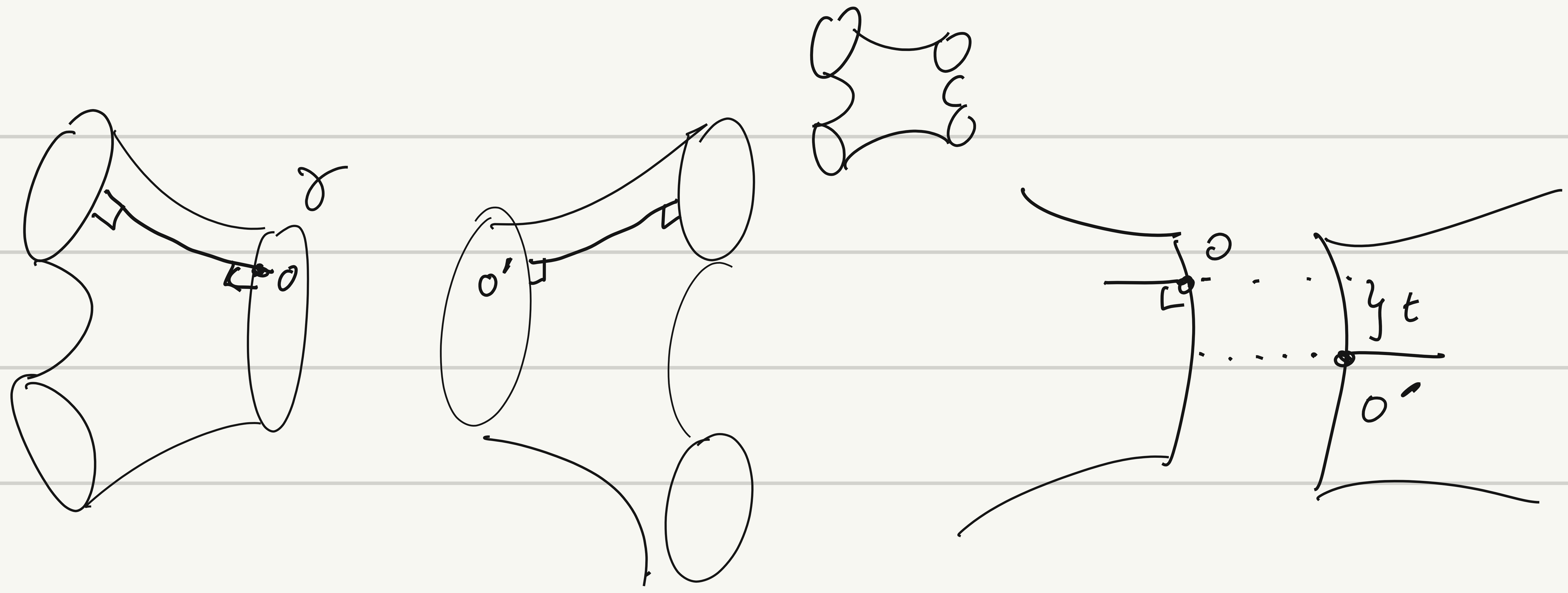
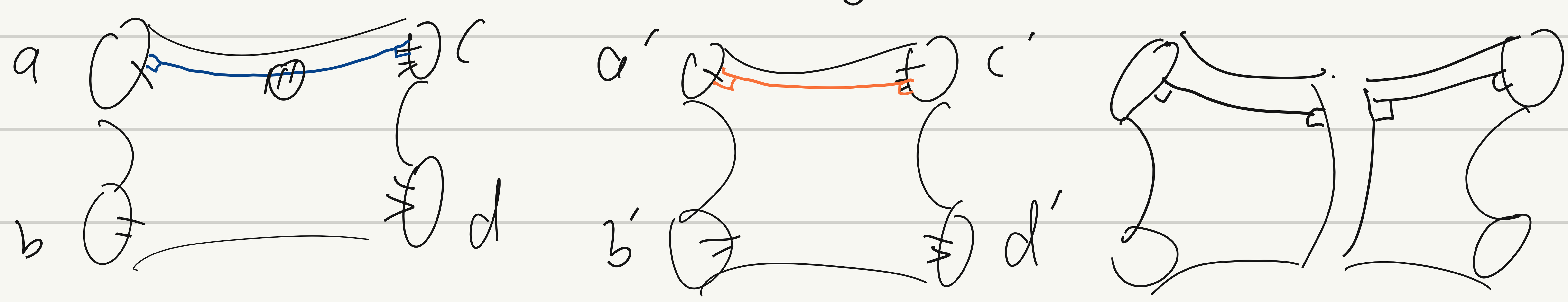
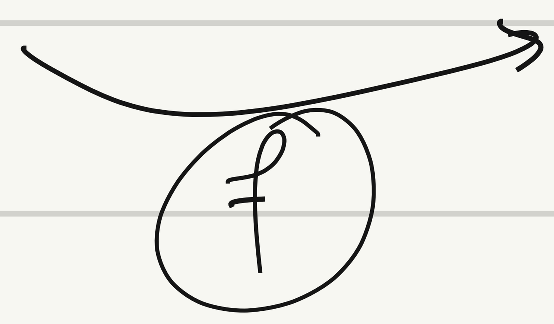


If $t \neq t' \in [0, l(\gamma))$, $S_{0,4} \not\cong S'_{0,4}$

generically.

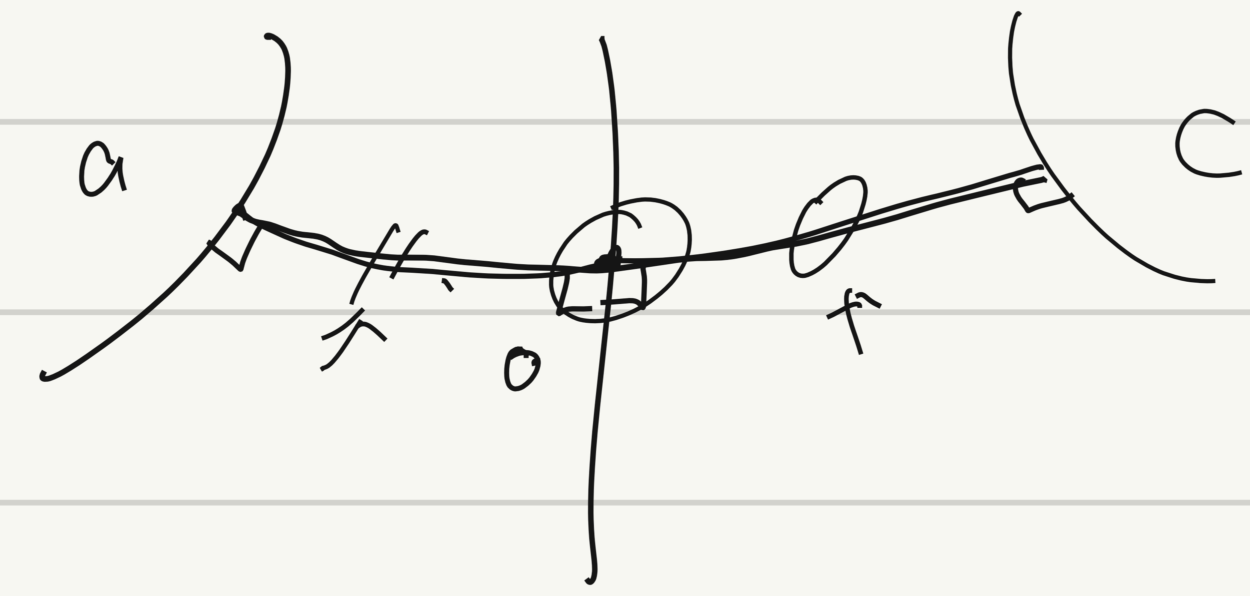




$a \neq b \neq c \neq d$

$\mathbb{R}^4$ $c = d.$

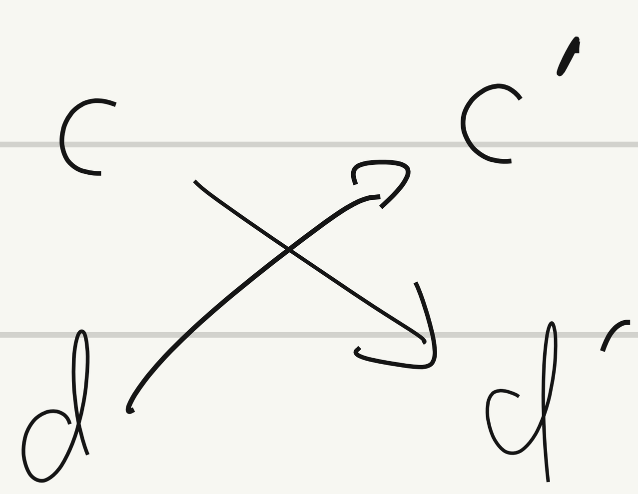
$a = a'$
$b = b'$
$c = c'$
$d = d'$

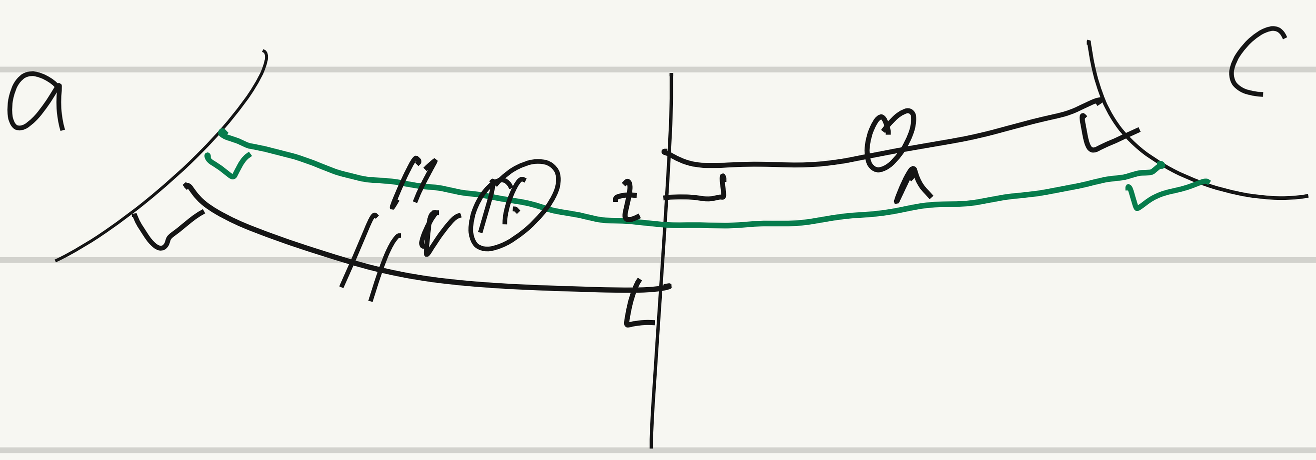


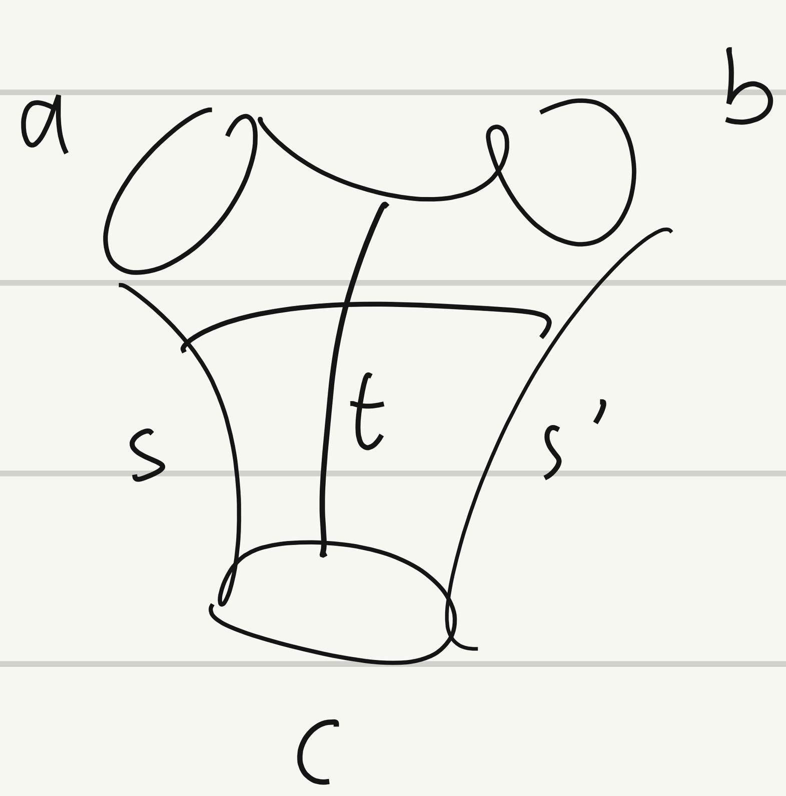
$\boxed{c = d}$

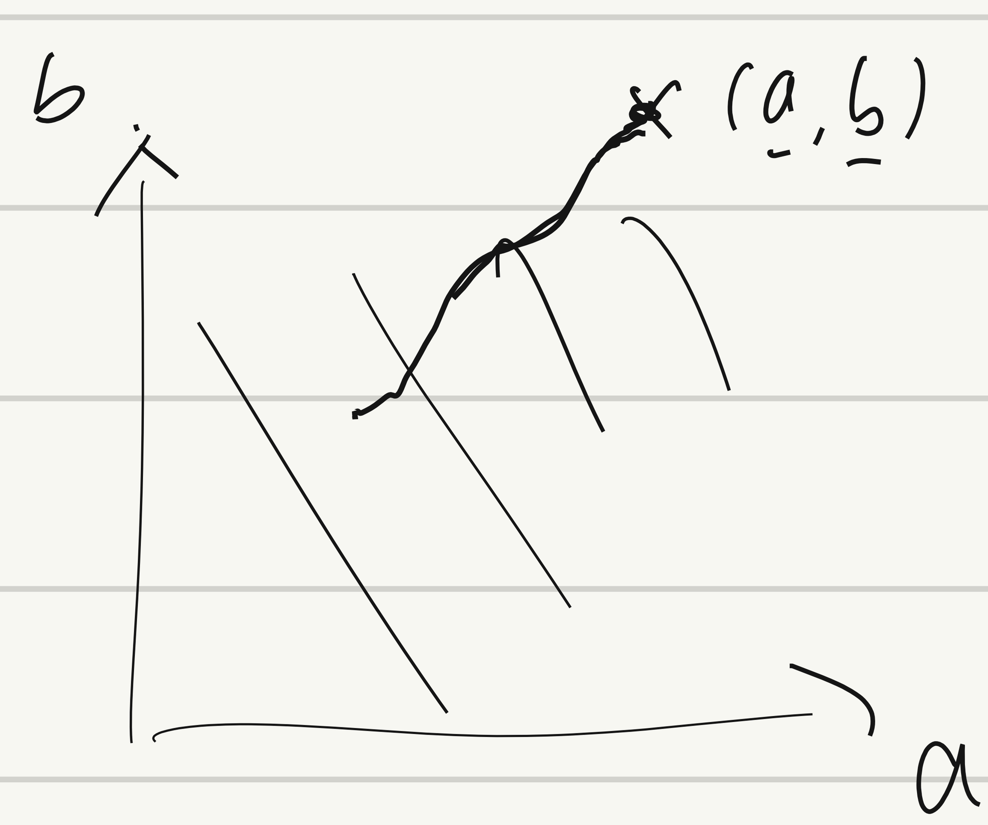
$$t \quad\quad t + \frac{l(\gamma)}{2}$$

$$S_{0,4} \cong S'_{0,4}$$





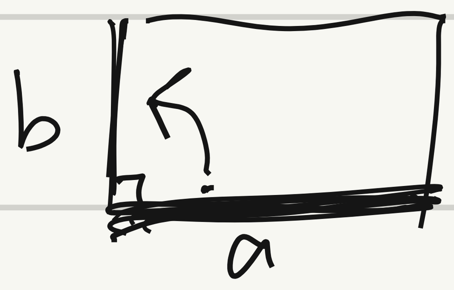




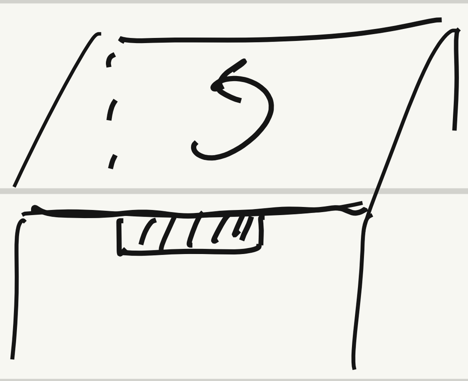
$\min\{s, s', t\} : \mathbb{R}^3_{>0} \to \mathbb{R}.$

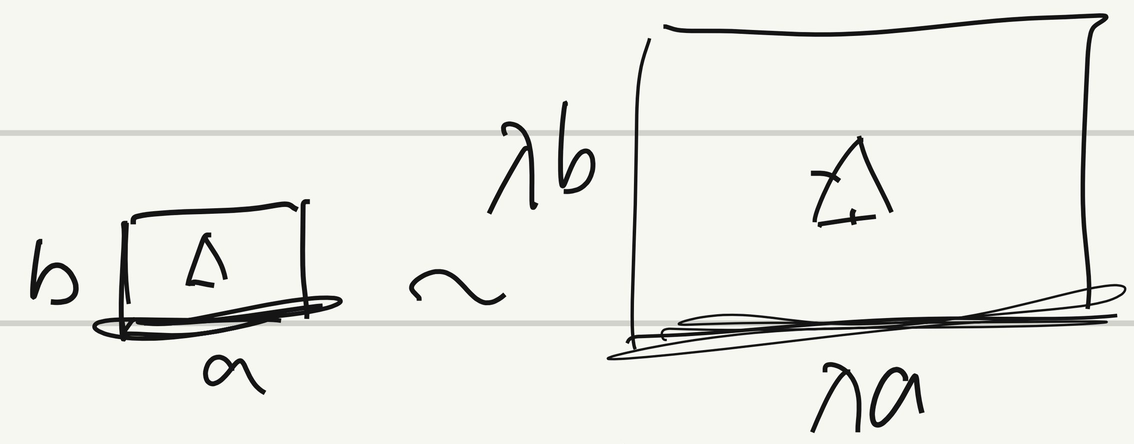
VIII Teichmüller Space and Mapping Class Group

1. Euclidean metric on parallelogram



① orientation
② marking → determined by (a, b)

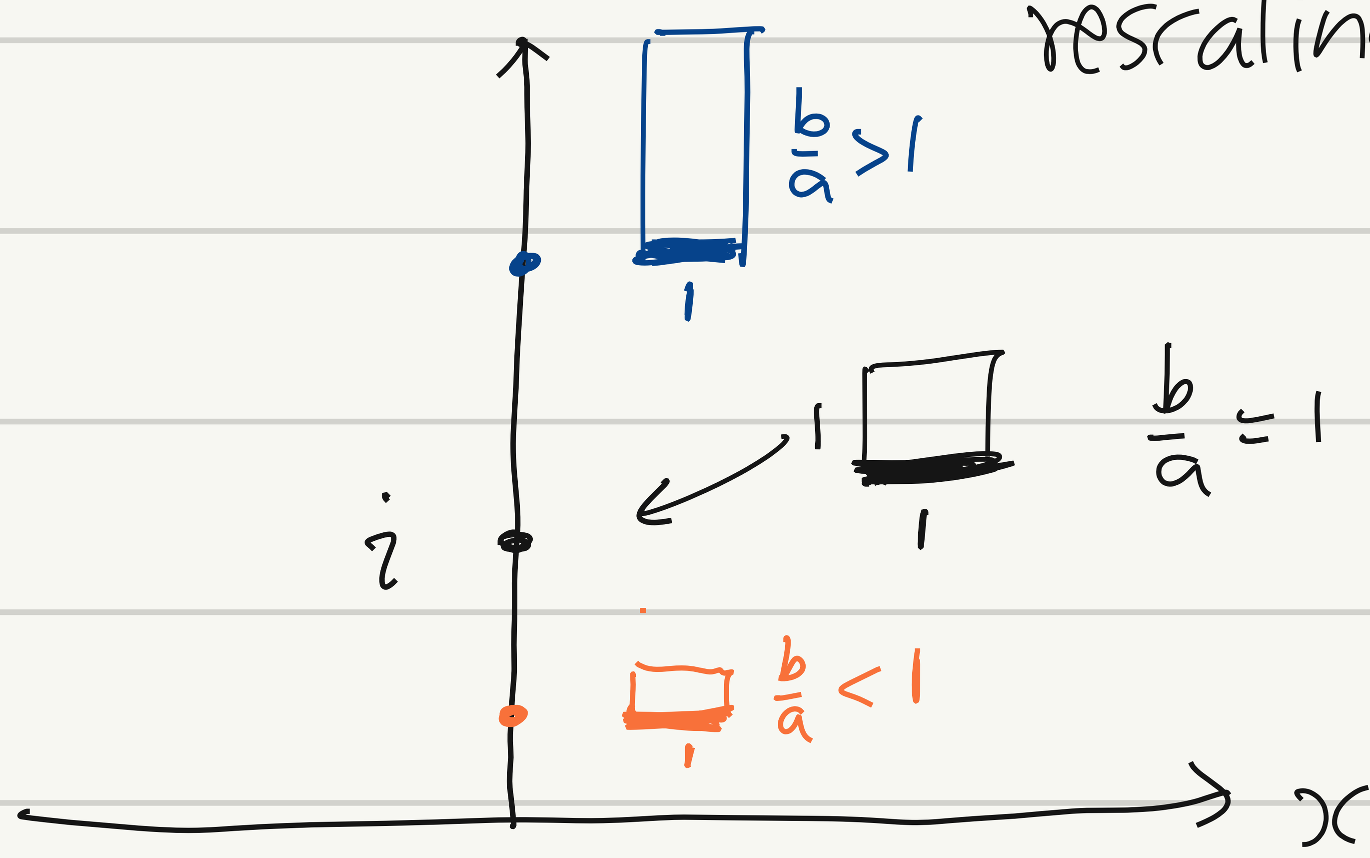




rescaling.

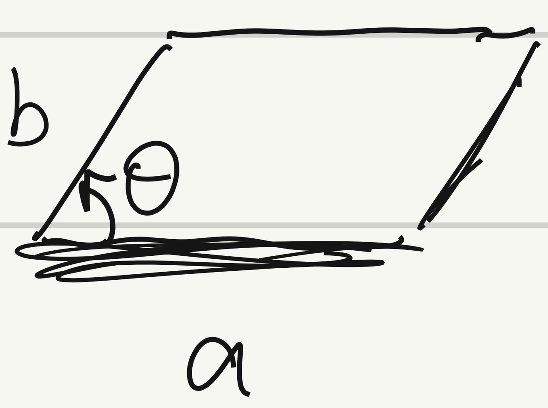
→ determined by $(1, \frac{b}{a})$

$\tau = \frac{b}{a} i$

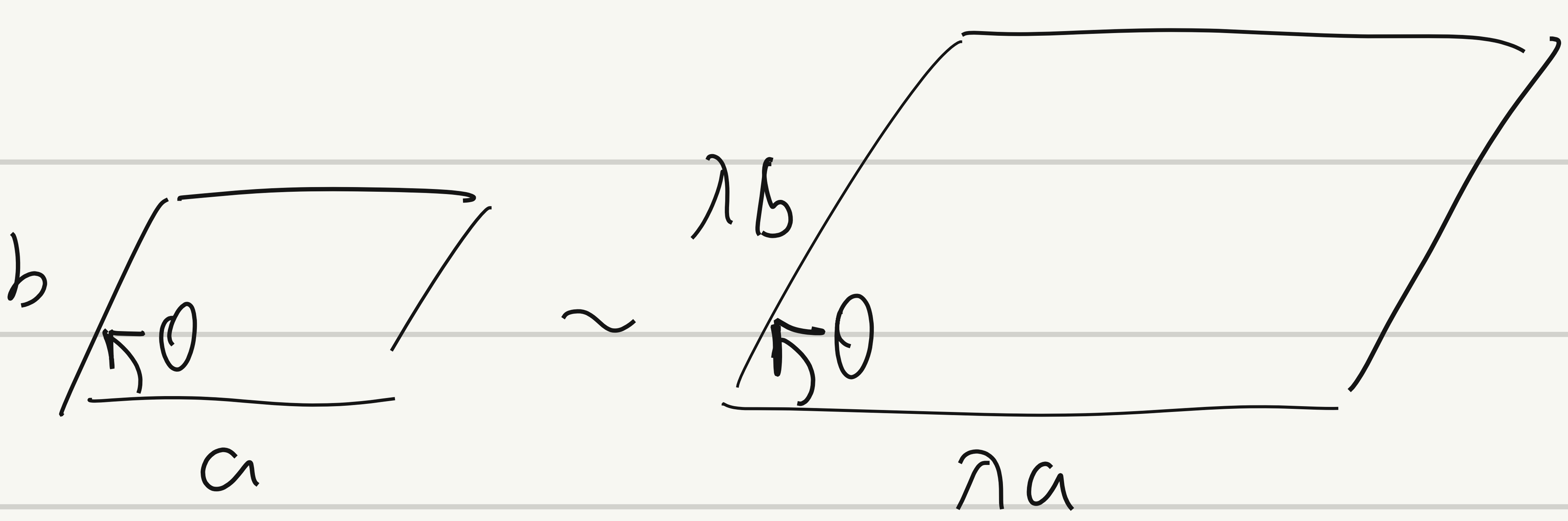


$\{$marked oriented Euclidean metric on $\square\}/\text{rescaling}$

$\cong \{\tau \in \mathbb{C} \mid \tau = iy \quad y > 0\}$



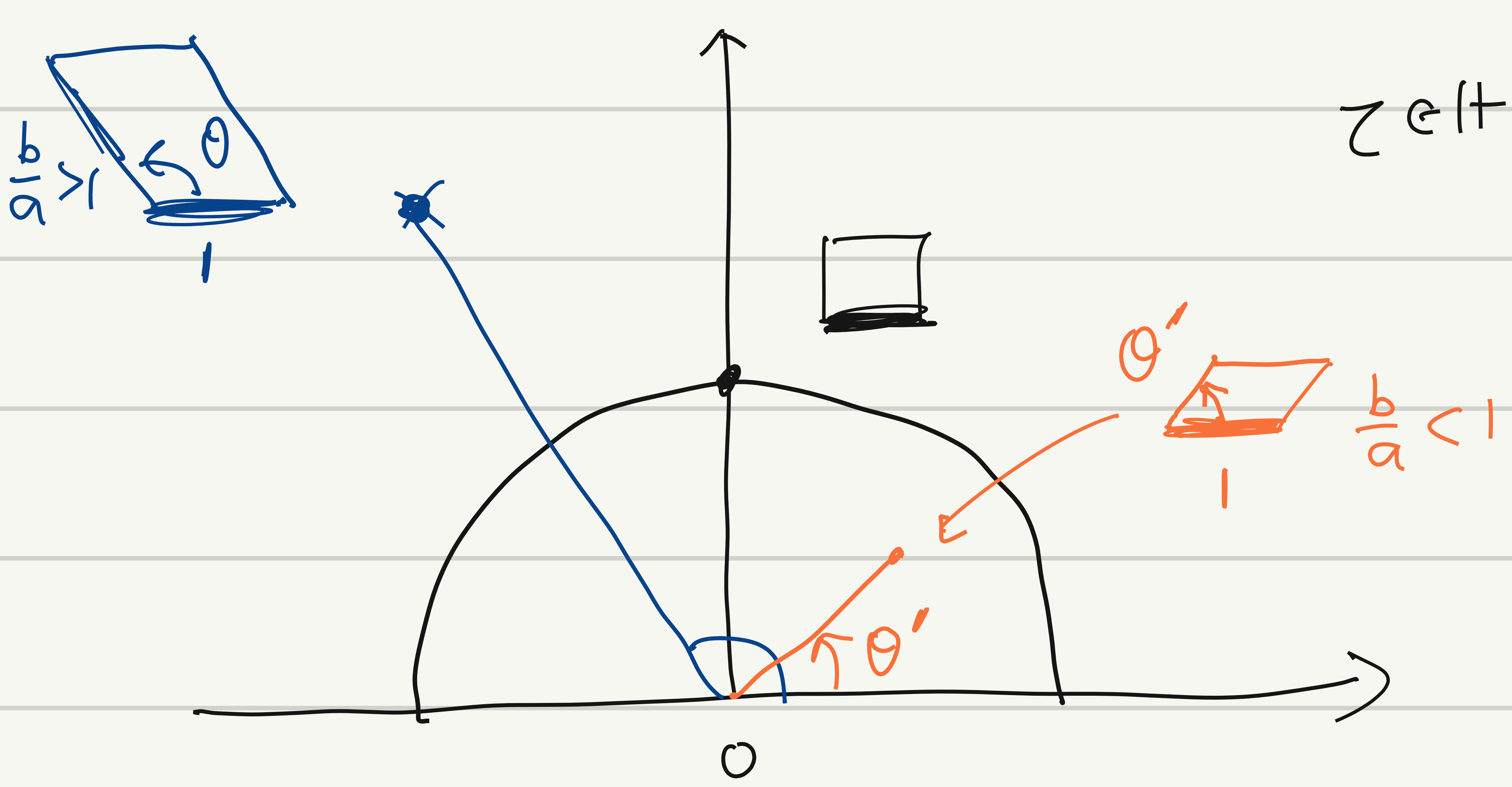
metric determined by (a, b, θ)



metric up to resaling determined by $(1, \frac{b}{a}, \theta)$, $\theta \in (0, \pi)$

$\tau \in \mathbb{H}$!

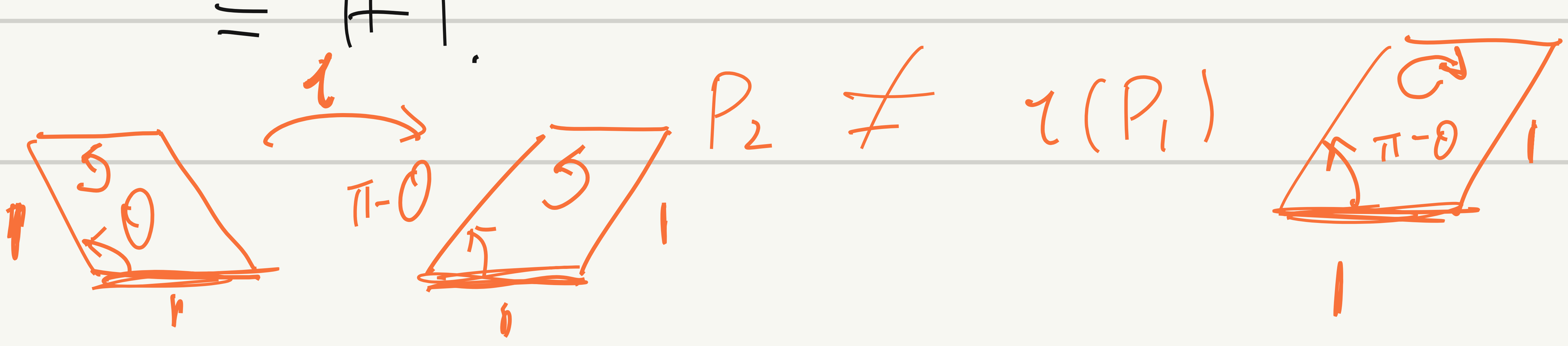
$\tau = \frac{b}{a} \cdot e^{i\theta}$



Conclusion: $\{$marked oriented metric Euclidean parallelogram$\}/\text{rescaling}$

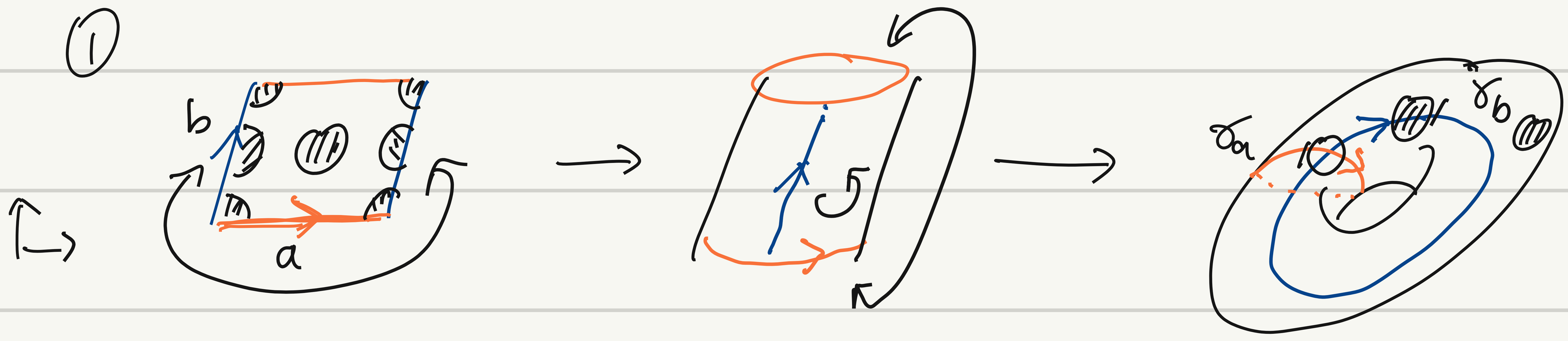
$\cong \mathbb{H}$!

Rmk. P_1 $\xrightarrow{\tau}$ $P_2 \neq \tau(P_1)$

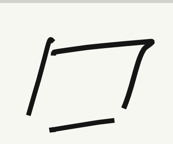


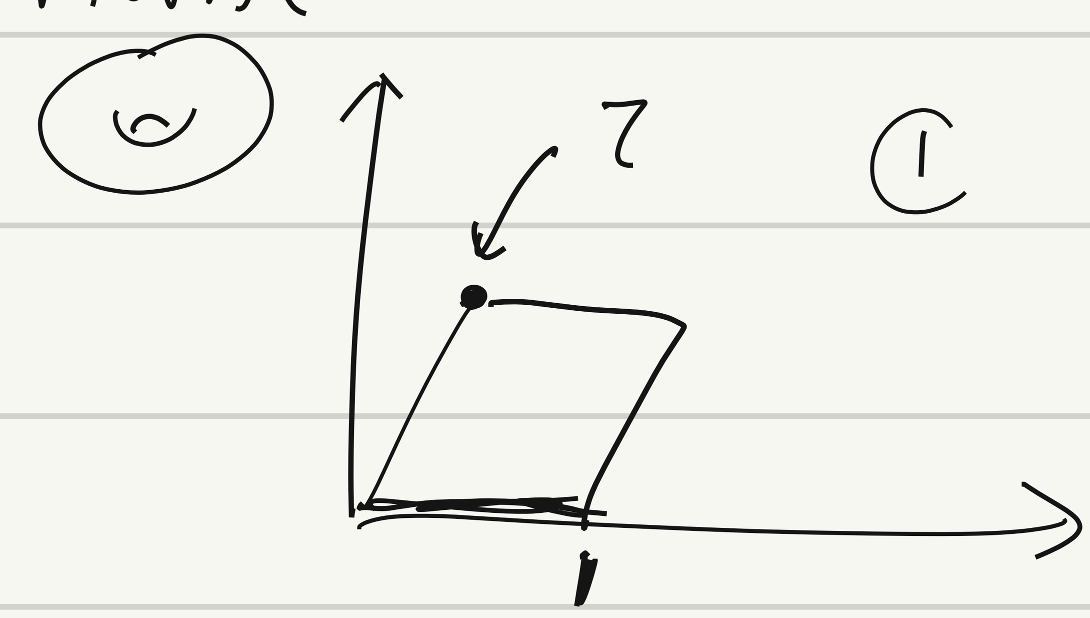
2. Flat torus:

①

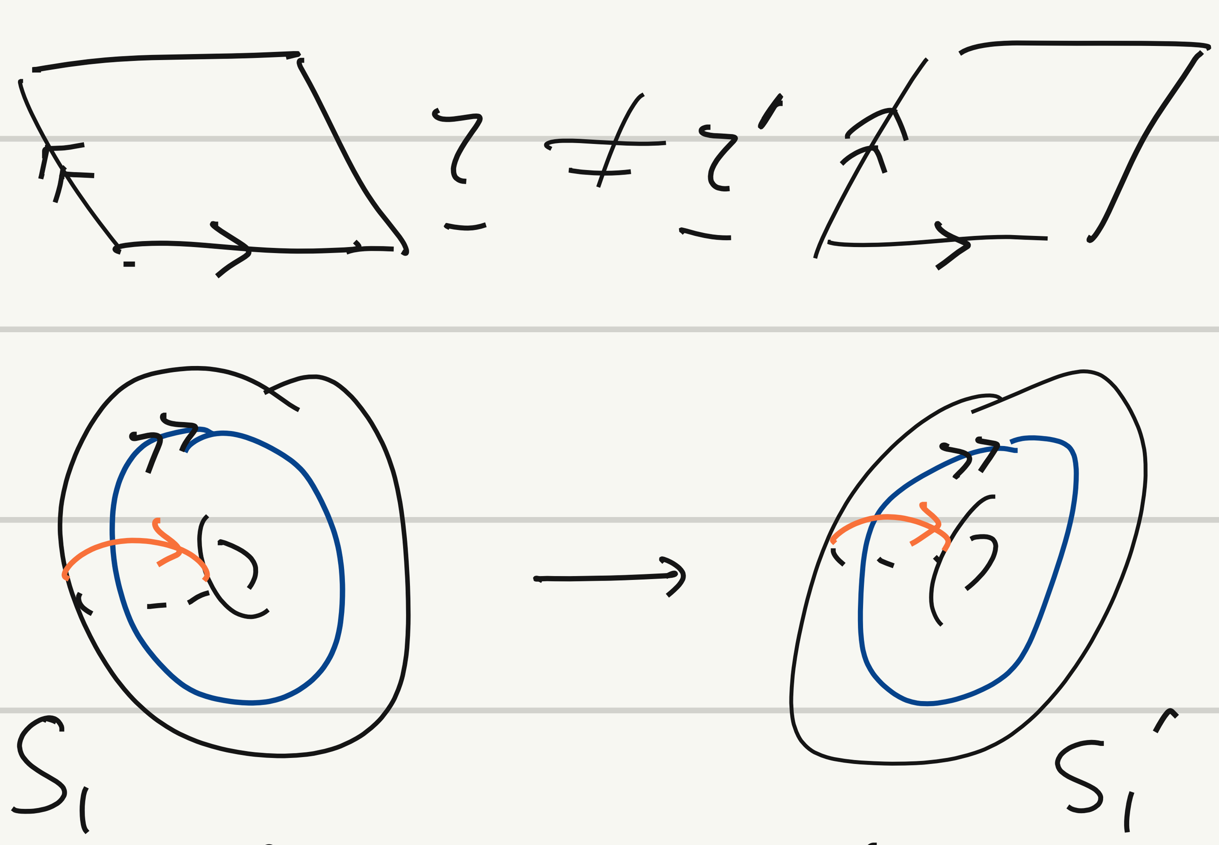


marked oriented Euclidean metric $\rightsquigarrow$ marked oriented Euclidean metric $\Big\}$ (γ_a, γ_b)





②

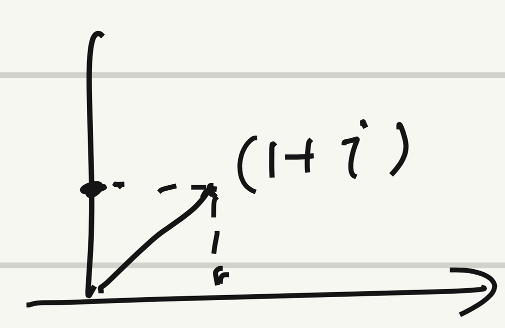


if $f: S_1 \to S_1'$

$f(\bigcirc) = \bigcirc$

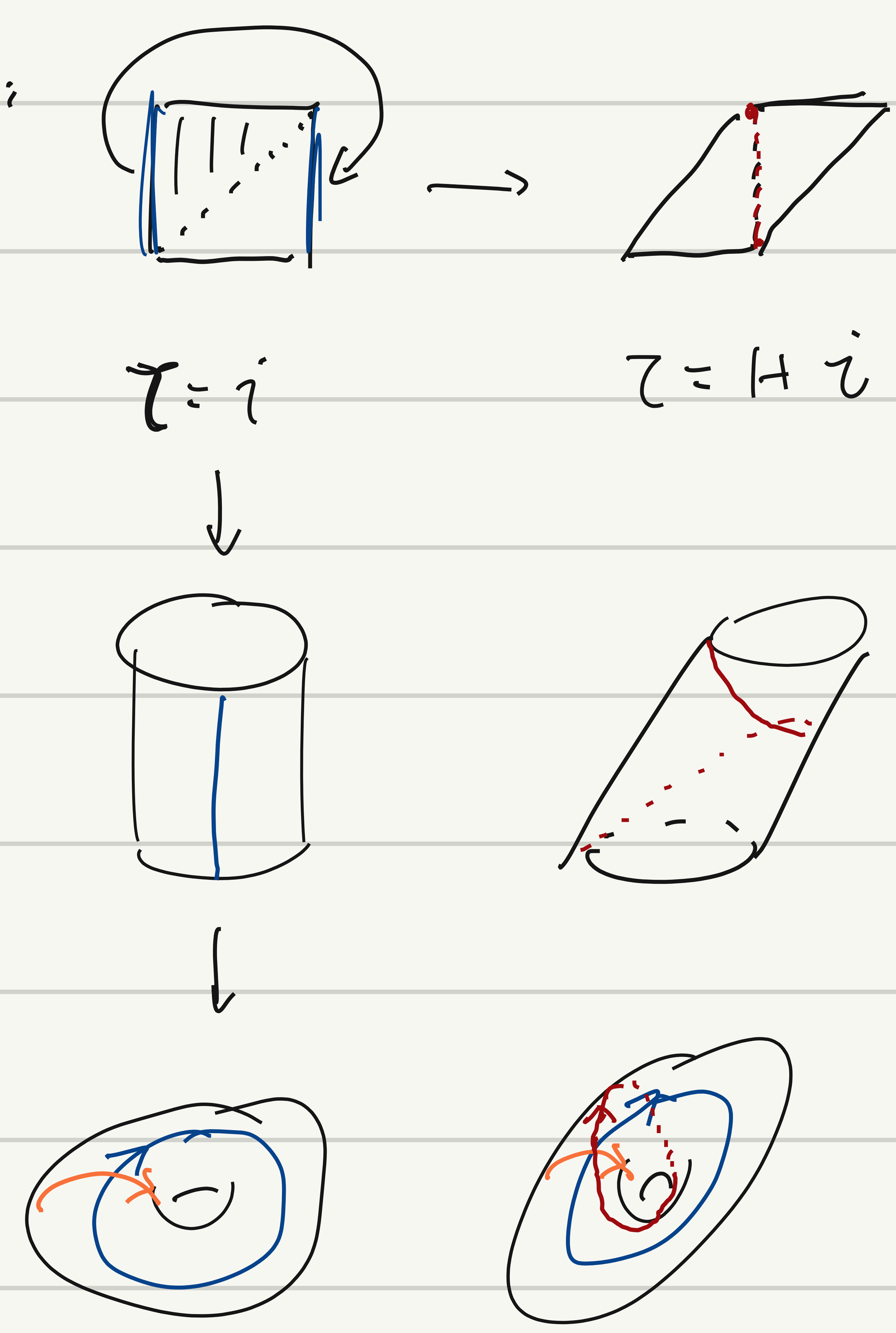
$f(\bigcirc) = \bigcirc$

$\underline{f}$ is not a isometry.



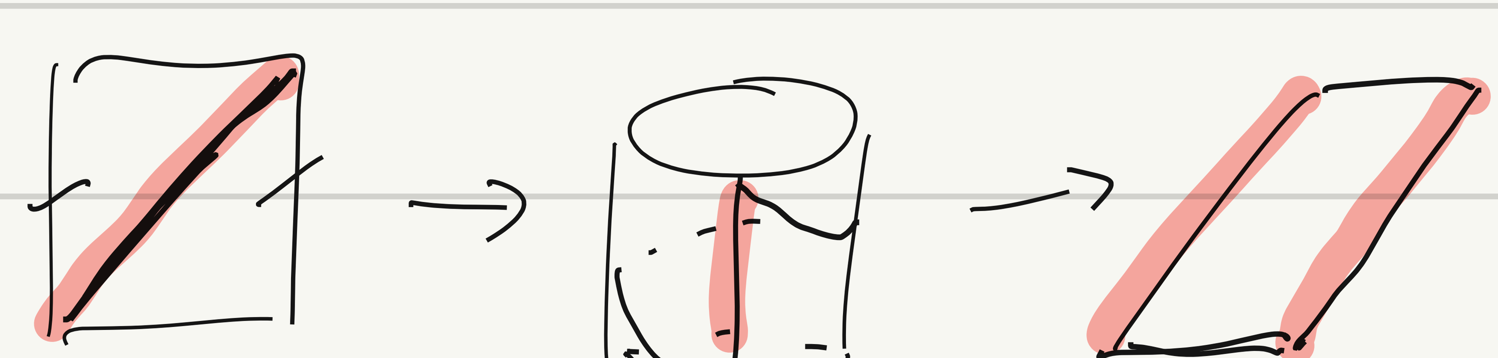
Q: ∃? isometry.

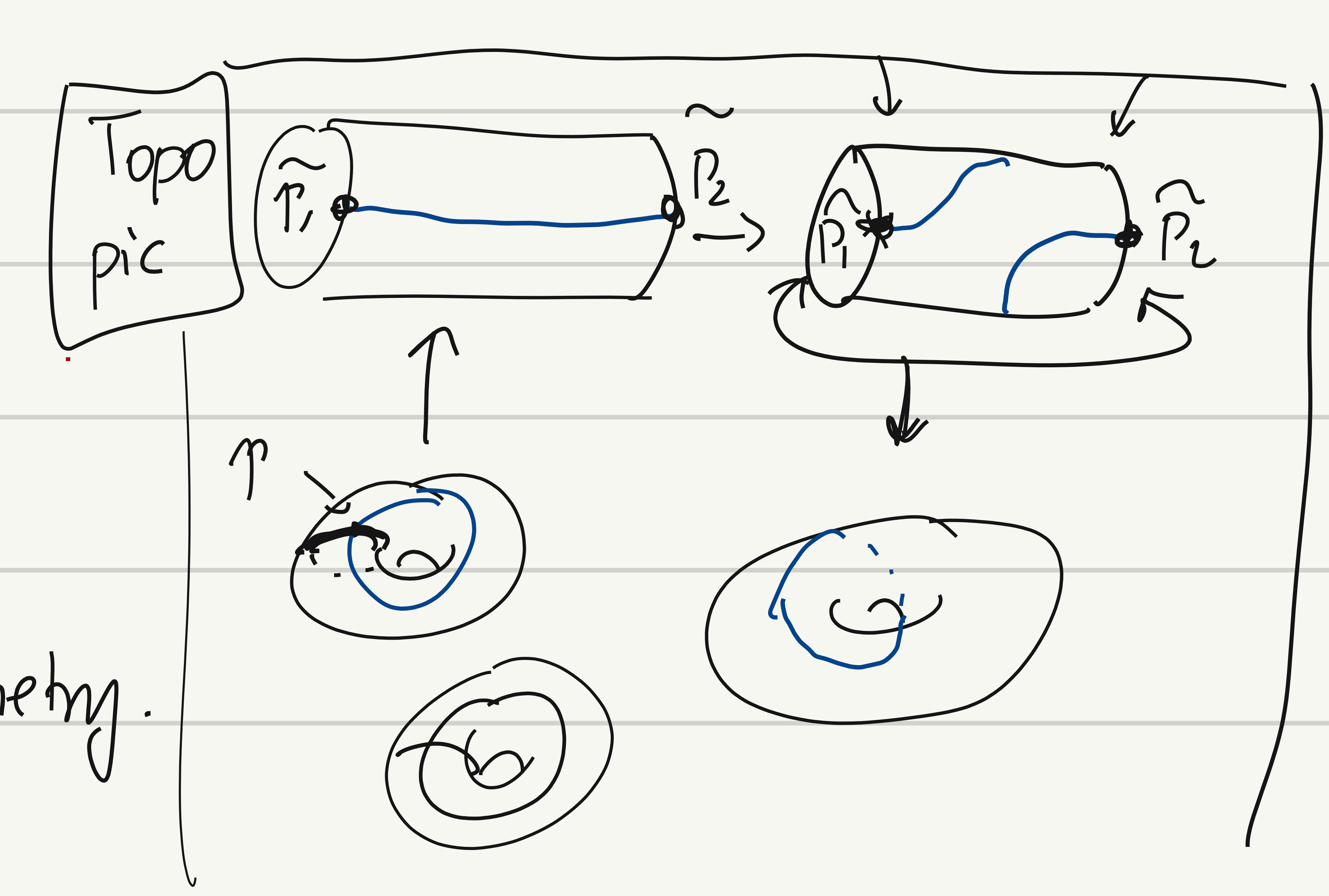
Ex:

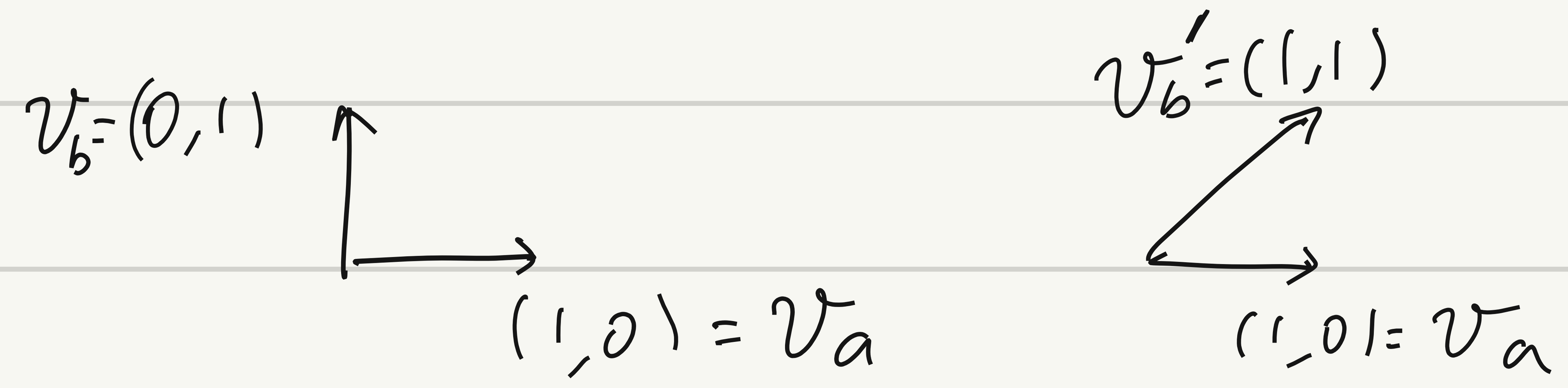


f_1: $\bigcirc \to \bigcirc$, $\bigcirc \to \bigcirc$ is isometry.

$f_1 \not\sim f$ not homotopic to each other.







$(v_a, v_b) \qquad (v_a, v_b')$

$v_a : \mathbb{R}^2 \to \mathbb{R}^2$
$(x,y) \mapsto (x,y) + v_a$

$v_b : \mathbb{R}^2 \to \mathbb{R}^2$
$(x,y) \mapsto (x,y) + v_b$

$\Gamma = \langle v_a, v_b \rangle \cong \mathbb{Z}^2$

$\Gamma' = \langle v_a, v_b' \rangle \cong \mathbb{Z}^2$, $v_b' = v_a + v_b$

$\Gamma = \Gamma' < \mathrm{Isom}^+(\mathbb{R}^2)$

$(\Gamma, \underbrace{(v_a, v_b)}_{=m}) \neq (\Gamma', \overbrace{(v_a, v_b')}^{=m'})$

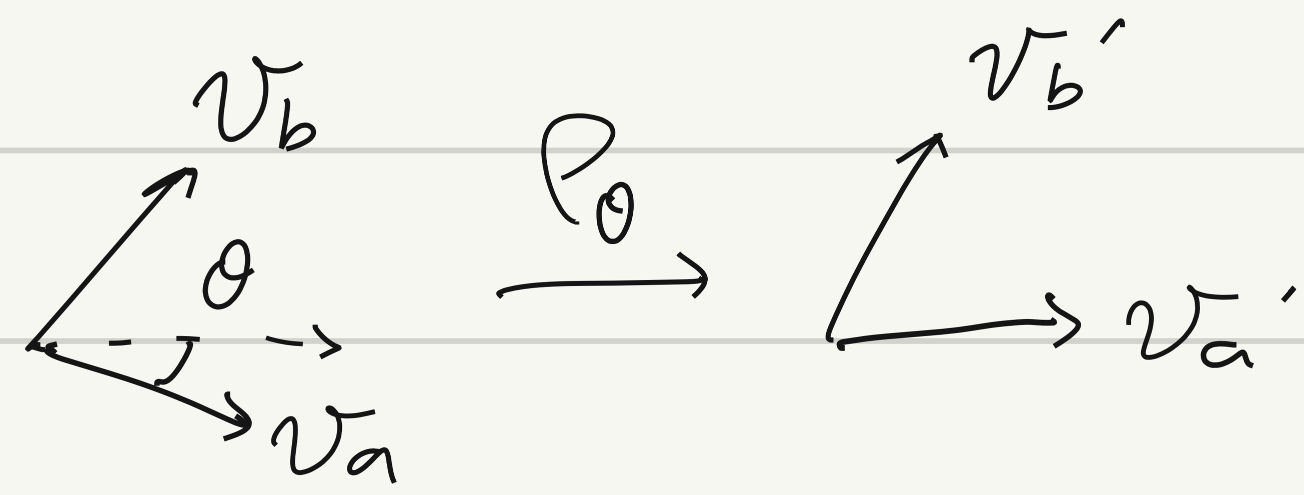
$\underline{\mathcal{T}(T)} := \{ (\Gamma, m) \mid \Gamma = \langle v_a, v_b \rangle, \ m = (v_a, v_b) \text{ marking} \} / \text{conj \& rescaling}$

discrete torsion free.

Teichmüller space of T

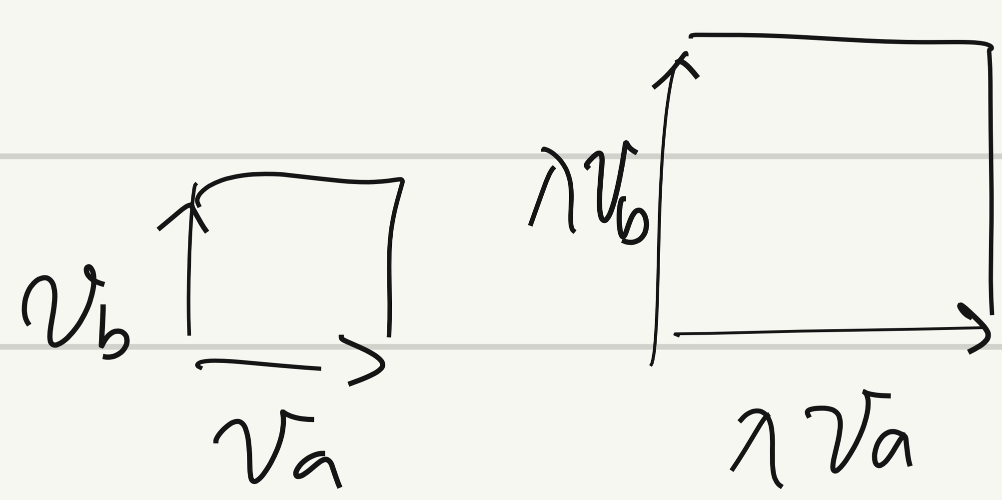
T : topo torus.

conj :



$v_a' = \rho_\theta v_a \rho_\theta^{-1} \qquad \Gamma' = \rho_\theta \Gamma \rho_\theta^{-1}$
$v_b' = \rho_\theta v_b \rho_\theta^{-1} \qquad \Gamma' \sim \Gamma$

rescaling :



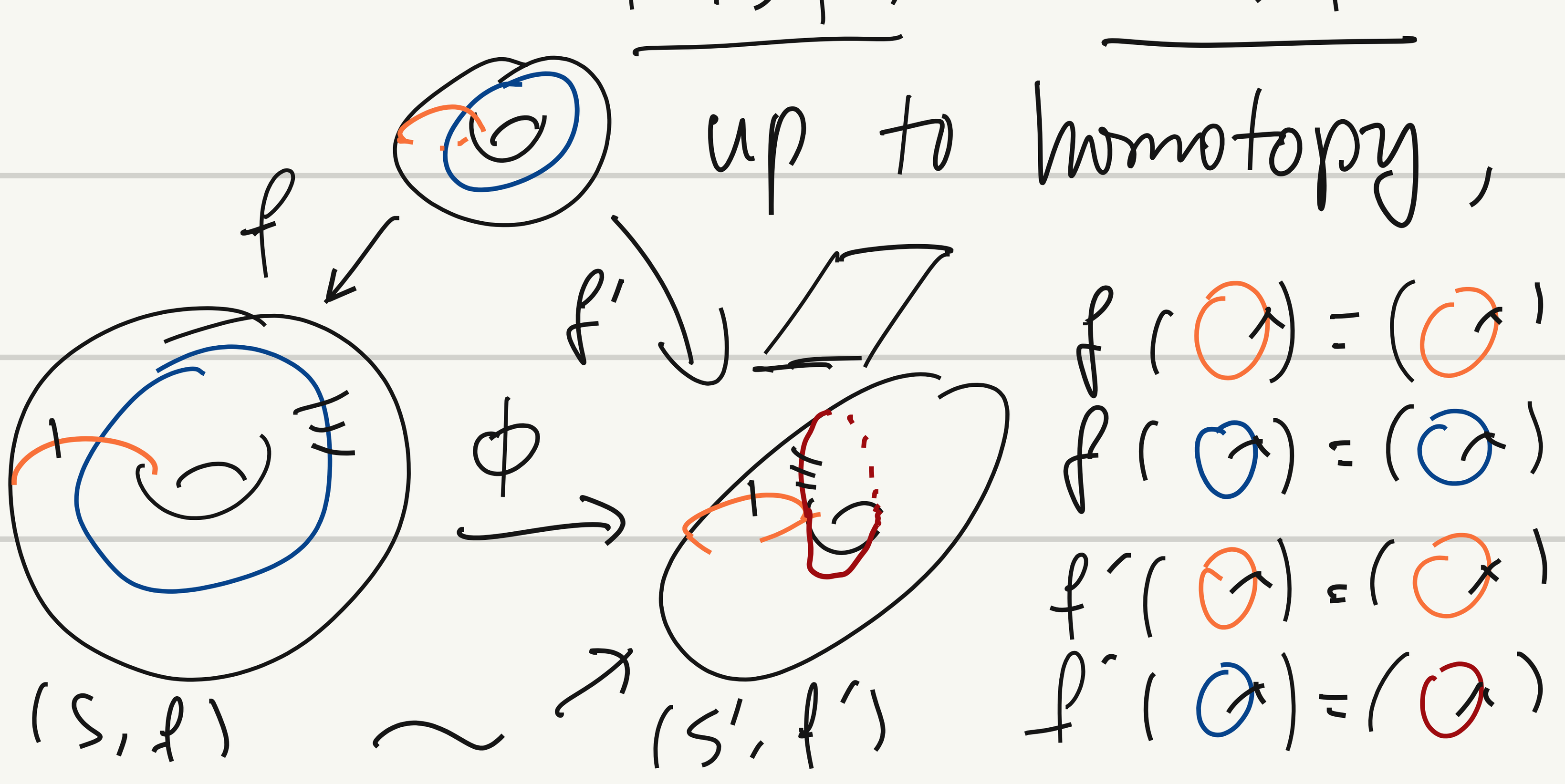
$\lambda > 0$

$\Gamma \sim \Gamma_\lambda = \langle \lambda v_a, \lambda v_b \rangle$

$\mathcal{T}(T) := \{ (S_1, f) \mid S_1 \text{ flat torus}, \ f : T \to S_1 \text{ orien preserving homeo} \} / \sim$

$\underline{(S_1, f)} \sim \underline{(S_1', f')}$ if $\exists \phi : S_1 \to S_1'$ isometry.

up to homotopy,



$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ S_1 \arrow[rr, "\phi"'] & & S_1' \end{tikzcd}$$

$\pi_1(T) = \langle a, b \mid \underset{\text{id.}}{[a, b]} \rangle$

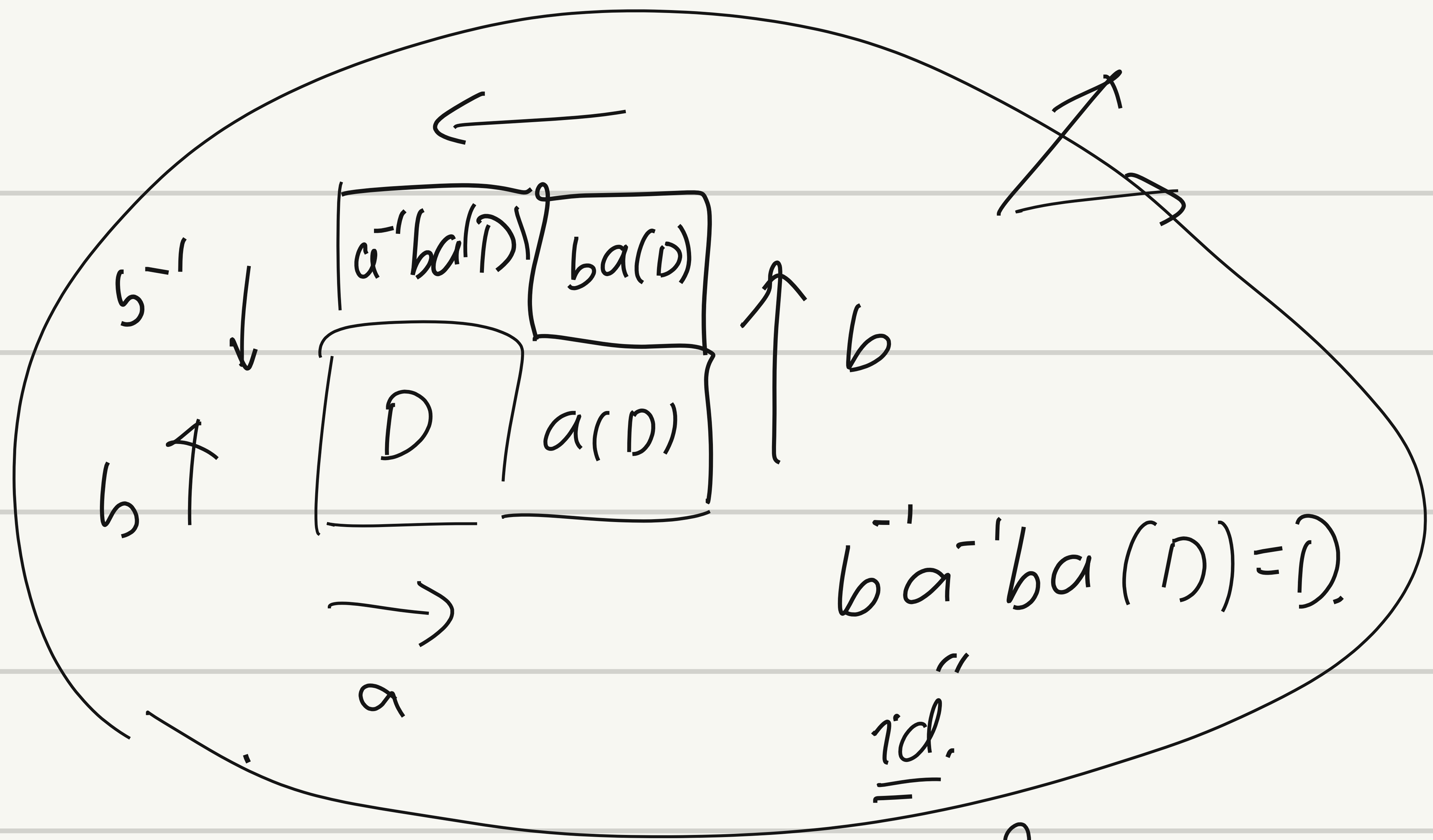
(S_1, f) $\quad \rho : \pi_1(T) \to \mathrm{Isom}^+(\mathbb{R}^2)$

$a \mapsto v_a$

$b \mapsto v_b.$

(Γ, m) $\quad \Gamma = \mathrm{Im}(\rho) < \mathrm{Isom}^+(\mathbb{R}^2)$

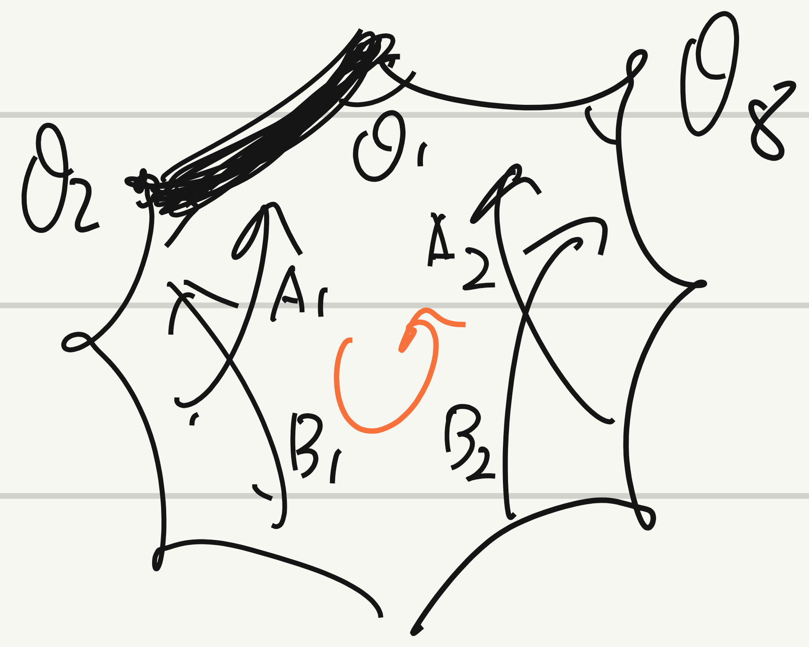
$m = (v_a, v_b)$

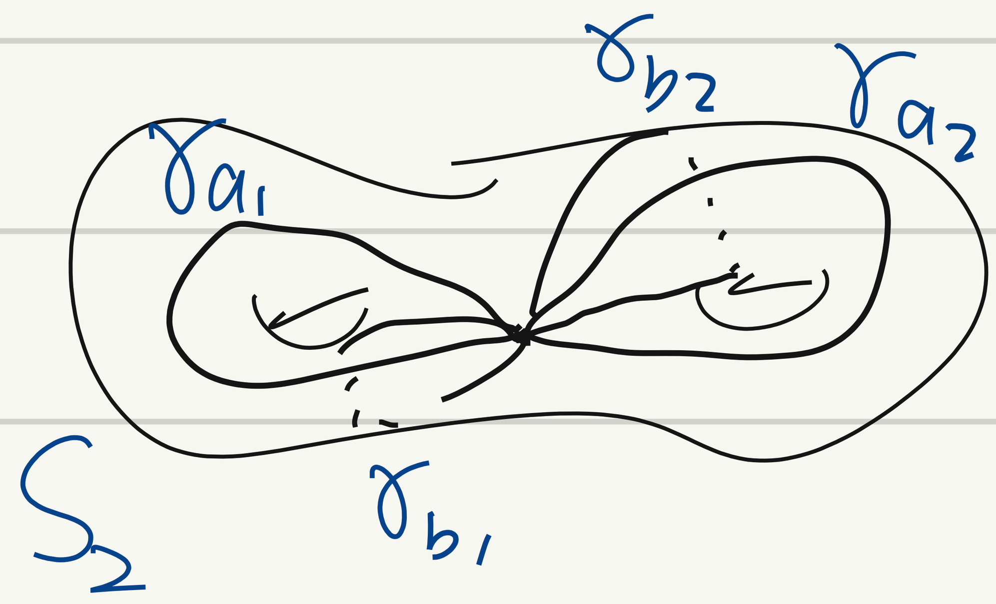


$\mathcal{T}(T) := \{ \rho : \underset{\text{inj.}}{\overset{(a,b)}{\pi_1(T)}} \to \mathrm{Isom}^+(\mathbb{R}^2)$ discrete faithful $\} /$ conj & rescaling

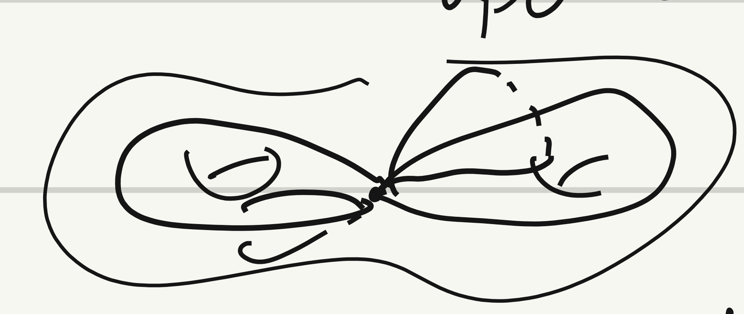
homomorphism.

3. Hyperbolic surface. (no more rescaling)





topo Σ_2



$\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] \rangle$

$\sum_{j=1}^{8} \theta_j = 2\pi$

(, side ordered, orientation)

$(\Gamma, (A, B, A_2, B_2))$

$(S_2, f : \Sigma_2 \to S_2)$

$a_1 \to \gamma_{a_1}$

$b_1 \to \gamma_{b_1}$

$a_2 \to \gamma_{a_2}$

$b_2 \to \gamma_{b_2}$

$(\rho : \pi_1(\Sigma_2) \to \mathrm{Isom}^+(\mathbb{H}))$

(a_1, b_1, a_2, b_2)

Σ_g topo surface of genus g

$\mathcal{T}(S_g) := \{ \Gamma = \langle A_1, B_1, A_2, B_2 \rangle$ discrete torsion free.

$\mathbb{H}/\Gamma \cong \Sigma_2$ homeo $\} /$ conj

$\exists M \in \mathrm{Isom}^+(\mathbb{H}) \quad M \Gamma M^{-1} \sim \Gamma$

$$\mathcal{T}(\Sigma_g) := \{ \rho : \underset{(a, b, a_2\, b_2)}{\pi_1(\Sigma_g)} \longrightarrow \mathrm{Isom}^+(\mathbb{H}) \cong PSL(2,\mathbb{R})$$

$$\text{discrete faithful} \} / \text{conj}.$$

$$\rho : \pi_1(\Sigma_g) \longrightarrow PSL(2,\mathbb{R})$$
$$a \longmapsto \rho(a).$$

$M \in PSL(2,\mathbb{R})$ $\quad M\rho M^{-1}: \pi_1(\Sigma_g) \xrightarrow{\rho} PSL(2,\mathbb{R}) \xrightarrow{M} PSL(2,\mathbb{R})$

$$a \longmapsto \rho(a) \longmapsto M\rho(a)M^{-1}$$

$$M\rho M^{-1} \sim \rho$$

$$\mathcal{T}(\Sigma_g) := \{ (\underline{\underline{S_g}}, f) \mid f : \Sigma_g \to S_g \text{ homeo} \} / \sim$$

marked hyp str. on Σ_g

$(S_g, f) \sim (S_g', f')$ if $\exists\, \phi : S_g \to S_g'$ isom

<u>up to homotopy</u>

$\boxed{\phi \circ f \sim f' \text{ homotopic.}}$

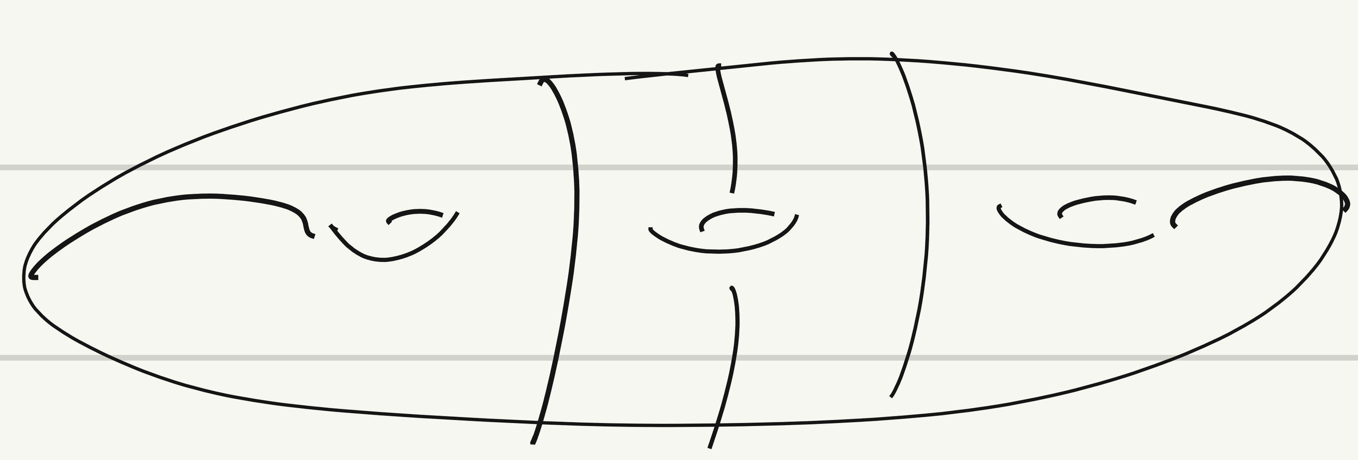
$$\begin{tikzcd} & \Sigma_g \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ S_g \arrow[rr, "\phi"'] & & S_g' \end{tikzcd}$$

$$\mathcal{M}(\Sigma_g) := \{ S_g \mid \text{hyp str on } \Sigma_g \} / \sim_{\text{isom}}$$

Riemann Moduli space.

$S_g \sim S_g'$ if $\exists\, \phi : S_g \to S_g'$ isometry.

Rmk. Complex structure.

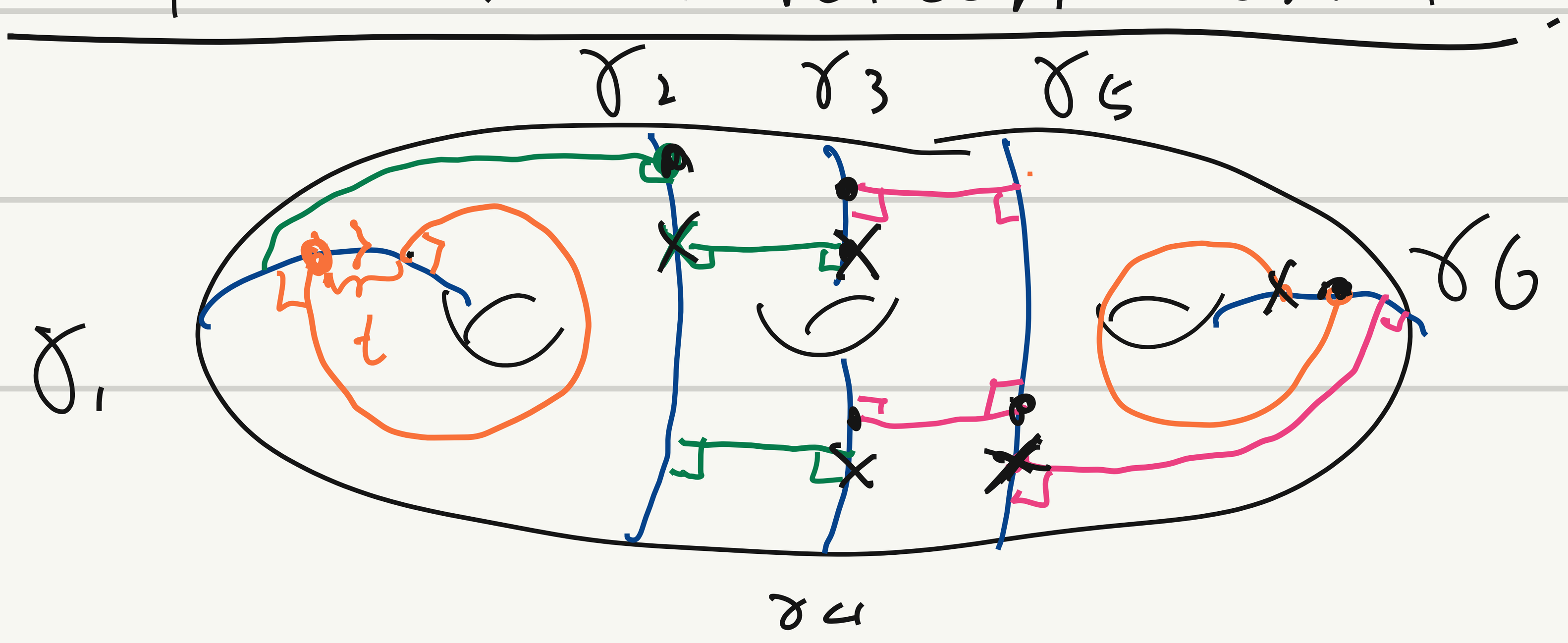


(l_3, t_3)

(l_1, t_1) (l_2, t_2) (l_5, t_5) (l_6, t_6)

(l_4, t_4)

<u>4. Fenchel - Nielsen coord</u>:



$FN: \mathcal{T}(\Sigma_g) \longrightarrow \mathbb{R}_{>0}^{3g-3} \times \mathbb{R}^{3g-3}$

$$(S, f) \longmapsto (\underbrace{l_1, \dots, l_{3g-3}}_{\text{length parameter}}, \underbrace{t_1, \dots, t_{3g-3}}_{\text{twist parameter}})$$

Rmk: depend on ① choice of pant decomposition

② base points on each boundary of pants.

Prop: $\mathcal{T}(\Sigma_g) \cong \mathring{\mathbb{B}}^{6g-6}$ $\boxed{\text{homeomorphic}}$

Rmk. Geometry on $\mathcal{T}(\Sigma_g)$ comes from measuring difference of marked hyp str.

(S_g, f) (S_g', f')

$$\begin{tikzcd} & \Sigma_g \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ S_g & & S_g' \end{tikzcd}$$

$\underline{f' \circ f^{-1}}: S_g \longrightarrow S_g'$

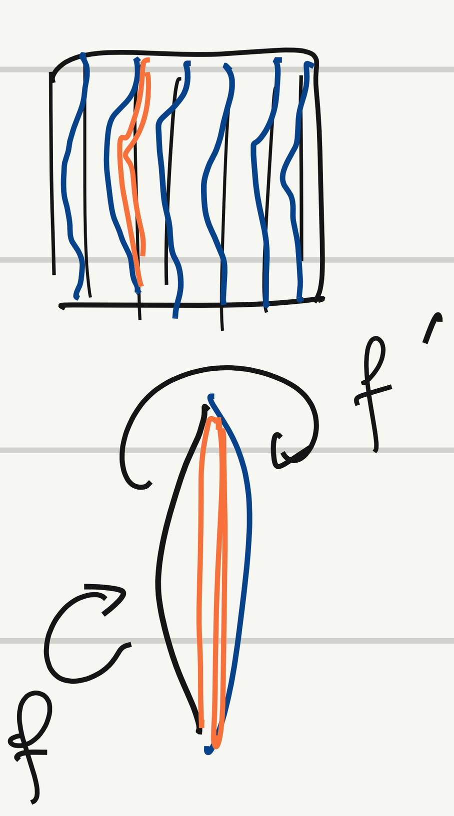
how different is $f' \circ f^{-1}$ from an isometry?

5. Mapping class group

Σ_g topo surface of genus g (closed, i.e. no ∂)
oriented.

$$MCG(\Sigma_g) \text{ or } Mod(\Sigma_g) := \mathrm{Homeo}^+(\Sigma_g) / \mathrm{Homeo}_0^+(\Sigma_g)$$

$\mathrm{Homeo}^+(\Sigma_g) = \{f: \Sigma_g \to \Sigma_g \mid \text{homeo orient preserving}\}$



$\boxed{\mathrm{Homeo}_0^+(\Sigma_g)} := \{f: \Sigma_g \to \Sigma_g \mid \text{homeo orient preserving}, f \sim id. \text{ homotopic}\}$

$f = id$

$f' \sim f$ homotopic.

$H: \Sigma_g \times [0,1] \longrightarrow \Sigma_g$ continuous.

(p, s)

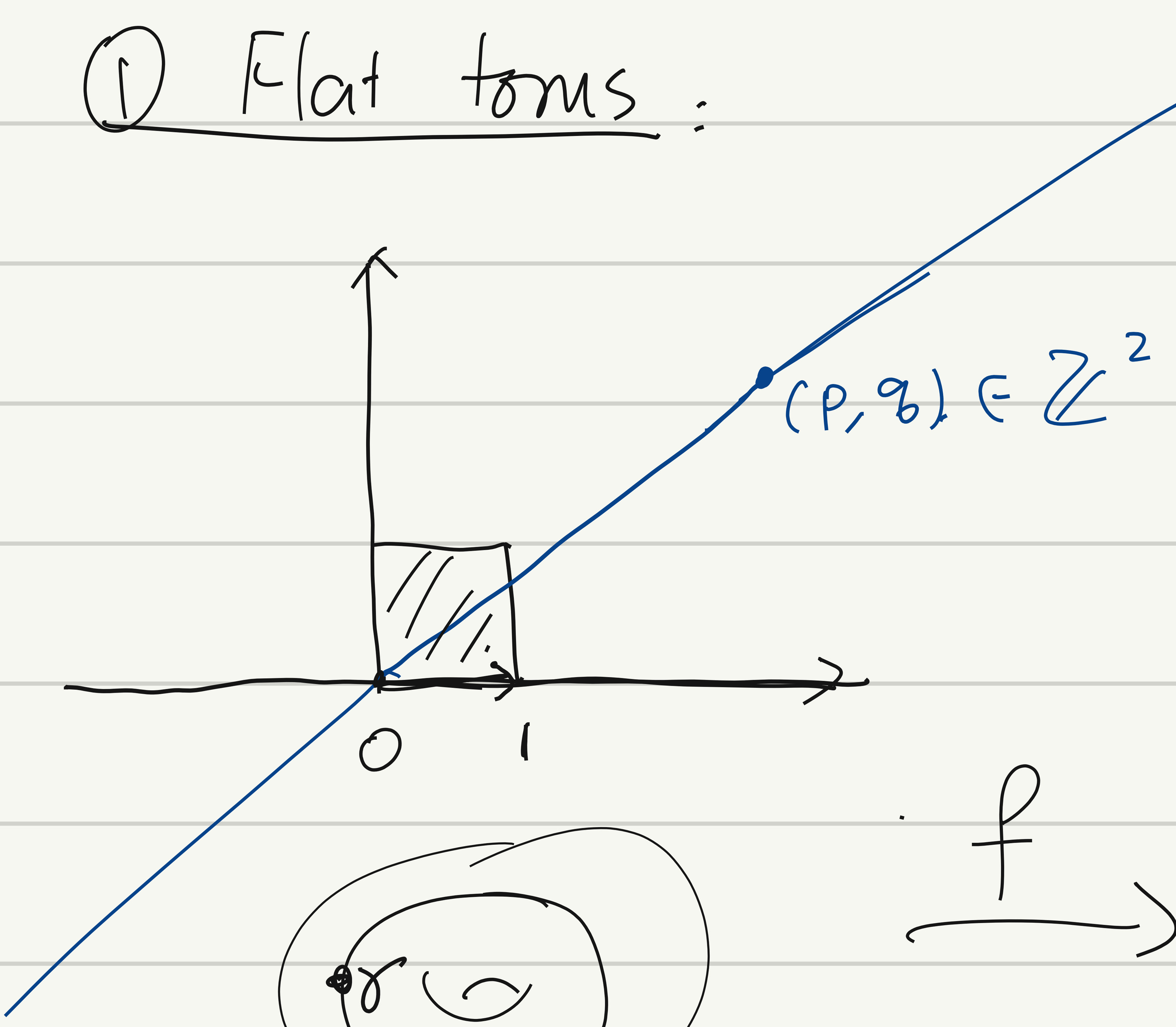
$H(p, 0) = f(p)$ $H(p, 1) = f'(p)$

$f \sim f'$ homotopic

$[f] \in \mathrm{Mod}(\Sigma_g)$

↳ homotopy class of f.

① <u>Flat torus</u>:

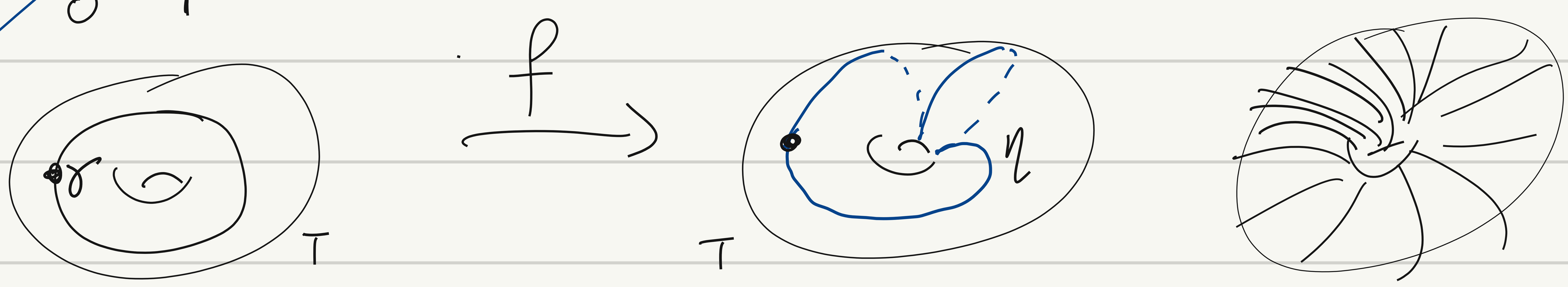


$\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = T$

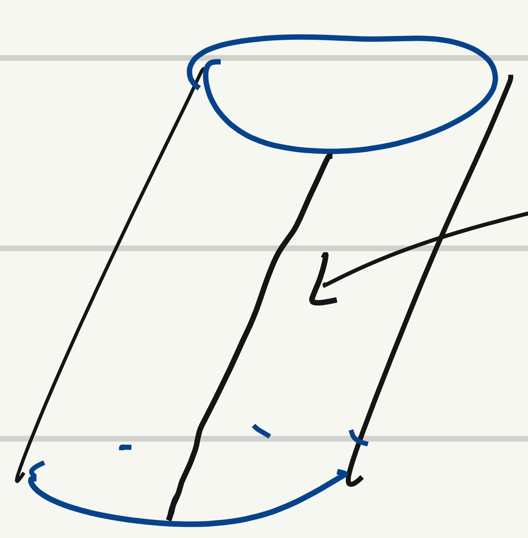
$\underline{L(x,y)}$ $\longrightarrow$ $\gamma_{(x,y)}$ geod in T

$\boxed{[(0,0)\ (x,y)]}$

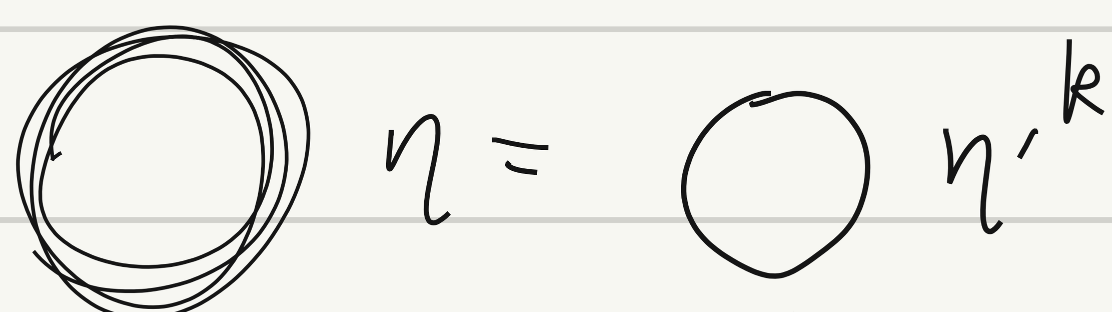
$\gamma_{(x,y)}$ closed. geod.
iff $(x,y) \in \mathbb{Z}^2$.



<u>Rmk</u>: All geod on T are simple.

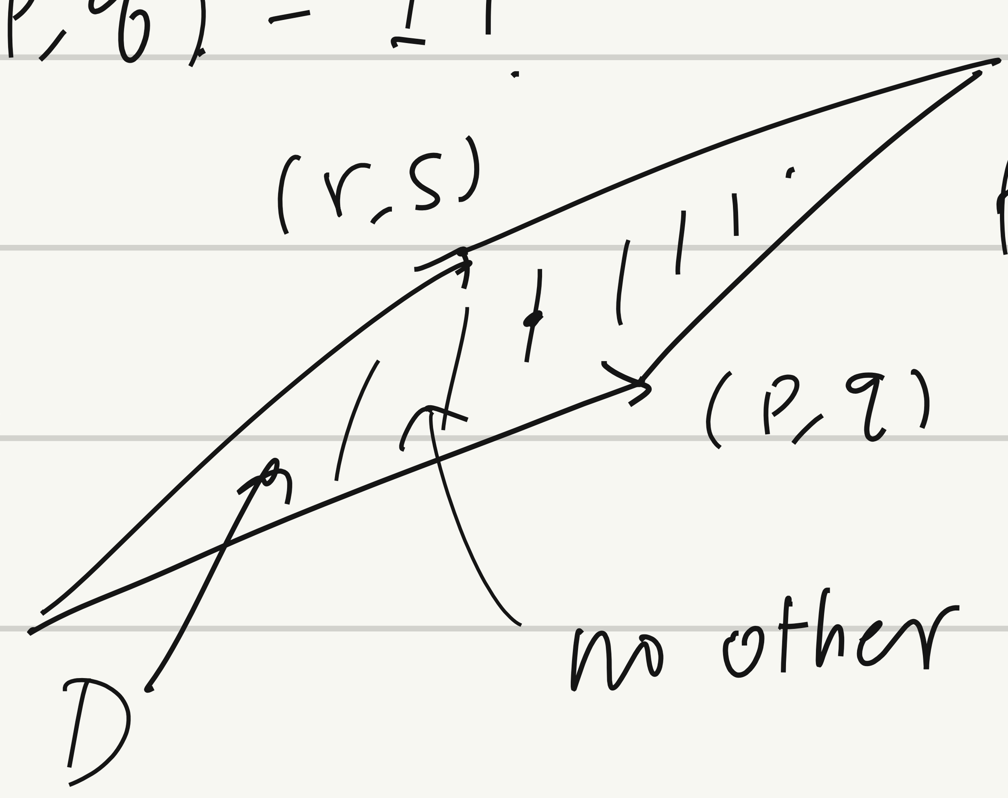


$L(r,s)$ $\quad (r,s) \in \mathbb{Z}^2$.



↳ primitive. (not a integer power of another geod)

<u>Prop</u>: η primitive. iff $\gcd(p,q) = \pm 1$.



fund domain of T.

$\gamma(p,q)$ $\gamma(r,s)$ primitive.

no other $(m,n) \in \mathbb{Z}^2$.

f: $(1,0) \longrightarrow (p,q)$ $\quad A = \begin{bmatrix} p & r \\ q & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$

$(0,1) \longrightarrow (r,s).$ $\quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$

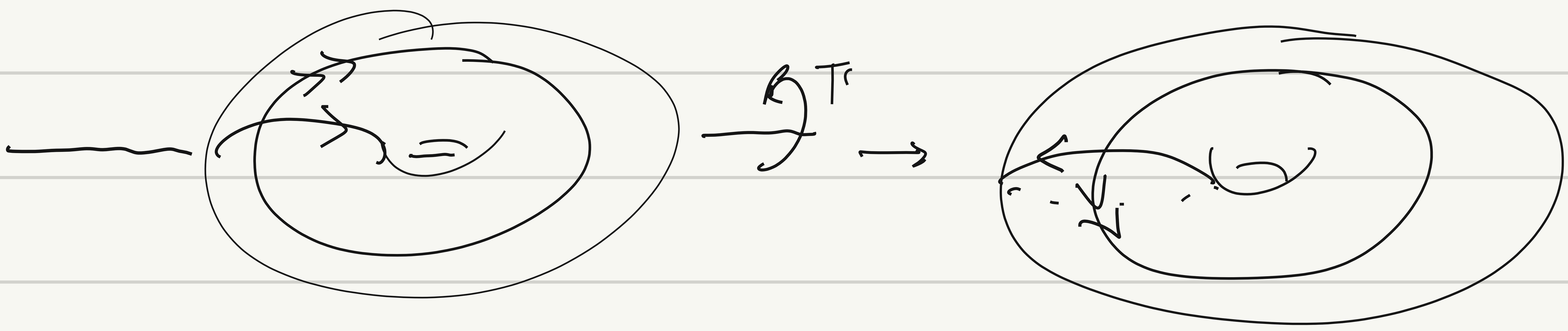
$\mathring{D}$ has no $(m,n) \in \mathbb{Z}^2 \Rightarrow \det A = ps - qr = 1.$

$A \in SL(2,\mathbb{Z})$

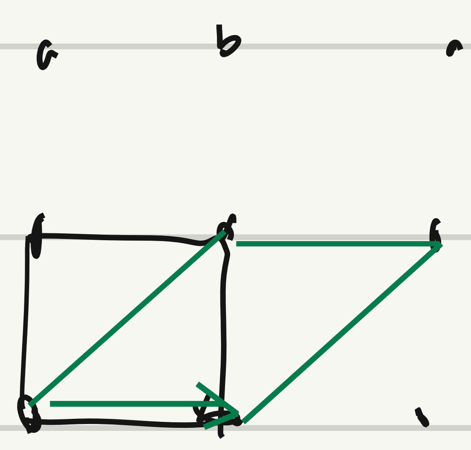
Prop: $\mathrm{Mod}(T) \cong SL(2,\mathbb{Z}) = \left\langle \underset{L}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}, \underset{R}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} \;\middle|\; (LR^{-1})^6 (LR^{-2})^4 \right\rangle$

$SL(2,\mathbb{Z}) := \left\{ \begin{bmatrix} p & r \\ q & s \end{bmatrix} \in M_2(\mathbb{Z}) \;\middle|\; ps - qr = 1 \right\}$

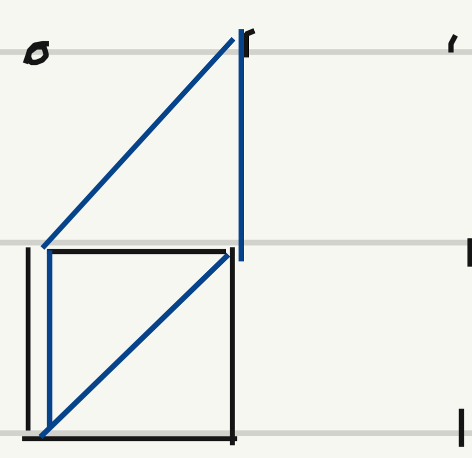
$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in SL(2,\mathbb{Z}) \rightsquigarrow$ elliptic involution.



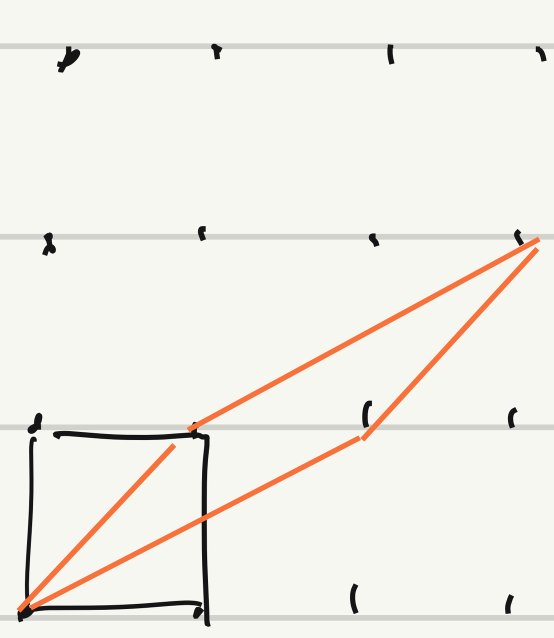
$L: \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$

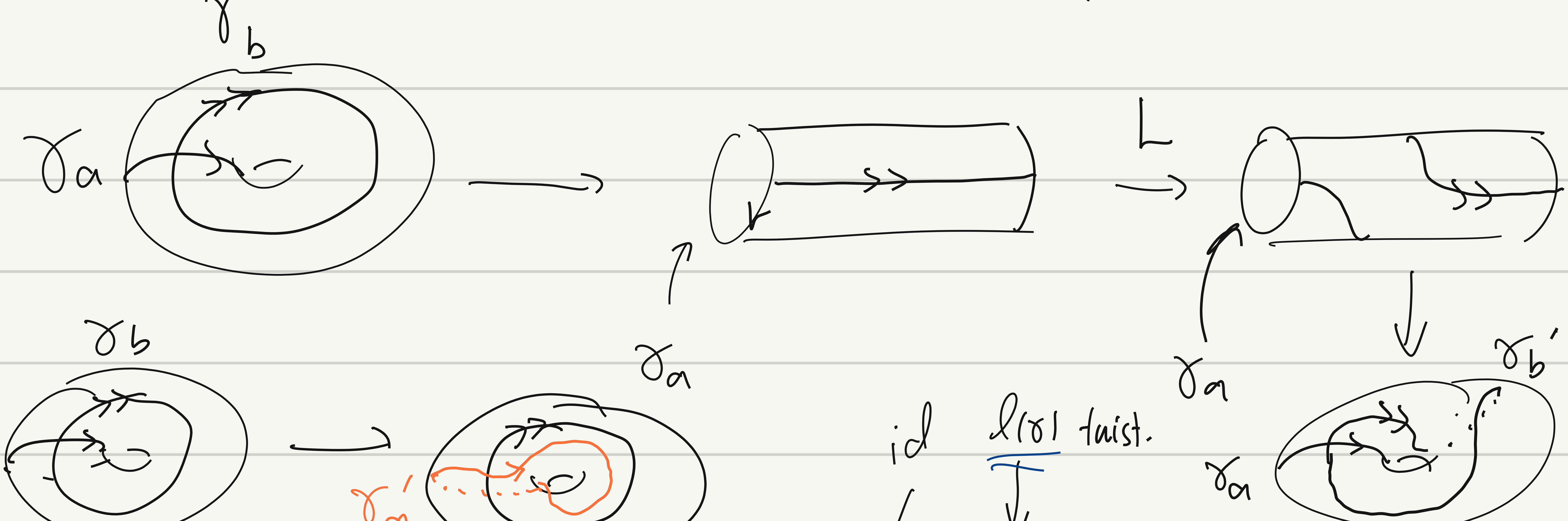


$R: \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x+y \end{bmatrix}$

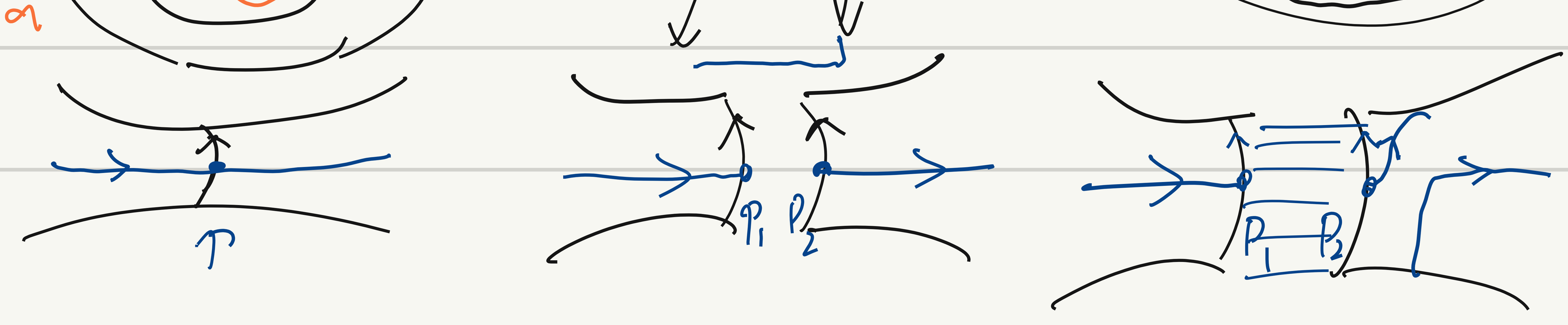


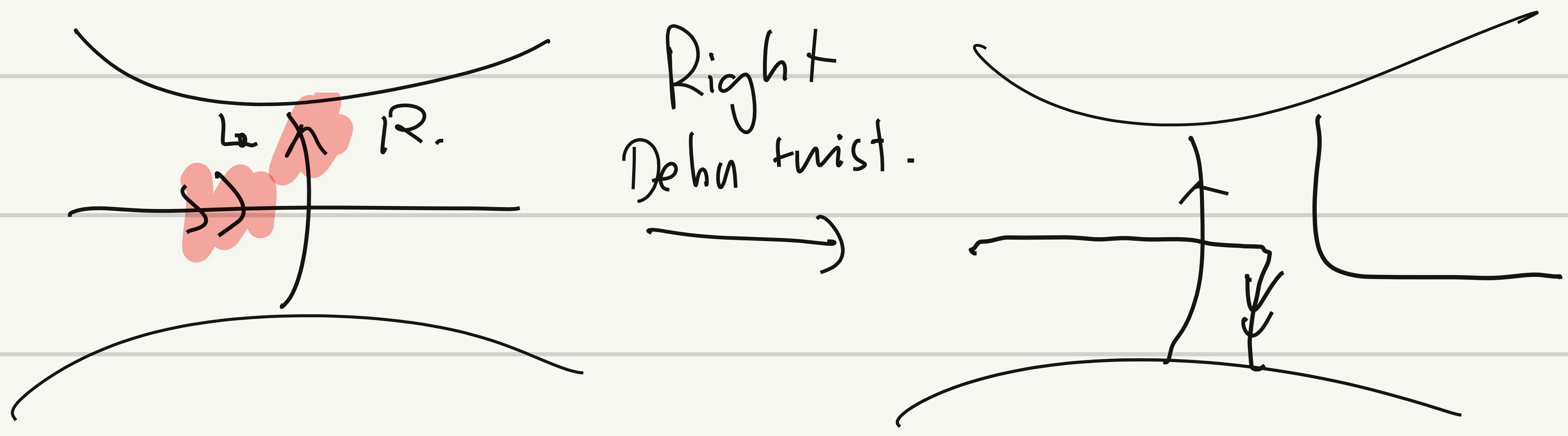
$LR: \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

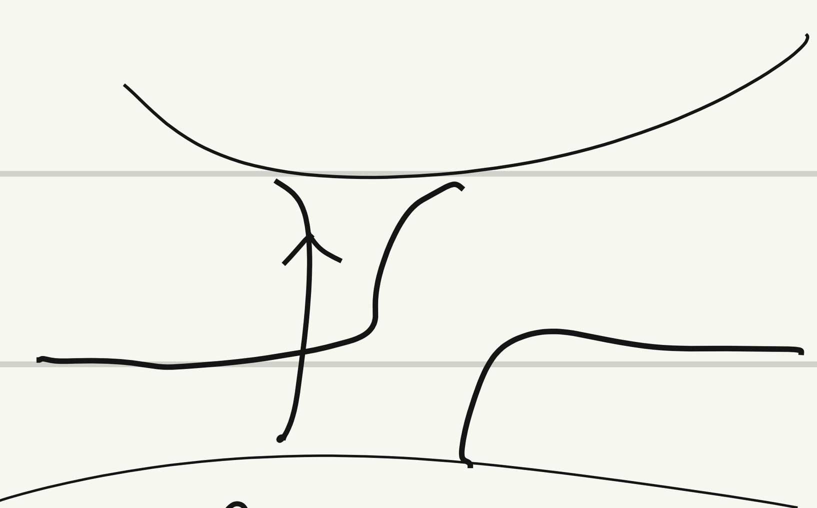




Dehn twist:
(Left)



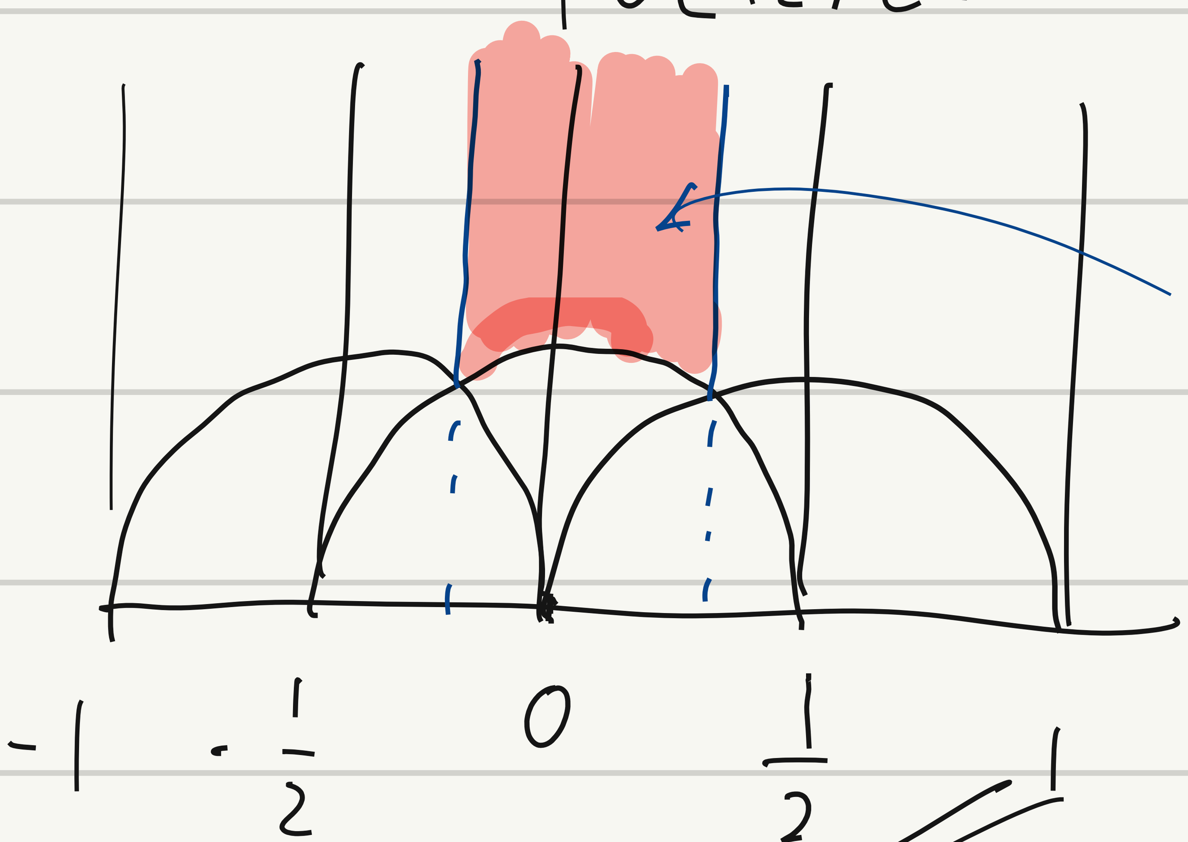




Prop: $\mathrm{Mod}(T) = \langle \underset{L}{\overset{\text{Right}}{D_a}}, \underset{R^{-1}}{\overset{\text{Left}}{D_b}} \mid D_aD_bD_a = D_bD_aD_b,\ (D_aD_b)^6 = 1 \rangle$

$\mathrm{Mod}(T)/\langle \text{ellip inv.} \rangle = \mathrm{Mod}^{\varepsilon}(T)$

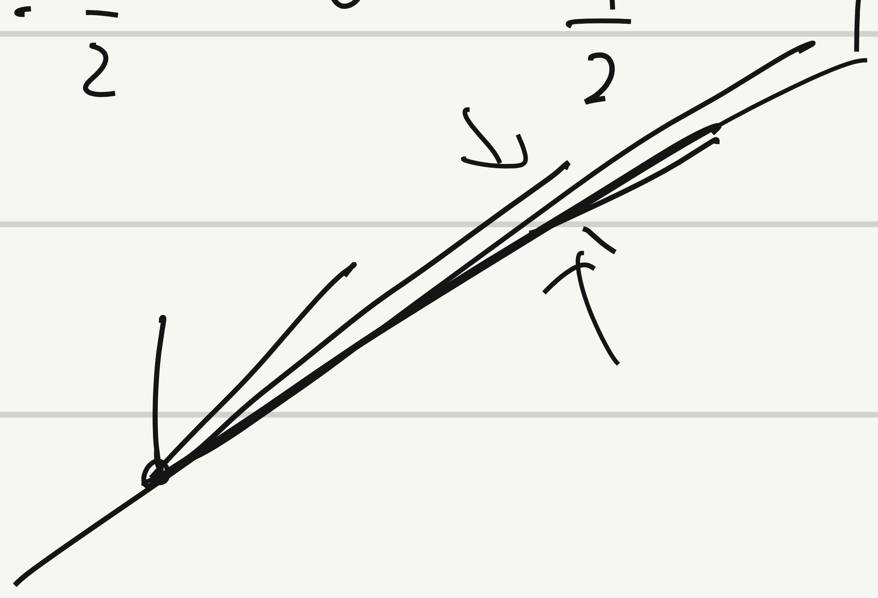
Prop: $\underset{\cong PSL(2,\mathbb{Z})}{\mathrm{Mod}^{\varepsilon}(T)} \circlearrowright \underset{\cong \mathbb{H}}{\mathcal{T}(T)}$ properly discontinuously



Fund domain for $PSL(2,\mathbb{Z})$

orbifold.

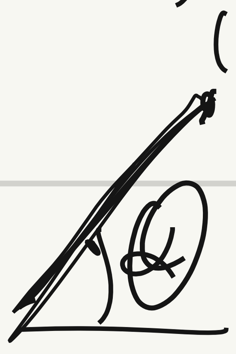
conic sing.



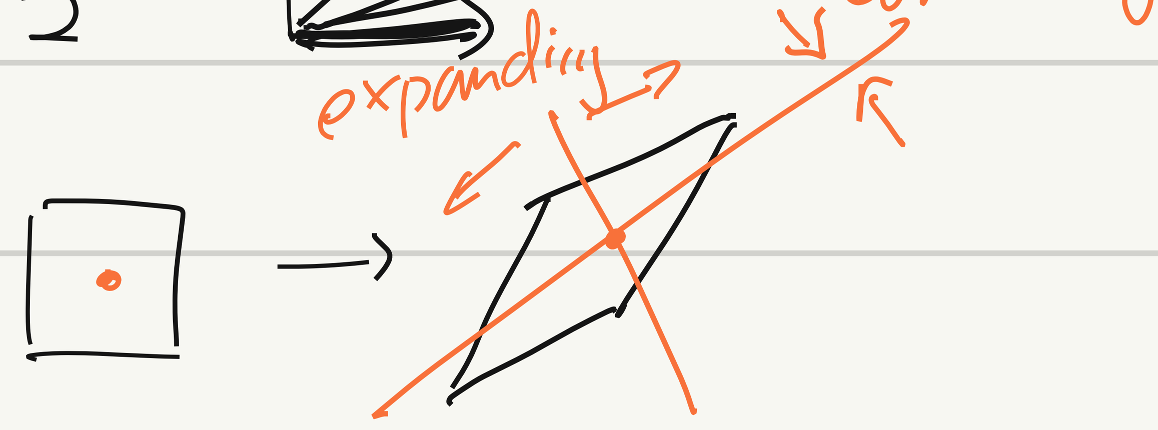
6. Classification of $[f] \in \mathrm{Mod}(S)$.

$\mathrm{Mod}(T)$ | $A \in SL(2,\mathbb{Z}) \setminus \{\pm I_2\}$ ① Dehn twist $|\mathrm{tr}\, A| = 2$

$\begin{bmatrix} m & n \\ p & q \end{bmatrix}$ (p,q) ② Anosov $|\mathrm{tr}\, A| > 2$



$\tan\theta = \alpha$. ③ finite order $|\mathrm{tr}\, A| < 2$.



Thm (Nielsen-Thurston classification)

$[f] \in \mathrm{Mod}(S_g)$ is one of the following 3 types

① reducible : ∞ order $\exists \gamma$ s.c.c. $\exists k \in \mathbb{N}_{>0}$ s.t. $f^k(\gamma) = \gamma$

② pseudo-Anosov : ∞ order $\forall \gamma$ s.c.c. $\forall k \in \mathbb{N}_{>0}$ $f^k(\gamma) \not\simeq \gamma$

not homotopic

③ finite order: $\exists k > 0$ s.t. $f^k = \mathrm{id}$.

Prop: $\mathrm{Mod}(S_g) \circlearrowright \mathcal{T}(\Sigma_g)$ prop. discont.

($g = 1, 2$ $\mathrm{Mod}(S_g) / \langle$ hyperelliptic inv $\rangle$.

$\mathrm{Mod}(S_g)$ finitely generated.

Thurston's work on surfaces.

A primer on MCG

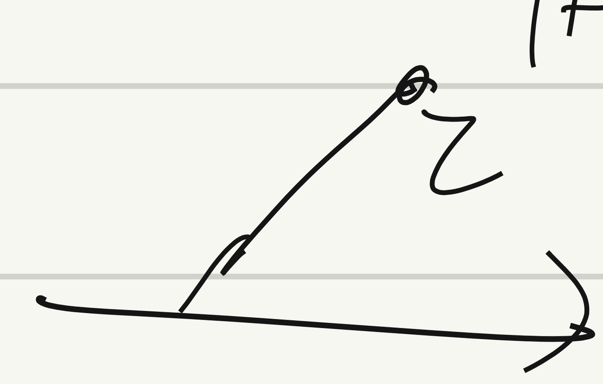
$\mathbb{C}$

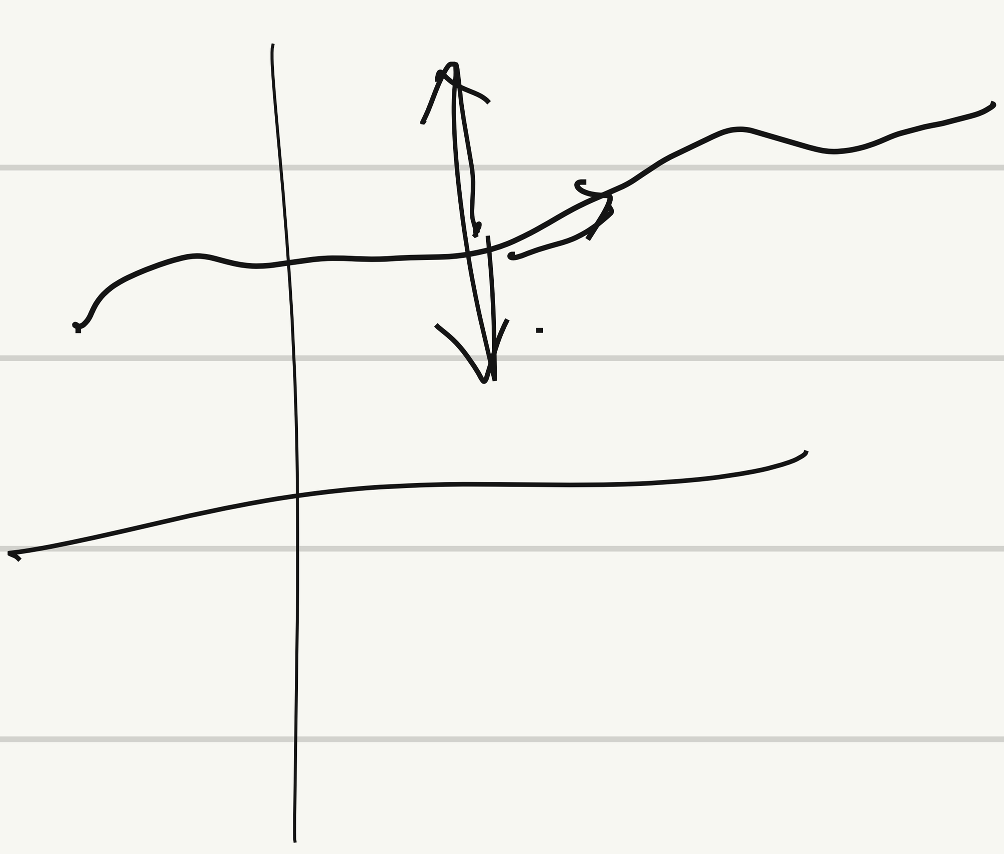
$\boxed{\tau = \frac{z_b}{z_a}}$

$\mathrm{Mod}(T) \circlearrowright \mathcal{T}(T)$

$SL(2, \mathbb{Z}) \circlearrowright \mathbb{H}$

$\mathbb{H}$





$\uparrow$ Möbius transf.

orientation reversing homeo

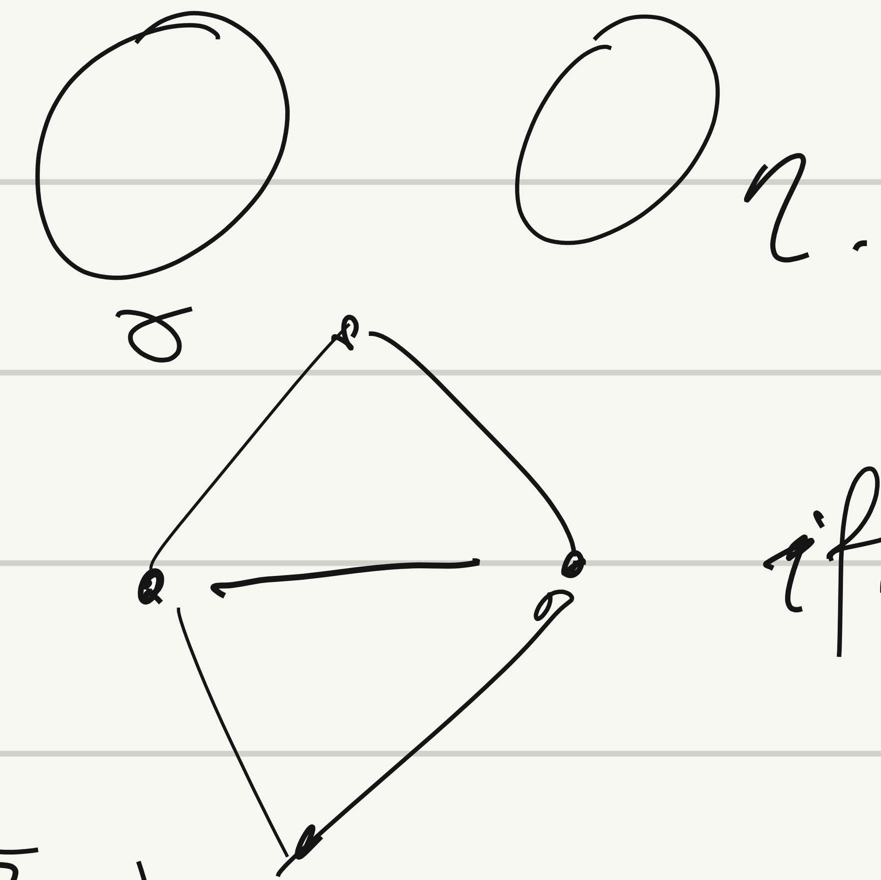
$\mathrm{Mod}^{\pm}(\Sigma_g)$ extended mcg.

$\mathcal{T}(\Sigma_g)$ WP-metric

Teichmüller metric.

$C(\Sigma_g)$ curve complex

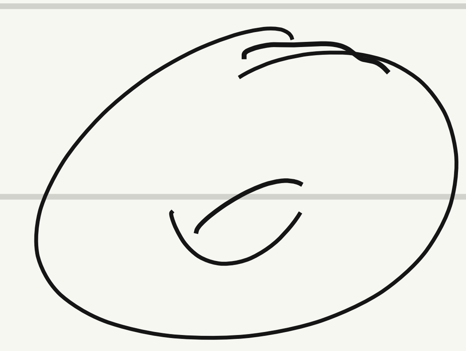
⋮



iff $i(\gamma, \eta) = 0$

isomorphism group

$= \mathrm{Mod}^{\pm}(\Sigma_g)$



$\mathbb{H}$

hyperbolic plane

$\mathcal{T}(T)$

$\to (\mathcal{T}(T), m_{Teich}) \cong (\mathbb{H}, d_{\mathbb{H}})$

Teichmüller metric.

$(\mathcal{T}(T), m_{Teich}) \rightsquigarrow (\mathcal{M}(T), m_{Teich})$

$(\mathcal{T}(T), m_{WP}) \not\cong (\mathbb{H}, d_{\mathbb{H}})$

WP-metric.

$(\mathcal{T}(T), m_{WP}) \rightsquigarrow (\mathcal{M}(T), m_{WP})$.