Introduction to K-stability @ Nankai by Yuchen Liu.

Lecture 5. Explicit K-stability I.

Thm (Kento Fujita '15) X K-semistable $\mathbb{Q}$-Fano variety. $\dim X = n$.

Then $(-K_X)^n \leq (-K_{\mathbb{P}^n})^n = (n+1)^n$.

"=" iff $X \cong \mathbb{P}^n$.

Pf. Recall Fujita-Li's valuative criteria:

$$X \text{ K-ss} \iff \beta_X(E) \geq 0 \text{ for all } E \text{ div}/X.$$

$$\beta_X(E) = A_X(E) - S_X(E) \iff A_X(E) \geq S_X(E).$$

$$E \overset{\text{div}}{\subseteq} Y \xrightarrow{\text{bir}} X$$

$$A_X(E) = 1 + \operatorname{coeff}_E(K_{Y/X})$$

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^{+\infty} \underline{\operatorname{vol}(-K_X - tE)}\, dt$$

Let $Y = Bl_x X \xrightarrow{\mu} X$, where $x \in X$ is a smooth pt.

E : exceptional divisor of μ.

$$A_X(E) = n.$$

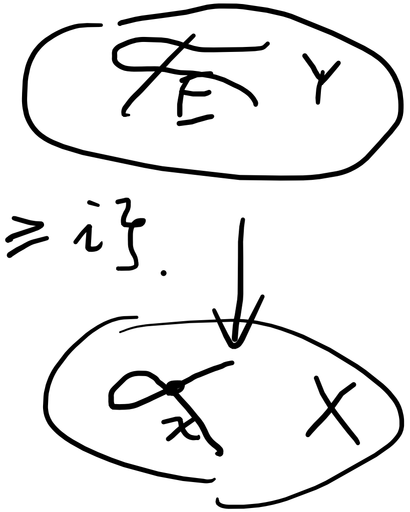
$$\operatorname{vol}(D) = \lim_{m \to \infty} \frac{h^0(mD)}{m^n/n!}.$$

Assume $mt \in \mathbb{Z}$.

$$h^0(m\mu^*(-K_X) - mtE) \leftarrow \text{growth tells us } \mathrm{vol}(-K_X - tE).$$

$$H^0(Y, m\mu^*(-K_X) - \underbrace{mt}_{= i} E) \cong H^0(X, \mathcal{O}_X(-mK_X)\cdot \mathfrak{m}_x^i)$$

$$= \{ s \in H^0(X, -mK_X) \mid \mathrm{mult}_x s \geq i \}.$$



short exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-mK_X)\cdot \mathfrak{m}_x^i \longrightarrow \mathcal{O}_X(-mK_X) \longrightarrow \mathcal{O}_X/\mathfrak{m}_x^i \longrightarrow 0.$$

long exact seq:

$$0 \to \underline{H^0(\mathcal{O}_X(-mK_X)\cdot \mathfrak{m}_x^i)} \longrightarrow H^0(\mathcal{O}_X(-mK_X)) \longrightarrow \mathcal{O}_X/\mathfrak{m}_x^i \longrightarrow H^1(\cdots).$$

$$\Rightarrow \quad h^0(\mathcal{O}_X(-mK_X)\cdot \mathfrak{m}_x^i) + \dim \mathcal{O}_X/\mathfrak{m}_x^i \geq h^0(-mK_X).$$

$(i = mt)$

Divide this inequality by $(m^n/n!)$, then take limit.

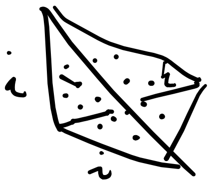
$$\Rightarrow \quad \mathrm{vol}(-K_X - tE) + \boxed{\lim_{m\to\infty} \frac{\dim \mathcal{O}_X/\mathfrak{m}_x^i}{m^n/n!}} \geq \mathrm{vol}(-K_X). \quad (1)$$

Since $x \in X$ is smooth, can "assume" $\mathcal{O}_X = \mathbb{C}[x_1, \dots, x_n]$

$\mathfrak{m}_x = (x_1, \dots, x_n)$

$$\mathcal{O}_X/\mathfrak{m}_x^i = \mathbb{C}\langle x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \mid \sum_{j=1}^n a_j < i \rangle.$$

$$\dim \mathcal{O}_X/\mathfrak{m}_x^i = \#\{(a_1, \dots, a_n) \in \mathbb{N}^n \mid \sum_{j=1}^n a_j < i\}.$$



$\Longrightarrow \quad \dim \mathcal{O}_X / \mathfrak{m}_x^i \sim$ volume of the simplex

$$\{(a_1, \dots, a_n) \in \mathbb{R}^n_{\geq 0} \mid \sum a_j \leq i\}$$

$$= i^n / n! = (mt)^n / n!$$

Hence $\displaystyle \lim_{m \to \infty} \frac{\dim \mathcal{O}_X / \mathfrak{m}_x^i}{m^n / n!} = t^n$. (2)

(1) & (2) $\Longrightarrow \quad \mathrm{vol}(-K_X - tE) \geq \mathrm{vol}(-K_X) - t^n$ (3)

$$\begin{tikzcd} & \mathrm{Bl}_0 \mathbb{P}^n \arrow[dl] \arrow[dr, Rightarrow] & \\ \mathbb{P}^n \arrow[rr, dashed] & & \mathbb{P}^{n-1} \end{tikzcd}$$

"=" iff $\varepsilon_x(E) = T_x(E) = T$

Seshadri   pseff threshold.

Seshadri const $\varepsilon_x(E) = \sup\{t \mid -K_X - tE \text{ nef}\}$ $\Rightarrow -K_X - T \cdot E$ semiample, not big.

$$S_X(E) = \frac{1}{\mathrm{vol}(-K_X)} \int_0^{+\infty} \mathrm{vol}(-K_X - tE)\, dt$$

$$\overset{(3)}{\geqslant} \frac{1}{\mathrm{vol}(-K_X)} \int_0^{\sqrt[n]{\mathrm{vol}(-K_X)}} \left(\mathrm{vol}(-K_X) - t^n\right) dt$$

$$= \frac{n}{n+1} \sqrt[n]{\mathrm{vol}(-K_X)} .$$

X K-ss

$\Rightarrow \beta_X(E) \geq 0$, i.e. $n = A_X(E) \geq S_X(E) \geq \frac{n}{n+1} \sqrt[n]{\mathrm{vol}(-K_X)}$.

$$\Rightarrow \quad \mathrm{vol}(-K_X) \leqslant (n+1)^n = \mathrm{vol}(-K_{\mathbb{P}^n}) .$$

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Rmk If X smooth, then $(-K_X)^n \leq (n+1)^n$ holds for $n \leq 3$
(classification)

Starting from $n=4$, $\exists$ smooth X & $(-K_X)^n > (n+1)^n$
[Batyrev] ...

If X singular, $X = \mathbb{P}(1,1,d)$ = cone over deg d RNC.

$$(-K_X)^2 = \frac{(d+2)^2}{d} \to +\infty .$$

§ Normalized volume (after Chi Li '15)

<u>Def</u>. $x \in X$ klt singularity

$$\begin{array}{ccc} E & \subseteq & Y \\ \downarrow & & \downarrow \\ x & \in & X \end{array} \quad \leftarrow \quad E \text{ div} / x \in X.$$

Log discrepancy: $A_X(E) = 1 + \mathrm{coeff}_E(K_{Y/X})$.

Volume: $a_i(E) = \{ f \in \mathcal{O}_{X,x} \mid \mathrm{ord}_E(f) \geq i \}$.

$$\mathrm{vol}_{x,X}(E) := \lim_{i \to \infty} \frac{\dim_{\mathbb{C}} \mathcal{O}_{X,x} / a_i(E)}{i^n / n!}.$$

generalization

(Hilbert - Samuel multiplicity).

Normalized volume

$$\widehat{\mathrm{vol}}_{x,X}(E) := A_X(E)^n \cdot \mathrm{vol}_{x,X}(E).$$

Local volume

$$\widehat{\mathrm{vol}}(x,X) := \inf_{E \text{ div}/x\in X} \widehat{\mathrm{vol}}_{x,X}(E).$$

$a_i(E_{\underline{w}}) = \langle x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \sum_j \alpha_j w_j \geq i \rangle$

$\mathrm{ord}_{E_{\underline{w}}}(x_j) = w_j.$

<u>Ex</u>. $0 \in \mathbb{A}^n$. $E_{\underline{w}}$ = exceptional div. of weighted blow up of weights $\underline{w} = (w_1, \cdots, w_n)$.

usual blow-up: $\underline{w} = (1, 1, \cdots, 1)$.

$$A_X(E_{\underline{w}}) = \sum_{j=1}^n w_j, \quad \mathrm{vol}_{x,X}(E_{\underline{w}}) = \left(\prod_{j=1}^n w_j\right)^{-1}.$$

$$\Rightarrow \quad \widehat{\mathrm{vol}}(E_{\underline{w}}) = \frac{\left(\sum_j w_j\right)^n}{\prod_j w_j} \geq n^n .$$

de Fernex - Ein - Mustaţă, Li : If $x \in X$ is smooth,

then $\widehat{\mathrm{vol}}$ is minimized by usual blow-up.

In other words, $\widehat{\mathrm{vol}}(0, \mathbb{A}^n) = n^n$.

<u>Ex</u>. $(x \in X) \cong (0 \in \mathbb{A}^n / G)$ $\quad$ G finite group.

$$\widehat{\mathrm{vol}}(x, X) = \frac{n^n}{|G|} \qquad \text{(Li-Xu)}.$$

Thm (L.) X $\mathbb{Q}$-Fano variety, K-semistable.

Then for every $x \in X$,

$$(-K_X)^n \leqslant \left(1+\frac{1}{n}\right)^n \widehat{\mathrm{vol}}(x, X).$$

Thm (Li–Xu). If (X, g) is a GH limit of KE Fano mfd.
($\Rightarrow$ X K-polystable)

volume density $\quad \Theta(x, X) = \lim_{r \to 0} \dfrac{\mathrm{vol}(B_g(x, r))}{\mathrm{vol}(B_{std}(0, r))} \in (0, 1]$.

Then $\widehat{\mathrm{vol}}(x, X) = n^n \cdot \Theta(x, X)$.

Every toric Fano mfd carries a Kähler-Ricci soliton metric

(come from limit of Kähler-Ricci flow).

KE: $\quad \mathrm{Ric}(\omega) = \omega.$

KR soliton: $\quad \mathrm{Ric}(\omega) = \omega + L_\xi \omega.$ $\quad \xi$ vector field.

<u>Thm</u> (Tian, Odaka–Spotti–Sun)

For cubic surfaces, K-stability $\Longleftrightarrow$ GIT stability.

(K-moduli $\cong$ GIT moduli).

<u>Strategy</u>: moduli continuity method.

<u>Step 1</u>. $\exists$ one K-stable cubic surface (Fermat cubic).

By openness, a general cubic surface is K-stable.

Step 2. M^K = K-moduli space of K-stable cubic surfaces & K-ps limits.

- We want to show every $X \in M^K$ is also a (possibly singular) cubic surface.

$-K_{X_t} = \mathcal{O}_{X_t}(1)$

$$X_t \overset{\mathbb{Q}\text{-Gor.}}{\rightsquigarrow} X \quad \Rightarrow \quad (-K_X)^2 = (-K_{X_t})^2 = 3.$$

cubic surface $\qquad$ $\mathbb{Q}$-Fano K-ss.

By volume comparison, $3 = (-K_X)^2 \leq \frac{9}{4} \widehat{\mathrm{vol}}(x, X)$ (any pt.)

(klt surf is quotient!) $(x \in X) \cong (0 \in \mathbb{A}^2/G_x) \Rightarrow \widehat{\mathrm{vol}}(x, X) = 4/|G_x|.$

$\Rightarrow |G_x| \leq 3.$

If $|G_x| = 2$, action is $(x,y) \mapsto (-x,-y)$

$G_x \cong \mathbb{Z}/2\mathbb{Z} \Rightarrow (x \in X) \cong (0 \in \mathbb{A}^2/\pm 1) \rightsquigarrow$ $\boxed{A_1\text{-singularity.}}$

If $|G_x| = 3$, actions are

$G_x \cong \mathbb{Z}/3\mathbb{Z}$ $\quad \xi = e^{\frac{2\pi i}{3}}$

$$\begin{cases} (x,y) \mapsto (\xi x, \xi y) & \boxed{\cancel{\tfrac{1}{3}(1,1)}} \\ (x,y) \mapsto (\xi x, \xi^{-1} y). & \boxed{A_2\text{-sing}} \end{cases}$$

Kollar-Shepherd-Barron: $\frac{1}{3}(1,1)$ is <u>not</u> $\mathbb{Q}$-Gorenstein smoothable.

$\Longrightarrow$ Any singularity on X is A_1 or A_2.

T. Fujita: If X is a klt del Pezzo surface,

$(-K_X)^2 = 3$, and X has only ADE singularities (du Val). $\Leftrightarrow$ K_X is Cartier.

Then $|-K_X| : X \hookrightarrow \mathbb{P}^3$ (very ample)

as a cubic surface

<u>step 3</u>. Construct map $\Phi : M^K \to M^{GIT}$.

$$\begin{array}{ccc} M^K & & \\ \Psi & & \\ [X] & \longmapsto & [X] \end{array}$$

$X \subseteq \mathbb{P}^3$ as a cubic surface

Paul-Tian criterion: For hypersurfaces,

K-stability $\Rightarrow$ GIT stability.

<u>Idea</u>. $X \subseteq \mathbb{P}^n$ hypersurface of deg d

$$\sigma : \mathbb{C}^* \longrightarrow SL(n+1)$$

$$\sigma(t) \cdot X \rightsquigarrow X_0 \quad \text{as } t \to 0$$

$\hookrightarrow$ test configuration $\mathcal{X}$.

$$\mathrm{Fut}(\mathcal{X}) = \mu^{\Lambda_{CM}}([X], \sigma)$$

Λ_{CM} is ample on $\mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))$

GIT weight

X K-ss $\Rightarrow \mathrm{Fut}(\mathcal{X}) \geq 0 \iff \mu([X], \sigma) \geq 0 \Rightarrow X$ is GIT semistable.

Φ is injective and birational.

Step 4. Since M^K and M^{GIT} are proper,

hence Φ is finite, injective, birational

$\Longrightarrow$ Φ is an isomorphism

Zariski's main thm.

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Thm. (L.-Xu) For cubic 3-folds, K-stability = GIT.

Thm (L.) True for cubic 4-folds too.

Pf (L.-Xu). Modify Step 2.

Goal. $X_t \overset{\mathbb{Q}\text{-Gor}}{\rightsquigarrow} X$ $\Rightarrow$ $X \subseteq \mathbb{P}^4$ as a cubic 3-fold.

cubic 3-fold $\qquad$ $\mathbb{Q}$-Fano K-ss.

$\mathcal{O}_{X_t}(1) \rightsquigarrow L$ Weil divisor, $\mathbb{Q}$-Cartier.

$-K_{X_t} = \mathcal{O}_{X_t}(2) \Rightarrow -K_X = 2L$, $(L^3) = 3$.

T. Fujita: If L is Cartier, then

$$|L| : X \hookrightarrow \mathbb{P}^4 \quad \text{very ample.}$$

image is a cubic 3-fold.

Assume to the contrary L is NOT Cartier at $x \in X$.

Take index 1 cover of $(x \in X) \xleftarrow{f} (\tilde{x} \in \tilde{X})$

$$L \longleftarrow \tilde{L} \quad \text{Cartier.}$$

$\xRightarrow{\text{Xu-Zhuang}}$

$$\widehat{\mathrm{vol}}(\tilde{x}, \tilde{X}) = \underset{\substack{\vee\!\mid \\ 2}}{\deg(f)} \cdot \widehat{\mathrm{vol}}(x, X).$$

By volume comparison.

$$\widehat{\mathrm{vol}}(x, X) \geq \frac{27}{64} \cdot (-K_X)^3 = \frac{27}{64} \cdot 24 = \frac{81}{8}.$$

$$\Rightarrow \widehat{\mathrm{vol}}(\tilde{x}, \tilde{X}) \geq 2\, \widehat{\mathrm{vol}}(x, X) \geq \frac{81}{4} = \underline{\underline{20.25}}.$$

☆ <u>Thm</u> (L.-Xu) If $y \in Y$ is a 3-fold non-smooth sing.

then $\widehat{\mathrm{vol}}(y, Y) \leq 16 = \widehat{\mathrm{vol}}\big(0 \in (x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0)\big)$
ODP.

$\left(\widehat{\mathrm{vol}}(0, \mathbb{A}^3) = 27 \right)$

By gap thm, $\tilde{x} \in \tilde{X}$ is smooth

and $\deg(f) = 2$. $f : \tilde{X} \to X$.

$\Rightarrow x \in X$ is a quotient sing of order 2.

Lefschetz thm $\Rightarrow$ L is Cartier at $x \in X$.

Cubic 4-fold. $X_t \rightsquigarrow X$

cubic 4 fold K-ss. Hope to prove L is Cartier.

$\mathcal{O}_{X_t}(1) \rightsquigarrow L$.

[L.-Xu] In any dim, $\widehat{\mathrm{vol}}(y, Y) \leq n^n$. "=" iff $y \in Y$ is sm.

$\Longrightarrow$ $2L$ is Cartier.

3-fold gap $\Longrightarrow$ L is Cartier away from finite set.

Do intersection for $2L$ & L. Ambro-Kawamata non-vanishing.

del Pezzo variety : X n-dim'l $\mathbb{Q}$-Fano.

$$-K_X = (n-1)L, \quad L \text{ ample } \underline{\underline{\text{Cartier}}}.$$

T. Fujita : $|L|$ has nice behavior.