G\cdot x_1}) & = & \varphi(\overline{G\cdot x_2}) \end{matrix}$$

· $G\cdot x_1$, $G\cdot x_2$ closed $G\cdot x_1 \cap G\cdot x_2 = \phi \Rightarrow \varphi(G\cdot x_1) \neq \varphi(G\cdot x_2)$ (check!)

$k[X]$ coordinate ring $= k[x_1, \dots, x_n]/I(X) \simeq \mathcal{O}_X(X)$

finitely generated k-alg.

$G \curvearrowright X \rightsquigarrow G \curvearrowright k[X]$ via

$f \in k[X]$, define

$$g\cdot f := f(g^{-1}\cdot)$$
$$\overset{!!}{} f^g$$

define $k[X]^G := \{ f \in k[X] : f^g = f \quad \forall g \in G \}$

Def 3.2 A good quotient of $G \curvearrowright X$ is G-invariant morphism of varieties $\varphi: X \to Y$ s.t.

(1) φ surjective & affine

(2) $\forall U \subset Y$ open, the pull-back

$$\varphi^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\varphi^{-1}(U))$$
$$f \longmapsto f\circ\varphi$$

(φ is G-inv. $\Rightarrow f\circ\varphi$ is G-inv. i.e. $Im(\varphi^*) \subset \mathcal{O}_X(\varphi^{-1}(U))^G$)

induces an isomorphism $\mathcal{O}_Y(U) \xrightarrow{\simeq} \mathcal{O}_X(\varphi^{-1}(U))^G$.

(3) $\forall W \subset X$ G-invariant closed $\Rightarrow$ $\varphi(W) \subset Y$ closed.

(4) $\forall W_1, W_2 \subset X$. G-inv. & closed s.t. $W_1 \cap W_2 = \phi$

$\Rightarrow \varphi(W_1) \cap \varphi(W_2) = \phi$.

Def 3.3 A geometric quotient of $G \curvearrowright X$ is a good quotient $\varphi : X \to Y$ which is also an orbit space.

$\uparrow$ $\forall y \in Y$ $\varphi^{-1}(y)$ is a single closed orbit.

The following property says good / geom. quotients are local w.r.t. the base.

Prop 3.4 (1) $\varphi: X \to Y$ good (resp. geometric) quotient of $G \curvearrowright X$

$\Rightarrow \forall U \subset Y$ open. $\varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \to U$ is good (resp. geometric) quotient for $G \curvearrowright \varphi^{-1}(U)$.

(2) $\varphi: X \to Y$ G-inv. morphism. if $\exists Y = \bigcup_i U_i$ U_i open covering of Y s.t. $\varphi|_{\varphi^{-1}(U_i)} : \varphi^{-1}(U_i) \to U_i$ is good (resp. geom.) quotient $\forall i$

$\Rightarrow \varphi: X \to Y$ is a good (resp. geom.) quotient.

Prop 3.5 Geometric quotient $\Rightarrow$ good quotient $\Rightarrow$ categorical quotient

$\not\Leftarrow$ $\not\Leftarrow$

$\uparrow$ very hard to construct.

pf (sketch) good $\Rightarrow$ categorical :

$\forall$ G-invariant morphism $f: X \to Z$ uniquely factors through $\varphi: X \to Y$

define set-theoretic map $g: Y \to Z$, $y \mapsto f(x)$

$$\begin{array}{ccc} & & Y \\ & \varphi \nearrow & \downarrow \exists! g \\ f: X & \to & Z \end{array}$$

x is any element in $\varphi^{-1}(y)$

Show: • well-defined.

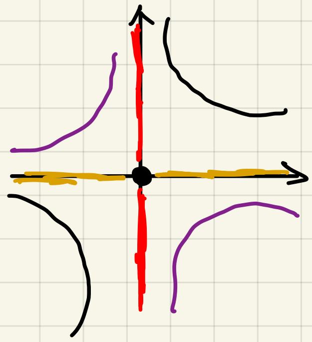
• g is a morphism of varieties.:

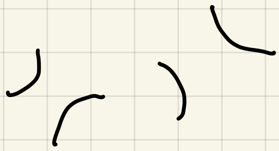
- continuous under Zariski top.
- $\forall\, V \subset Z$ affine open. $g|_{g^{-1}(V)} : g^{-1}(V) \to V$ is a morphism.

□

Ex 1) $\mathbb{G}_m \curvearrowright \mathbb{A}^2_k = \mathrm{Spec}(k[x,y])$

$t.(x,y) := (t^{-1}x,\ ty)$

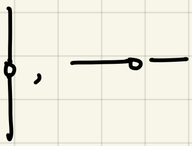




closed • closed

• $\subseteq$ 

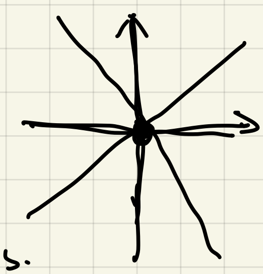
$\subseteq$

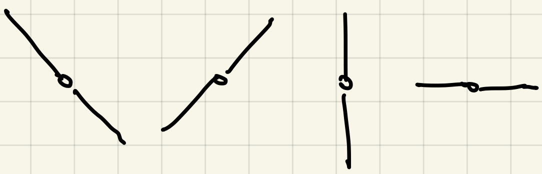
⇝ orbits  , • will be identified.

$\Rightarrow\ \varphi : \mathbb{A}^2_k \to \mathbb{A}^1_k\ (= \mathrm{Spec}\, k[xy])$ good quotient

$(x,y) \mapsto xy$ not geometric quotient

2) $\mathbb{G}_m \curvearrowright \mathbb{A}^2_k$ via $t.(x,y) = (tx, ty)$





non-closed orbits.

• closed $\subseteq$ closure of all orbits $\Rightarrow$ all orbits will be identified.

$\Rightarrow\ \varphi : \mathbb{A}^2_k \to pt$

3) $\mathbb{G}_m \curvearrowright \mathbb{A}^2_k \setminus \{0\}$. via $t.(x,y) := (tx, ty)$

⇝ all orbits are closed

⇝ geom. quotient $\varphi : \mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k$.

$G \curvearrowright X \rightsquigarrow G \curvearrowright k[X] \rightsquigarrow k[X]^G$

Q: Is $k[X]^G$ finitely generated?

Thm 3.6 (Nagata) G reductive $\Rightarrow$ $k[X]^G$ is finitely generated k-alg.

Def 3.7 The affine GIT quotient is $\varphi: X \to X/\!/G := \mathrm{Spec}(k[X]^G)$ induced from $k[X]^G \hookrightarrow k[X]$

Thm 3.8 $\varphi: X \to X/\!/G$ is good quotient. Hence categorical quotient.

Pf. Denote $Y := X/\!/G$

(1) $\varphi: X \to Y$ is G-invariant:

if not. $\exists\, g \in G$ s.t. $\varphi(g\cdot x) \neq \varphi(x)$

$\Rightarrow \exists\, f \in \mathcal{O}_Y(Y)$ s.t. $f(\varphi(gx)) \neq f(\varphi(x))$

i.e. $(\varphi^* f)(gx) \neq (\varphi^* f)(x)$.

contradicts to $\varphi^*(\mathcal{O}_Y(Y)) \subset \mathcal{O}_X(X)^G$.

(2) $\forall\, W_1, W_2 \subset X$ closed disjoint G-inv. subsets.

then $\overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \phi$

[closed disjoint G-invariant subsets are distinguished by G-invariant functions,
i.e. $\exists\, f \in k[X]^G$. s.t. $f(W_1) = 1$. $f(W_2) = 0$.

$k[X]^G \simeq k[Y]$ view. $f \in k[Y] \Rightarrow f(\varphi(W_1)) = 1$

$f(\varphi(W_2)) = 0$

$\Rightarrow \varphi(W_1) \cap \varphi(W_2) = \phi$

(3) $\forall\, W \subset X$ closed G-invariant. $\Rightarrow \varphi(W) \subset Y$ closed.

if not. $\exists\, y \in \overline{\varphi(W)} \setminus \varphi(W)$. $W_1 := W$ $W_2 := \varphi^{-1}(y)$

$\Rightarrow W_1, W_2$ are closed, disjoint, G-inv.

$\overset{(2)}{\Rightarrow}$ $\overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \phi$ i.e. $\overline{\varphi(W_1)} \cap \{y\} = \phi$ ⚡

▨

$G \curvearrowright X \rightsquigarrow \varphi : X \to X/\!/G = \mathrm{Spec}(\mathbb{k}[X]^G)$.

Q: What do the $\mathbb{k}$-pts $X/\!/G(\mathbb{k})$ represent?

Def 3.9 (1) $x \in X$ is <u>polystable</u> if $G \cdot x \subset X$ is closed.

(2) $x \in X$ is <u>stable</u> if
- $G \cdot x \subset X$ closed
- $\dim(G_x) = 0$

(3) $x_1, x_2 \in X$ are called <u>S-equivalent</u> if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi$ denoted as $x_1 \sim_S x_2$.

↗ Seshadri

X^{ps} (resp. X^s) $\subset X$: sets of polystable (resp. stable) points.

Thm 3.10

(1) $X^s \subset X$ open & G-invariant.

(2) $\varphi(X^s) \subset X/\!/G$ open & $\varphi^{-1}(\varphi(X^s)) = X^s$

(3) $\varphi|_{X^s} : X^s \to \varphi(X^s)$ geometric quotient.

Thm 3.11 (Hilbert-Mumford criterion)

$x \in X$. let $O_x \subset \varphi^{-1}(\varphi(x))$ the unique closed orbit.

$\Rightarrow \exists$ 1-PS $\lambda : \mathbb{G}_m \to G$ s.t.

$\lim_{t \to 0} \lambda(t) \cdot x$ exists $\in O_x$

1-PS: 1-parameter subgroup is a group homomorphism $\lambda : \mathbb{G}_m \to G$

In particular.

$x \in X^s \Leftrightarrow \lim_{t\to 0} \lambda(t)\cdot x$ does not exist for $\forall$ non-trivial 1-PS $\lambda: G_m \to G$

pf:

"$\Rightarrow$": if $\exists$ non-trivial $\lambda: G_m \to G$ s.t. $\lim_{t\to 0} \lambda(t)\cdot x =: y$ exists,

$\Rightarrow y \in \overline{G\cdot x} = G\cdot x \Rightarrow y \in X^s$ since $x \in X^s$

but G_y contains $\lambda(G_m)$ $\rightsquigarrow$ $\dim(G_y) > 0$

contradicts to $y \in X^s$.

"$\Leftarrow$" if $\lim_{t\to 0} \lambda(t)\cdot x$ does not exist for $\forall$ non-trivial 1-PS,

$\Rightarrow G\cdot x$ closed.

(if not. $O_x \subset \overline{G\cdot x} \setminus G\cdot x$ by $\lim_{t\to 0} \lambda(t)\cdot x' \in O_x$ for trivial λ

$\Rightarrow g\cdot x \in O_x$ for some $g \in G$.)

G_x finite: otherwise it contains a non-trivial 1-PS. impossible.

$\boxtimes$

Cor 3.12 The following sets of k-pts are 1:1 correspondence (bijective)

$$X /\!/ G(k) \simeq X^{ps}(k)/G(k) \simeq X(k)/\sim_S$$

pf:

(1) $\forall$ $\overline{G\cdot x}$ contains a unique closed orbit.

(2) $\forall x_1, x_2$. $\varphi(x_1) = \varphi(x_2) \Leftrightarrow \overline{G\cdot x_1} \cap \overline{G\cdot x_2} \neq \emptyset$

$\Leftrightarrow x_1 \sim_S x_2$

$\boxtimes$