Introduction to partial differential equations

1. What is partial differential equation (PDE)?

PDE is an equation for unknown multi-variable function that involves partial derivatives.

The order of a PDE is the order of highest derivative that occurs in it.

Examples. Let u be a function of x_1 and x_2

(1). $\frac{\partial u}{\partial x_1} \cdot \frac{\partial u}{\partial x_2} + u = 0$ (1st order)

(2) $\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0$ (2nd order)

(3) $\frac{\partial^3 u}{\partial x_1^3} - \frac{\partial^2 u}{\partial x_1 \partial x_2} + 5 \cdot \frac{\partial u}{\partial x_1} = 10$ (3rd order)

Remark. (1) Ordinary differential equation (ODE) is an equation for an unknown single-variable function that involves ordinary derivatives.

(2) For PDE and ODE, the methods are totally different.

2. Some geometric motivations

Many geometric problems can be transformed into PDEs.

Examples. (1) Calabi's conjecture

(2) Yamabe problem

(3) Minkowski Problem

(4) Minimal surface problem

(5) Prescribed curvature problem

$$\text{Geometric Problems} \rightsquigarrow \text{PDEs} \begin{cases} (1)\ \text{Existence} \\ (2)\ \text{Uniqueness} \\ (3)\ \text{Regularity} \end{cases}$$

3. Main contents in this mini-course

The theory of linear 2nd order elliptic PDEs

Main goals: Dirichlet Problem $\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$

Laplace's equation $\rightsquigarrow$ Poisson's equation $\rightsquigarrow$ General equation.

Method: Continuity method

Tools: Maximum principle, Schauder's estimates

Remark. The above method and tools are still widely used in the study of non-linear 2nd order elliptic PDEs.

Lecture 1. Laplace's equation (I)

1. Notations

$\mathbb{R}^n$: n-dimensional Euclidean space.

$x \in \mathbb{R}^n$. $\quad x = (x_1, \cdots, x_n)^T \quad |x| = \sqrt{x_1^2 + \cdots + x_n^2}$

$B_r(x_0) = \{ x \in \mathbb{R}^n \mid |x - x_0| < r \}. \quad B_r = B_r(0)$

$\partial_i = \frac{\partial}{\partial x_i}. \quad \partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \quad \Delta = \sum_{i=1}^n \partial_{ii}$

$\Rightarrow \Delta u =$ trace of Hessian matrix $(\partial_{ij} u)$

Definition. (1) Laplace's equation $\Delta u = 0$

(2) Let u be a C^2 function.

- u is said to be subharmonic if $\Delta u \geq 0$.
- ——————— superharmonic if $\Delta u \leq 0$
- ——————— harmonic if $\Delta u = 0$

Example (1) 2nd order Polynomials.

$$u(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + c. \quad a_{ij} = a_{ji}$$

$$\Rightarrow \partial_{ij} u = 2 a_{ij}$$

Write $A = (a_{ij}) \Rightarrow \Delta u = 2 \operatorname{tr} A$.

Hence. $\operatorname{tr} A \geq 0 \iff u$ is subharmonic

$\operatorname{tr} A \leq 0 \iff u$ is superharmonic

$\operatorname{tr} A = 0 \iff u$ is harmonic

(2) Let f be a holomorphic function on $\Omega \subset \mathbb{C}$.

Then $\operatorname{Re} f$ and $\operatorname{Im} f$ are both harmonic.

Write $f = u + \sqrt{-1}\, v$. Cauchy-Rieman equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\Rightarrow \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

Similarly $\Rightarrow \Delta v = 0$.

2. Fundemental solutions

Laplace's equation is invariant under orthogonal transformation.

Lemma. Suppose that u is harmonic and $A = (a_{ij})$ is an orthogonal matrix ($AA^T = I_n$). Then $v(x) := u(Ax)$ is harmonic.

Proof. Write $y = Ax$

Chain Rule $\Rightarrow$
$$\frac{\partial v}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial y_k} a_{ki}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x_i \partial x_j} = \sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial y_k \partial y_l} a_{ki} a_{lj}$$

$$\Rightarrow \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right) = A^T \cdot \left(\frac{\partial^2 u}{\partial y_k \partial y_l}\right) \cdot A$$

$$\Rightarrow \Delta v = \operatorname{tr}\left(A^T \cdot \left(\frac{\partial^2 u}{\partial y_k \partial y_l}\right) \cdot A\right)$$

$$= \operatorname{tr}\left(\frac{\partial^2 u}{\partial y_k \partial y_l} \underbrace{A \cdot A^T}_{AA^T = I_n}\right) = \Delta u$$

#

Motivated by the above lemma. let us find radial solution

$\Longleftrightarrow$ u depends only on $r=|x|$

$\Longleftrightarrow$ $u(x)=f(r)$ where f is a single-variable function.

Goal: Apply $\Delta u=0$ to solve f

$$\partial_{ii}u=\partial_i\left(f'(r)\cdot\partial_i r\right)=f''(r)\cdot(\partial_i r)^2+f'(r)\,\partial_{ii}r.$$

$$r=\sqrt{x_1^2+\cdots+x_n^2}\Rightarrow\partial_i r=\frac{x_i}{r}.\quad\partial_{ii}r=\frac{1}{r}-\frac{x_i^2}{r^3}.$$

$$\Rightarrow\partial_{ii}u=f''(r)\cdot\frac{x_i^2}{r^2}+f'(r)\cdot\left(\frac{1}{r}-\frac{x_i^2}{r^3}\right)$$

$$\Rightarrow 0=\Delta u=f''(r)+f'(r)\left(\frac{n}{r}-\frac{1}{r}\right)$$

ODE: $$f''(r)+f'(r)\cdot\frac{n-1}{r}=0$$

Multiply r^{n-1} $\Rightarrow$ $$r^{n-1}f''(r)+(n-1)r^{n-2}f'(r)=0$$

$$\Rightarrow\left(r^{n-1}f'(r)\right)'=0$$

$$\Rightarrow r^{n-1}f'(r)=C$$

$$\Rightarrow f(r)=\begin{cases}C_1\log r+C_2 & n=2\\ C_1 r^{2-n}+C_2 & n\geq 3.\end{cases}$$

Definition. Fundamental solution.

$$\Gamma(x)=\begin{cases}\frac{1}{2\pi}\log|x| & n=2\\ \frac{1}{n(2-n)\omega_n}|x|^{2-n} & n\geq 3.\end{cases}$$

where ω_n denotes the volume of unit ball in $\mathbb{R}^n$.
i.e., $\omega_n=|B_1|$

Remark. $|B_r|=\omega_n\cdot r^n$ $\quad|\partial B_r|=|\partial B_1|\cdot r^{n-1}$

$$|B_1|=\int_0^1|\partial B_r|\,dr=|\partial B_1|\cdot\int_0^1 r^{n-1}\,dr$$

$$\Rightarrow|B_1|=\frac{1}{n}|\partial B_1|.\Rightarrow|\partial B_1|=n\cdot\omega_n.$$

Lemma (1) Γ is defined on $\mathbb{R}^n \setminus \{0\}$.

(2) Γ has a singularity at 0

However. Γ is L^1-integrable near 0. i.e. $\Gamma \in L^1(B_1)$.

(3) Γ is harmonic in $\mathbb{R}^n \setminus \{0\}$: $\Delta \Gamma = 0$ in $\mathbb{R}^n \setminus \{0\}$

(4) For any $r>0$. $\displaystyle\int_{\partial B_r} \frac{\partial \Gamma}{\partial \vec{n}}\, ds = 1$

where $\vec{n}$ denotes the unit outward normal to ∂B_r.

Proof. (1) (3) are trivial.

Suppose $n \geq 3$. ($n=2$. Similarly)

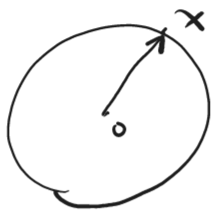
(2) $$\int_{B_1} \Gamma(x)\,dx = \int_0^1 \int_{\partial B_r} \frac{1}{n(2-n)\omega_n} |x|^{2-n}\, dx\, dr$$

$$= \int_0^1 \frac{1}{n(2-n)\omega_n} r^{2-n} \cdot |\partial B_r| \cdot dr$$

$$\left(|\partial B_r| = |\partial B_1| \cdot r^{n-1} = n\omega_n \cdot r^{n-1} \right)$$

$$= \int_0^1 \frac{1}{2-n} \cdot r\, dr = \frac{1}{2(2-n)}.$$

(4) Fix $x \in \partial B_r$. $\vec{n} = \dfrac{x}{|x|}$



$$\frac{\partial \Gamma}{\partial \vec{n}} = \nabla \Gamma \cdot \vec{n} = \sum_{i=1}^{n} \frac{1}{n\omega_n} |x|^{1-n} \frac{x_i}{|x|} \cdot \frac{x_i}{|x|} = \frac{|x|^{1-n}}{n\omega_n}.$$

$$\Rightarrow \int_{\partial B_r} \frac{\partial \Gamma}{\partial \vec{n}}\, dS = \frac{r^{1-n}}{n\omega_n} \cdot |\partial B_r| = 1.$$ #

3. Integral formulas.

$\Omega \subseteq \mathbb{R}^n$: bounded domain (connected open set) with C^1 boundary.

$\vec{n} = (n_1, \cdots, n_n)$. unit outward normal to $\partial\Omega$.

- Divergence theorem.

$\vec{F} = (F_1, \cdots, F_n)$: C^1 vector field in $\overline{\Omega}$.

$$\int_\Omega \operatorname{div} \vec{F}\, dx = \int_{\partial\Omega} \vec{F} \cdot \vec{n} \cdot ds \quad (1)$$

where $\operatorname{div} \vec{F} = \displaystyle\sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$

Remark. $u \in C^1(\bar{\Omega})$ choose $\vec{F} = (0, \cdots, \overset{i\text{-th}}{u}, \cdots, 0)$

$$(1) \Rightarrow \int_\Omega \partial_i u \, dx = \int_{\partial\Omega} u \cdot n_i \, dS$$

Integration by parts.

- Green's formula

$u, v \in C^2(\bar{\Omega})$

Take $\vec{F} = u \cdot \nabla v$ in (1) $\Rightarrow \int_\Omega \mathrm{div}(u \nabla v) dx = \int_{\partial\Omega} u \cdot \nabla v \cdot \vec{n} \, dS$

1st version of Green's formula: $\int_\Omega (u \Delta v + \nabla u \cdot \nabla v) dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}} dS$ (2)

$u \leftrightarrow v$ $\int_\Omega (v \Delta u + \nabla u \cdot \nabla v) dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \vec{n}} dS$ (3)

(2) − (3) $\Rightarrow$ 2nd version of Green's formula.

$$\int_\Omega (u \Delta v - v \Delta u) dx = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}} \right) dS \quad (4)$$

3rd version Green's formula.

Take $u = 1$ in (4) $\Rightarrow \int_\Omega \Delta v \, dx = \int_{\partial\Omega} \frac{\partial v}{\partial \vec{n}} dS$

(or take $\vec{F} = \nabla v$ in (1))

4. Green's identity

Theorem. $\Omega \subseteq \mathbb{R}^n$ bounded domain with C^1 boundary

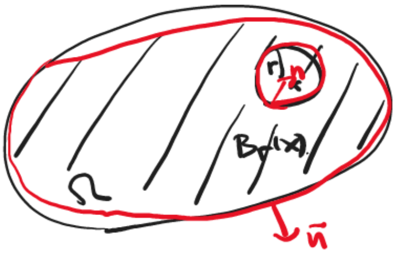
$u \in C^2(\bar{\Omega})$. $\forall x \in \Omega$

$$u(x) = \int_\Omega \Gamma(x-y) \Delta_y u(y) dy - \int_{\partial\Omega} \left[\Gamma(x-y) \frac{\partial u}{\partial \vec{n}_y}(y) - u(y) \frac{\partial \Gamma}{\partial \vec{n}_y}(x-y) \right] dS_y$$

Proof. Fix $x \in \Omega$. Write $\Gamma(y) = \Gamma(x-y)$.

choose $r > 0$ sufficiently small such that $B_r(x) \subset \Omega$

$\Rightarrow \Gamma$ is smooth in $\Omega \setminus B_r(x)$



Applying 2nd version of Green's formula to u and Γ in $\Omega \setminus B_r(x)$

$$\int_{\Omega\setminus B_r(x)} (\Gamma \Delta u - u \Delta_y \Gamma)\, dy = \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial \vec{n}} - u \frac{\partial \Gamma}{\partial \vec{n}_y}\right) dS_y$$
$$+ \int_{\partial B_r(x)} \left(\Gamma \frac{\partial u}{\partial \vec{n}} - u \frac{\partial \Gamma}{\partial \vec{n}_y}\right) dS_y$$

Recall $\Delta_y \Gamma = 0$ in $\Omega \setminus B_r(x)$

$$\int_{\Omega\setminus B_r(x)} \Gamma \Delta u\, dy = \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial \vec{n}} - u \frac{\partial \Gamma}{\partial \vec{n}_y}\right) dS + \int_{\partial B_r(x)} \left(\Gamma \frac{\partial u}{\partial \vec{n}} - u \frac{\partial \Gamma}{\partial \vec{n}_y}\right) dS$$

Let $r \to 0$. $u \in C^2(\bar{\Omega})$. $\Gamma \in L^1(B_1(x))$

$$\int_{\Omega} \Gamma \Delta u\, dy = \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial \vec{n}} - u \frac{\partial \Gamma}{\partial \vec{n}}\right) dS.$$
$$+ \lim_{r\to 0} \int_{\partial B_r(x)} \left(\Gamma \frac{\partial u}{\partial \vec{n}} - u \frac{\partial \Gamma}{\partial \vec{n}}\right) dS$$

when $n \geq 3$ ($n=2$ similarly)

Step 1. Compute Γ and $\frac{\partial \Gamma}{\partial \vec{n}_y}$ on $\partial B_r(x)$

$$\Gamma = \Gamma(x-y) = \frac{1}{n(2-n)\omega_n} |y-x|^{2-n} \Rightarrow \Gamma = \frac{r^{2-n}}{n(2-n)\omega_n} \text{ on } \partial B_r(x) \quad (1)$$

$$\frac{\partial \Gamma}{\partial y_i} = \frac{1}{n(2-n)\omega_n} \times (2-n) \times |y-x|^{1-n} \cdot \frac{y_i - x_i}{|y-x|} = \frac{1}{n\omega_n} \cdot \frac{y_i - x_i}{|y-x|^n}$$

$$\vec{n}_y = -\frac{y-x}{r} \Rightarrow n_i = -\frac{y_i - x_i}{r}$$

$$\frac{\partial \Gamma}{\partial \vec{n}_y} = \sum_{i=1}^{n} \frac{\partial \Gamma}{\partial y_i} \cdot n_i = -\sum_{i=1}^{n} \frac{1}{n\omega_n r} \frac{|y_i - x_i|^2}{|y-x|^n}$$
$$= -\frac{1}{n\omega_n r^{n-1}} \text{ on } \partial B_r(x)$$

(2)

Step 2. Compute $\lim_{r\to 0}\int_{\partial B_r(x)} (\Gamma \frac{\partial u}{\partial \vec{n}} - u\frac{\partial \Gamma}{\partial \vec{n}})dS$

$$\left|\int_{\partial B_r(x)} \Gamma \frac{\partial u}{\partial \vec{n}} ds\right| \overset{(1)}{=} \left|\frac{r^{2-n}}{n(2-n)\omega_n}\int_{\partial B_r(x)} \frac{\partial u}{\partial \vec{n}} ds\right|$$

$\left|\frac{\partial u}{\partial \vec{n}}\right| \le |\nabla u|$

$$\le \frac{r^{2-n}}{n(2-n)\omega_n}|\partial B_r(x)| \cdot \max_{\partial B_r(x)}|\nabla u|$$

$$= \frac{r}{n-2}\max_{\partial B_r(x)}|\nabla u|$$

$$\longrightarrow 0 \quad \text{as } r \to 0.$$

$$-\int_{\partial B_r(x)} u\frac{\partial \Gamma}{\partial \vec{n}} dS \overset{(2)}{=} \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)dS_y$$

(Taylor's formula)
$$= \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} [u(x) + O(r)]dS_y$$

$$= u(x) + O(r)$$

$$\longrightarrow u(x) \quad \text{as } r \to 0.$$

Recall that

$$\int_\Omega \Gamma \Delta u\, dy = \int_{\partial\Omega} (\Gamma \frac{\partial u}{\partial \vec{n}} - u\frac{\partial \Gamma}{\partial \vec{n}})dS + \underbrace{\lim_{r\to 0}\int_{\partial B_r(x)} (\Gamma \frac{\partial u}{\partial \vec{n}} - u\frac{\partial \Gamma}{\partial \vec{n}})dS}_{\substack{\parallel \\ 0\, +\, u(x)}}$$

$$\Rightarrow \quad u(x) = \int_\Omega \Gamma \Delta u\, dy - \int_{\partial\Omega} (\Gamma \frac{\partial u}{\partial \vec{n}} - u\frac{\partial \Gamma}{\partial \vec{n}})dS \qquad \#$$

Remark: Take $u=1$ in Green's identity.

$$\forall\, x\in\Omega. \qquad \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \vec{n}}(x-y)\,dS = 1.$$