Recall: · Strong maximum principle

· Apply maximum principle to establish some a priori estimates

key: construct suitable auxiliary function.

e.g. $w = \gamma_1 w_1 + \gamma_2 w_2$. $w_1 = R^{-\beta}(R^2-|x|^2)^\beta$ $w_2 = R^\beta(e^A - e^{\frac{Ax_1}{d}})$.

Lecture 5. Poisson's equation (I)

Ω: domain in $\mathbb{R}^n$. $\Delta u = f$ in Ω is called Poisson's equation.

Observation: If $u \in C^2(\Omega)$, then $\Delta u \in C(\Omega)$

Question: Given $f \in C(\Omega)$, is there a C^2-solution of Poisson's equation?

Answer: No. $\exists f \in C(\Omega)$ s.t. $\Delta u = f$ does not admit any C^2-solution.

$\Rightarrow$ For Poisson's equation $\Delta u = f$. $C(\Omega)$ is not a "good" space.

2. Hölder spaces

Ω: bounded domain in $\mathbb{R}^n$. $\alpha \in (0,1)$.

f: a function defined in Ω.

Definition. (1) For any $\Omega' \subset\subset \Omega$. $\exists C_{\Omega'}$ (may depend on Ω')

s.t. $|f(x) - f(y)| \le C_{\Omega'}|x-y|^\alpha$ $\forall x, y \in \Omega'$.

Then f is called locally Hölder continuous with respect to α in Ω. Denote the set of all such functions by $C^\alpha(\Omega)$

(2) $\exists C$ s.t.

$|f(x) - f(y)| \le C|x-y|^\alpha$ $\forall x, y \in \Omega$.

Then f is called uniformly Hölder continuous with respect to α in Ω. Denote the set of all such functions by $C^\alpha(\bar{\Omega})$.

Remark. · If $f \in C^\alpha(\Omega)$, then $f \in C(\Omega)$

· If $f \in C^\alpha(\bar{\Omega})$, then f can be extended continuously to $\bar{\Omega}$.

Definition. (1) For $f \in C^\alpha(\bar\Omega)$. define its Hölder semi-norm by

$$[f]_{C^\alpha(\bar\Omega)} = [f]_{C^\alpha(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} = \sup_{\substack{x,y\in\bar\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

(2). For $k \in \mathbb{Z}_{\geq 0}$ define

$$C^{k,\alpha}(\Omega) = \{ u \in C^k(\Omega) \mid \nabla^k u \in C^\alpha(\Omega) \}$$

$$C^{k,\alpha}(\bar\Omega) = \{ u \in C^k(\bar\Omega) \mid \nabla^k u \in C^\alpha(\bar\Omega) \}$$

For $u \in C^{k,\alpha}(\bar\Omega)$. define

$$\|u\|_{C^k(\Omega)} = \sum_{i=0}^{k} \sup_\Omega |\nabla^i u| .$$

$$\|u\|_{C^{k,\alpha}(\Omega)} = \|u\|_{C^k(\Omega)} + [\nabla^k u]_{C^\alpha(\Omega)}$$

Remark. $(C^k(\bar\Omega), \|\cdot\|_{C^k(\Omega)})$ and $(C^{k,\alpha}(\bar\Omega), \|\cdot\|_{C^{k,\alpha}(\Omega)})$

are Banach spaces.

3. Newtonian Potential.

Ω: bounded domain in $\mathbb{R}^n$. $f \in L^\infty(\Omega)$

Definition. The Newtonian potential of f in Ω is defined by

$$W_{\Omega,f}(x) = \int_\Omega \Gamma(x-y) f(y)\, dy \qquad x \in \mathbb{R}^n$$

where Γ is the fundamental solution

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log|x| & n=2 \\ \frac{1}{n(2-n)\omega_n} |x|^{2-n} & n \geq 3 \end{cases}$$

Remark. Recall $\Gamma(x-\cdot)$ is L^1-integrable near x (cf. Lecture 1).

$\Rightarrow W = W_{\Omega,f}$ is well-defined in $\mathbb{R}^n$.

Recall: Green's identity

$$u(x) = \int_\Omega \Gamma(x-y)\, \Delta u(y)\, dy - \int_{\partial\Omega} \left[\Gamma(x-y) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial \Gamma}{\partial n}(x-y) \right] dS_y$$

Newtonian Potential W is the first term of RHS of Green's identity.

↓

(right hand side)

Lemma. $\Omega = B_1$. $f \in L^\infty(B_1)$.

(1) $w \in C^1(B_1)$ & $\partial_i w(x) = \int_{B_1} \partial_{x_i}\Gamma(x-y) f(y)\,dy$

(2) $\|w\|_{C^1(B_1)} \le C(n) \|f\|_{L^\infty(B_1)}$

Proof. (1). Set $V_i(x) = \int_{B_1} \partial_{x_i}\Gamma(x-y) f(y)\,dy$ $\quad x \in \bar{B}_1$

$|\partial_{x_i}\Gamma(x-y)| \le C(n) \cdot |y-x|^{1-n} \underset{(*)}{\Rightarrow} \partial_{x_i}\Gamma(x-y) \in L^1(B_1)$

$\exists R>0$. s.t. $B_R(x) \subset\subset B_1$

$$\int_{B_1} |\partial_i\Gamma(x-y)|\,dy = \underbrace{\int_{B_R(x)} |\partial_i\Gamma(x-y)|\,dy}_{} + \underset{+\infty}{\underset{\wedge}{\int_{B_1\setminus B_R(x)} |\partial_i\Gamma(x-y)|\,dy}}$$

$$(*) \Rightarrow \text{"}\underline{\quad}\text{"} \le C(n) \int_{B_R(x)} |y-x|^{1-n}\,dy$$

$$= C(n) \int_0^R \int_{\partial B_r(x)} |y-x|^{1-n}\,dS_y \cdot dr$$

$$= C(n) \int_0^R n\omega_n r^{n-1} \cdot r^{1-n}\,dr$$

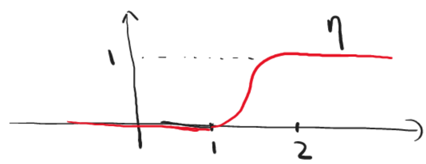
$$< +\infty$$

$\Rightarrow \partial_{x_i}\Gamma(x-y) \in L^1(B_1) \Rightarrow V_i$ is well-defined

Step 1. Construct smooth approximation w_ε of w

Choose $\eta \in C^\infty(\mathbb{R})$ s.t.

$$\begin{cases} 0 \le \eta \le 1 & \text{in } \mathbb{R} \\ \eta \equiv 0 & \text{in } (-\infty, 1] \\ \eta \equiv 1 & \text{in } [2, +\infty) \\ 0 \le \eta' \le 100 & \text{in } \mathbb{R} \end{cases}$$



$\forall\, \varepsilon > 0$. define

$w_\varepsilon(x) = \int_{B_1} \Gamma \cdot \eta_\varepsilon \cdot f(y)\,dy$ where $\Gamma = \Gamma(x-y)$. $\eta_\varepsilon = \eta\left(\frac{|x-y|}{\varepsilon}\right)$

$\left[w(x) = \int_{B_1} \Gamma \cdot f(y)\,dy \right]$ $\quad\Downarrow$

$w_\varepsilon \in C^\infty(\bar{B}_1)$

$$\partial_i w_\varepsilon(x) = \int_{B_1} \partial_{x_i}(\Gamma\,\eta_\varepsilon) f(y)\,dy.$$

Step 2. As $\varepsilon \to 0$. $w_\varepsilon \to w$ in $C^0(\bar{B}_1)$ &

$\partial_i w_\varepsilon \to V_i$ in $C^0(\bar{B}_1)$

(1) follows from Step 2.

Only prove $n \geq 3$. ($n=2$ similarly)

$$|w_\varepsilon(x) - w(x)| \leq \left| \int_{B_1} \Gamma \cdot (\eta_\varepsilon - 1) f(y)\, dy \right|$$

$$\leq \|f\|_{L^\infty(B_1)} \cdot \int_{B_1} |\Gamma \cdot (1-\eta_\varepsilon)| \, dy$$

$$\leq \|f\|_{L^\infty(B_1)} \cdot \int_{B_{2\varepsilon}(x)} |\Gamma| \, dy$$

$$= \|f\|_{L^\infty(B_1)} \cdot \int_{B_{2\varepsilon}(x)} \frac{1}{n(n-2)\omega_n} |y-x|^{2-n} \, dy$$

$$= \|f\|_{L^\infty(B_1)} \cdot C(n) \cdot \int_0^{2\varepsilon} r \, dr$$

$$= \|f\|_{L^\infty(B_1)} \cdot C(n) \quad 2\varepsilon^2 \longrightarrow 0$$

$$|\partial_i w_\varepsilon(x) - V_i(x)| \leq \|f\|_{L^\infty(B_1)} \int_{B_{2\varepsilon}(x)} |\partial_{x_i}(\Gamma \eta_\varepsilon - \Gamma)| \, dy$$

$$= \|f\|_{L^\infty(B_1)} \int_{B_{2\varepsilon}(x)} |\partial_{x_i}(\Gamma(1-\eta_\varepsilon))| \, dy$$

$$\leq \|f\|_{L^\infty(B_1)} \int_{B_{2\varepsilon}(x)} \left(|\partial_{x_i}\Gamma| + \frac{100}{\varepsilon} |\Gamma| \right) dy$$

$$\leq C(n) \|f\|_{L^\infty(B_1)} \cdot \varepsilon \longrightarrow 0$$

$\Rightarrow$ Step 2 ✓

(2). $\forall\, x \in B_1$

$$|w(x)| \leq \|f\|_{L^\infty} \int_{B_1} |\Gamma| \, dy \leq C(n) \|f\|_{L^\infty}$$

$$|\partial_i w(x)| \leq \|f\|_{L^\infty} \int_{B_1} |\nabla \Gamma| \, dy \leq C(n) \|f\|_{L^\infty}$$

$$\Rightarrow \|w\|_{C^1(B_1)} \leq C(n) \|f\|_{L^\infty(B_1)}$$

#

Lemma. $\Omega = B_1$. $f \in C^\alpha(\bar{B}_1)$ for some $\alpha \in (0,1)$.

(1). $w \in C^2(B_1)$ &

$$\partial_{ij} w(x) = \int_{B_1} \partial_{x_i x_j} \Gamma(x-y) \cdot (f(y) - f(x))\, dy$$
$$- f(x) \int_{\partial B_1} \partial_{x_j} \Gamma(x-y) n_i \, dS_y \quad (*)$$

& $\Delta w = f$

(2) $\|w\|_{C^2(B_{1/2})} \le C(\alpha, n) \|f\|_{C^\alpha(B_1)}$.

Proof. (1) Set $v_{ij}(x) = $ RHS of $(*)$

$f \in C^\alpha(\bar{B}_1) \Rightarrow |f(y) - f(x)| \le [f]_{C^\alpha(B_1)} \cdot |y-x|^\alpha$.

$\Rightarrow |\partial_{x_i x_j}\Gamma(x-y) \cdot (f(y)-f(x))| \le C(n) \cdot |x-y|^{-n} \cdot [f]_{C^\alpha(B_1)} |x-y|^\alpha$

$= C(n) \cdot [f]_{C^\alpha(B_1)} \cdot |x-y|^{\alpha-n} \in L^1(B_1)$

$\left[\int_{B_R(x)} |x-y|^{\alpha-n} dy = \int_0^R C(n) \cdot r^{n-1} \, r^{\alpha-n} \, dr < \infty \right]$

$\Rightarrow v_{ij}$ is well-defined in B_1

Note that $|x-y|^{-n} \notin L^1(B_1)$ so $f \in C^\alpha(\bar{B}_1)$ is important!

Step 1. Smooth approximation $v_{j,\varepsilon}$ of $\partial_j w$.

$\forall \varepsilon > 0$. define $v_{j,\varepsilon}(x) = \int_{B_1} \partial_{x_j}\Gamma \cdot \eta_\varepsilon \cdot f(y) dy$

Where η_ε is the same fuction in the previos lemma.

$$\partial_i v_{j,\varepsilon}(x) = \int_{B_1} \partial_{x_i} (\partial_{x_j}\Gamma \cdot \eta_\varepsilon) f(y) dy$$
$$= \int_{B_1} \partial_{x_i} (\partial_{x_j}\Gamma \cdot \eta_\varepsilon)(f(y) - f(x)) dy$$
$$+ f(x) \int_{B_1} \partial_{x_i} (\partial_{x_j}\Gamma \cdot \eta_\varepsilon) dy$$
$$= \int_{B_1} \partial_{x_i} (\partial_{x_j}\Gamma \cdot \eta_\varepsilon)(f(y) - f(x)) dy$$
$$- f(x) \int_{B_1} \partial_{y_i} (\partial_{x_j}\Gamma \cdot \eta_\varepsilon) dy$$

$$\Big[\ \partial_{x_j}\Gamma\cdot\eta_\varepsilon \text{ depends only on } (x-y).$$

$$\Rightarrow \partial_{x_i}(\partial_{x_j}\Gamma\cdot\eta_\varepsilon) = -\partial_{y_i}(\partial_{x_j}\Gamma\cdot\eta_\varepsilon)\ \Big]$$

$$= \int_{B_1}\partial_{x_i}(\partial_{x_j}\Gamma\cdot\eta_\varepsilon)(f(y)-f(x))\,dy$$

$$- \int_{\partial B_1}\partial_{x_j}\Gamma\cdot\eta_\varepsilon\cdot n_i\,dS_y$$

Step 2. $\forall r\in(0,1)$ as $\varepsilon\to 0$. $V_{j,\varepsilon}\to\partial_j W$ $\quad\partial_i V_{j,\varepsilon}\to V_{ij}$ in $C^0(\overline{B_r})$.

$$|V_{j,\varepsilon}(x)-\partial_j W(x)| \le \|f\|_{L^\infty(B_1)}\cdot\int_{B_{2\varepsilon}(x)}|\partial_{x_j}\Gamma|(1-\eta_\varepsilon)\,dy$$

$$\le C(n)\cdot\|f\|_{L^\infty(B_1)}\cdot\varepsilon \longrightarrow 0$$

when $\varepsilon<\frac{1}{2}(1-r)$ $\Rightarrow$ $\eta_\varepsilon=1$ on ∂B_1

$$|\partial_i V_{j,\varepsilon}(x)-V_{ij}(x)| \le \int_{B_{2\varepsilon}(x)}|\partial_{x_i}(\partial_{x_j}\Gamma(1-\eta_\varepsilon))\cdot(f(y)-f(x))|\,dy$$

$$\le [f]_{C^\alpha(B_1)}\cdot\int_{B_{2\varepsilon}(x)}\left(|\partial_{x_ix_j}\Gamma|+\frac{100}{\varepsilon}|\partial_{x_j}\Gamma|\right)|y-x|^\alpha\,dy$$

$$\le [f]_{C^\alpha(B_1)}\cdot\int_{B_{2\varepsilon}(x)}\left[C(n)|y-x|^{-n}+\frac{100}{\varepsilon}C(n)|y-x|^{1-n}\right]|y-x|^\alpha$$

$$\le C(n,\alpha)\,[f]_{C^\alpha(B_1)}\cdot\varepsilon^\alpha \longrightarrow 0 \quad \text{as } \varepsilon\to 0.$$

$\Rightarrow$ step α $\Rightarrow$ $W\in C^2(B_1)$ & $\partial_{ij}W=V_{ij}$.

Step 3 $\quad \Delta W=f.$

$$\Delta W(x)=\int_{B_1}\Delta_x\Gamma(x-y)(f(y)-f(x))\,dy - f(x)\int_{\partial B_1}\partial_{x_i}\Gamma(x-y)n_i(y)\,dS_y$$

$$= 0 + f(x)\int_{\partial B_1}\partial_{y_i}\Gamma(x-y)\,n_i(y)\,dS_y$$

$$= f(x)\cdot\underbrace{\int_{\partial B_1}\frac{\partial\Gamma}{\partial\vec{n}_y}(x-y)\,dS_y}_{=1}$$

$$= f(x)$$

(cf. Lecture 1)

(2). $\forall x \in B_{1/2}$

$$|\partial_{ij} W(x)| = \left| \int_{B_1} \partial_{x_i x_j} \Gamma(x-y)(f(y)-f(x))\,dy - f(x) \int_{\partial B_1} \partial_{x_i} \Gamma(x-y)\, n_j \, dS_y \right|$$

$$\leq C(n)\, [f]_{C^\alpha(B_1)} \int_{B_1} |y-x|^{\alpha-n}\, dy$$

$$+ C(n)\, \|f\|_{L^\infty(B_1)} \int_{\partial B_1} |y-x|^{1-n}\, dS_y$$

$$\left(|x|<\tfrac{1}{2}\right) \quad \leq C(\alpha,n)\, [f]_{C^\alpha(B_1)} + C(n)\, \|f\|_{L^\infty(B_1)}$$

$$\leq C(\alpha,n)\, \|f\|_{C^\alpha(B_1)}$$

Combining this with $\|W\|_{C^1(B_1)} \leq C(n)\, \|f\|_{L^\infty(B_1)}$

$$\Longrightarrow \quad \|W\|_{C^2(B_{1/2})} \leq C(\alpha,n)\, \|f\|_{C^\alpha(B_1)} \qquad \#$$