Introduction to K-stability @ Nankai by Yuchen Liu.

Lecture 3. K-moduli theory I.

§ Motivation

Moduli space: classifying space of algebraic objects (varieties, vector bundles, ...) of certain kind.

Ex. (moduli of curves).

$g \geq 2$: genus of curve.

M_g = moduli space of genus g (smooth, proj.) curves.

g : topological invariant of the curve.

Every pt on M_g represents an isomorphic class of algebraic/complex structure on a genus g top. surface.

M_g is a quasi-projective variety, of dim $= 3g-3$,

M_g only has quotient singularities.

Defect: M_g is not compact/proper.

* In AG, we'd like to compactify moduli spaces.

- Even if you care about smooth objects only, singular objects are helpful.

- Intersection theory on moduli spaces is the foundation for many areas (Enumerative geometry, arithmetic geometry).

M_g has a canonical choice of compactification

Deligne-Mumford compactification $\overline{M_g}$.

Every pt on $\overline{M_g}$ represents a <u>stable</u> curve.

<u>Def</u>. C is a stable curve if

1. C is reduced connected scheme of f.t. $/\mathbb{C}$, pure dim 1.
2. C has only nodal singularities $\{xy=0\} \subseteq \mathbb{C}^2$.
3. K_C is ample.

Thm (Deligne-Mumford)

$\overline{M}_g$ is a projective variety of dim $3g-3$,
it has quotient singularities.

(As a stack, $\overline{\mathcal{M}}_g$ is smooth.

$\boxed{\overline{\mathcal{M}}_g \xrightarrow{\text{coarse moduli space}} \overline{M}_g}$

But certain curves in M_g or $\overline{M}_g$ has non-trivial Aut.
$\hookrightarrow$ singularities in M_g or $\overline{M}_g$).

Moduli functor $\mathcal{M}_g : \mathrm{Sch} \longrightarrow \mathrm{Groupoid}$

$$B \longmapsto \left\{ \mathcal{C} \to B \;\middle|\; \begin{array}{l}\text{smooth curve fibration} \\ \forall \text{fiber has genus } g\end{array} \right\}$$

$\mathcal{M}_g(B) = \mathrm{Hom}(B, \mathcal{M}_g)$.

① Abstract approach

First show $\overline{\mathcal{M}}_g$ is a (Artin) stack with certain properties.

- boundedness
- Openness. (locally closedness)

Then show $\overline{\mathcal{M}}_g$ is separated (Hausdorff) and proper (compact) using valuative criterion.

E.g. (properness) $\mathcal{C}^\circ \to B^\circ$ family of stable curves / punctured curve

then $\exists$ extension (after base change) $\mathcal{C} \to B$ stable curve fibration.

Finally, show projectivity.

② Explicit approach. $C \xrightarrow{|mK_C|} \mathbb{P}^{N_m}$

Embed every stable curve into a fixed $\mathbb{P}^N$ (boundedness)

Take Hilbert scheme $\mathcal{H}$ of $\mathbb{P}^N$, which is projective.

Next use GIT to study $\mathcal{H} /\!\!/ PGL(N+1)$.

Then show GIT quotient is independent of m.

For higher dim varieties, ① is better than ②

E.g. Even for alg. surfaces, ② may not stabilize.

K-moduli theory for Fano varieties

parametrize K-semistable / K-polystable Fano varieties.

Firstly, we want to show boundedness.

Boundedness: A class of varieties $\{X\}$ is bounded

if $\exists$ uniform N, d s.t. $X \hookrightarrow \mathbb{P}^N$

and its degree in $\mathbb{P}^N \leq d$.

Roughly, boundedness is saying finiteness of topological types.

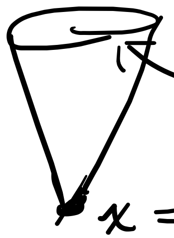
A bounded family has finitely many volumes.

Classically. For smooth Fano manifolds.

- del Pezzo surfaces : 10 families
- Fano 3-folds : 105 families.
- All Fano manifolds in any fixed dimension n are bounded (Kollár-Miyaoka-Mori, Campana).

singular. may not bounded

Ex. $\mathbb{P}(1,1,d)$ $\mathbb{Q}$-Fano var.



deg d RNC in $\mathbb{P}^d$.

$x = [0,0,1]$.

$\mathcal{O}_x$ has embedding dimension $= d+1 . \leq N$. if $\mathbb{P}(1,1,d) \hookrightarrow \mathbb{P}^N$.

$$-K_{\mathbb{P}(1,1,d)} = \mathcal{O}(d+2).$$

$$\left(-K_{\mathbb{P}(1,1,d)}\right)^2 = \mathcal{O}(d+2)^2 = \frac{(d+2)^2}{d} \longrightarrow \infty \quad \text{as } d \to \infty.$$

- <u>Hacon-McKernan-Xu</u>.

  X $\mathbb{Q}$-Fano variety, its Gorenstein index = minimum $r \in \mathbb{N}$ s.t. rK_X is Cartier.

  All $\mathbb{Q}$-Fano var. of dim n and Gorenstein index $\leq r_0$ all bounded.

- Birkar: Fix $\varepsilon > 0$

All $\mathbb{Q}$-Fano variety X with $\mathrm{mld}(X) \geq \varepsilon$ of dim $= n$ are bounded.

$$\mathrm{mld}(X) = \min\{A_X(E) \mid E \text{ divisor over } X\}.$$

For $X = \mathbb{P}(1,1,d)$,

$$\text{Gor. index} = \begin{cases} d/2 & \text{if } d \text{ even} \\ d & \text{if } d \text{ odd.} \end{cases}$$

$$\mathrm{mld}(X) = \frac{2}{d}.$$

Theorem 1 (Jiang) Fix $n \in \mathbb{N}$, $V_0 \in \mathbb{Q}_{>0}$

All K-semistable $\mathbb{Q}$-Fano variety X with $\dim X = n$ and $(-K_X)^n \geq V_0$ are bounded.

K-ss $\iff$ $\delta \geq 1$.

Theorem 2 (Jiang). n, V_0, $\alpha_0 \in \mathbb{Q}_{>0}$

All $\mathbb{Q}$-Fano var. X with $\dim X = n$, $(-K_X)^n \geq V_0$, $\alpha(X) \geq \alpha_0$ are bounded.

Ex. $\alpha(\mathbb{P}(1,1,d)) = \dfrac{1}{2+d} \to 0$ as $d \to \infty$.

<u>First pf</u>. (Jiang) use Birkar's techniques in BAB conj.

<u>Second pf</u> (Xu-Zhuang) If $\alpha(X) \geq \alpha_0$, $(-K_X)^n \geq v_0$,

then Gorenstein index has an upper bound

depending only on α_0, v_0, & n. use [L.'18] for local

to global volume comparison. $\xRightarrow{HMX}$ bounded.

<u>Ex</u>. [Johnson - Kollár] $\exists$ a series of weighted hypersurfaces

((K-~~stable~~) X_i s.t. $\alpha(X_i) = 1$, $\mathrm{vol}(X_i) \to 0$.

$\dim(X_i) = 2$

§ Openness

By boundedness, all K-ss $\mathbb{Q}$-Fano X of $\dim = n$ & $\mathrm{vol} = V$ are bounded, i.e. $\exists\, m = m(n, V)$ s.t.

$$|-mK_X| \text{ is very ample}, \qquad X \hookrightarrow \mathbb{P}^N$$

To construct moduli space, we want to pick up a Zariski open subset $\mathcal{U} \subseteq \mathrm{Hilb}(\mathbb{P}^N)$ s.t. $\mathcal{U}$ parametrizes all K-ss Fano.

Then study $\mathcal{U}/PGL(N+1)$.

Thm (Blum-L.-Xu)

In a family of $\mathbb{Q}$-Fano varieties $\mathcal{X} \to T$, the locus $\{t \in T \mid \mathcal{X}_t \text{ is K-semistable (K-stable)}\}$ is a Zariski open subset of T.

· It suffices to show:

① $t \longmapsto \delta(\mathcal{X}_t)$ is lower semicontinuous.

② $t \longmapsto \delta(\mathcal{X}_t)$ only takes finitely many values.

Thm. Every <u>general</u> Fano hypersurface $(f=0)$ in $\mathbb{P}^{n+1}$ of $\overset{d}{\overset{\shortparallel}{\deg}} \geq 3$ is K-stable. $(d \leq n+1)$.

$f \in H^0(\mathbb{P}^{n+1}, \mathcal{O}(d))$

Pf. It suffices to show $\exists$ one K-stable Fano hypersurface for $\forall$ $d \geq 3$.

Tian: Fermat hypersurface $\overset{X}{\overset{\shortparallel}{x_0^d + x_1^d + \cdots + x_{n+1}^d = 0}}$ is K-stable.

Automorphism, covering. $X \xrightarrow{d:1} \left(\mathbb{P}^n, \frac{d-1}{d} \cdot D\right)$

$D = (x_1^d + \cdots + x_{n+1}^d) \subseteq \mathbb{P}^n$.

① [Blum-L.]

δ-invariant is a generalization of lct.

lower semicontinuity comes from lct.

Use filtration space on flag variety is proper.

Transform δ to invariants of filtrations.

$\Rightarrow \{t \in T \mid \delta(\mathcal{X}_t) > 1\}$ is Zariski open.

uniform K-stable.

(2) $\mathcal{X} \longrightarrow T$ family of $\mathbb{Q}$-Fano var.

$\{ \delta(\mathcal{X}_t) \mid t \in T \}$ is a finite set, assuming every fiber has $\delta \leq 1$.

We use Birkar's boundedness of complements.

<u>Def</u>. X $\mathbb{Q}$-Fano variety.

Δ is a ($\mathbb{Q}$-)complement of X

if $\Delta \geq 0$ is a $\overset{\mathbb{Q}\text{-Cart!}}{\mathbb{Q}}$-divisor

$K_X + \Delta \sim_{\mathbb{Q}} 0$

and (X, Δ) is log canonical.

Δ is an N-complement

if in addition, $N(K_X + \Delta) \sim 0$.

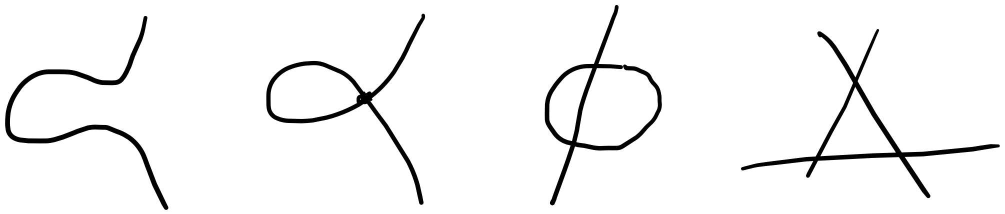
<u>Ex</u>. $X = \mathbb{P}^2$, classify 1-complements, i.e.

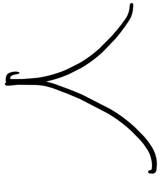
$K_X + \Delta \sim 0$, $\Delta^{\geq 0}$ is a $\mathbb{Z}$-divisor.

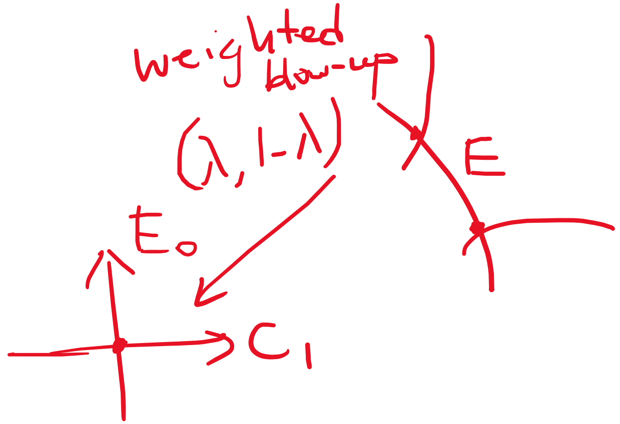
$K_X = \mathcal{O}(-3) \Rightarrow \Delta$ is a cubic curve.

(X, Δ) is log canonical. $\Rightarrow \Delta$ has nodal singularities

Δ :



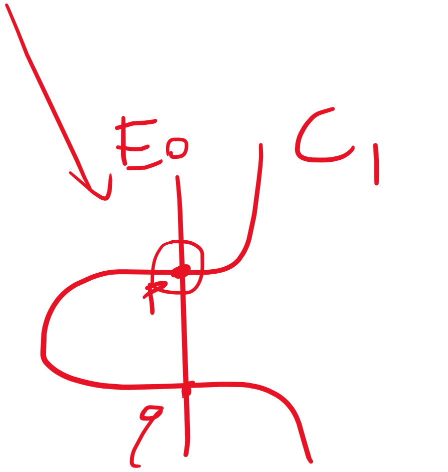
Δ cannot be  $(y^2 = x^3)$



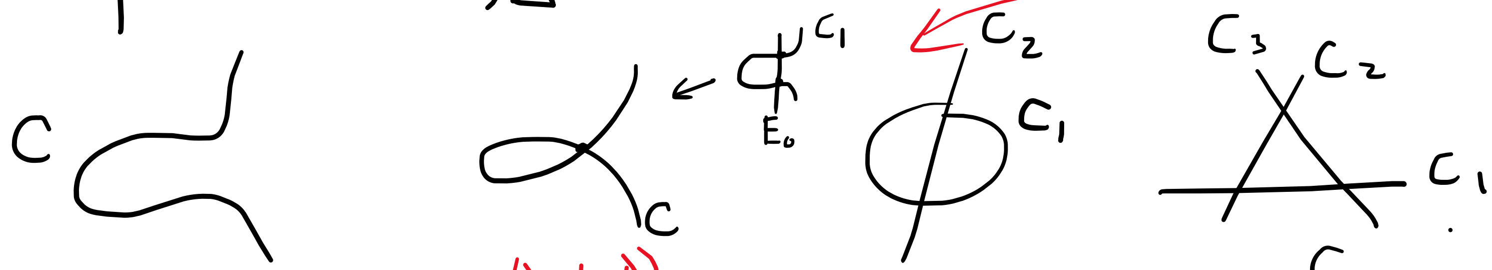
$$\mathrm{lct}(\mathbb{P}^2, (y^2 = x^3)) = \frac{5}{6} < 1.$$

Def. E is a prime divisor over X.

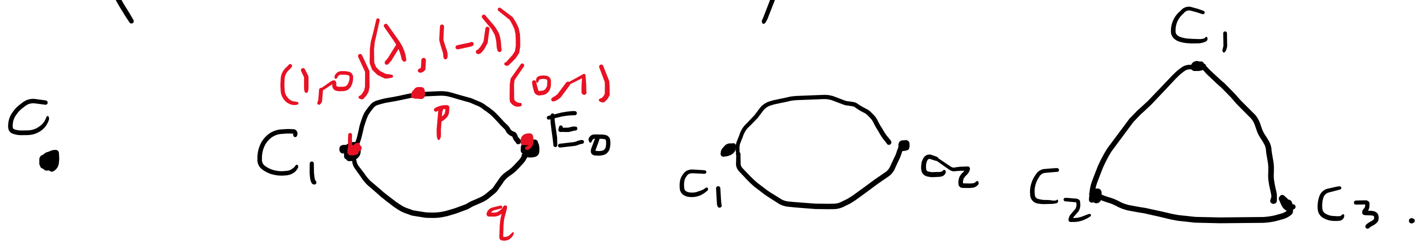
We say E is a lc place of (X, Δ) if $A_{X,\Delta}(E) = 0$.



Complements:



lc places:



Recall : $$\delta(X) = \inf_{E \text{ div}/X} \frac{A_X(E)}{S_{\cancel{X}}(E)}.$$

<u>Step 1</u>. $$\delta(\mathcal{X}_t) = \inf_{\substack{E \text{ lc place} \\ \text{of } \mathbb{Q}\text{-comp.}}} \frac{A_X(E)}{S_X(E)}.$$

<u>Step 2</u>. Use Birkar's bddness of complements

$\exists N$, s.t. $$\delta(\mathcal{X}_t) = \inf_{\substack{E \text{ lc place} \\ \text{of } N\text{-comp } \Delta_t}} \frac{A_X(E)}{S_X(E)}.$$

in particular, Δ_t is bounded.

<u>step 3</u>. By boundedness of Δ_t,

analyze functions $(t, E) \longmapsto A_{\mathcal{X}_t}(E)$ piecewise linear

$\longmapsto S_{\mathcal{X}_t}(E)$ (concave) continuous.

$E \in LCP(\mathcal{X}_t, \Delta_t)$: finite Δ-complex.

Show only finitely many functions show up as t varies T.

(S : invariant in each stratum uses invariance of logplurigenera)

$$\begin{array}{c} (X, \Delta) \\ \uparrow \mu \\ Y \end{array}$$

(X, Δ) : log canonical pair.

Y log resolution.

$$K_Y + \Delta_Y = \mu^*(K_X + \Delta)$$

$$\mathrm{LCP}(X, \Delta) = \underline{\underline{\text{dual complex}}} \text{ of } \Delta_Y^{=1}.$$

divisor $\longmapsto$ pt

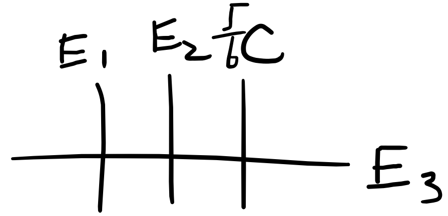
codim 2 $\longmapsto$ edge

codim 3 $\longmapsto$ face

$\vdots$

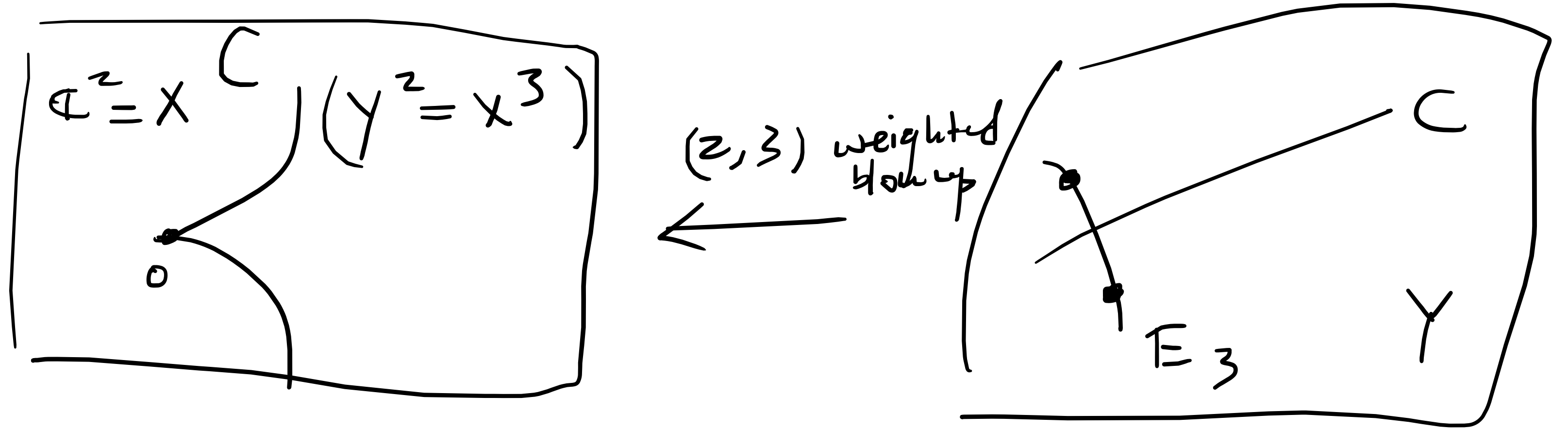
<u>Ex</u>. $X = \mathbb{P}^2$, $\Delta = \frac{5}{6}(y^2 = x^3)$

Y = 3 blow-ups of X.



$\mathrm{LCP}(X, \Delta) = \{E_3\}$.

$\mathrm{Supp}(\Delta_Y) = E_1 \cup E_2 \cup E_3 \cup C$, $\Delta_Y^{=1} = E_3$.



$$E_3 \cong \mathbb{P}(2,3)$$

$$= \left(\mathbb{P}^1, \tfrac{1}{2}[0] + \tfrac{2}{3}[\infty]\right)$$

$v = \mathrm{ord}_{E_3}$ valuation on $\mathbb{C}^2$.

$$v(x) = 2, \quad v(y) = 3.$$

$$f = \sum c_{ij} x^i y^j, \qquad v(f) = \min\{2i + 3j \mid c_{ij} \neq 0\}.$$

$$Y = \mathrm{Proj}_{\mathbb{C}^2} \bigoplus_{m=0}^{\infty} I_m, \qquad I_m = \{f \in \mathcal{O}_{\mathbb{C}^2,0} \mid v(f) \geq m\}.$$