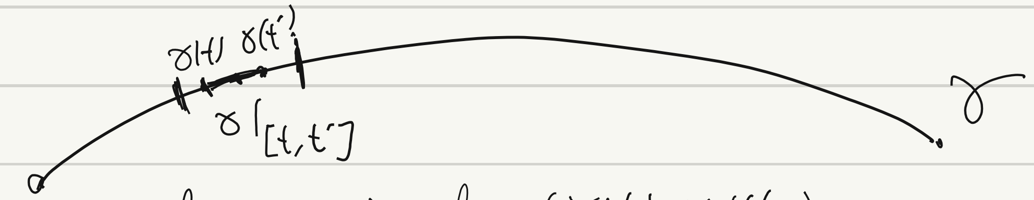
Introduction to hyperbolic surfaces II

4. Geodesic:

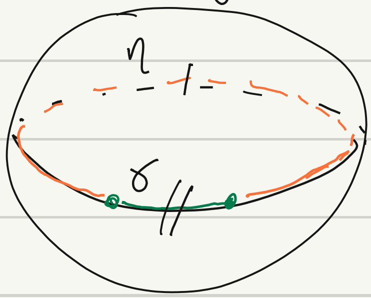
Def: A path $\gamma : [a,b] \to \mathbb{H}$ is a geod if if it locally minimizes the distance.

i.e. $\forall c \in [a,b]$, $\exists \varepsilon > 0$ s.t. $\forall [t, t'] \subset [c-\varepsilon, c+\varepsilon] \cap [a,b]$ we have $\ell_{\mathbb{H}}(\gamma|_{[t,t']}) = d_{\mathbb{H}}(\gamma(t), \gamma(t'))$.

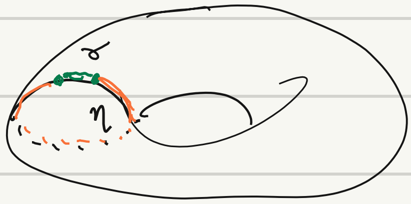


$$\ell_{\mathbb{H}}(\gamma|_{[t,t']}) = d_{\mathbb{H}}(\gamma(t), \gamma(t'))$$

Rmk: "locally minimize"



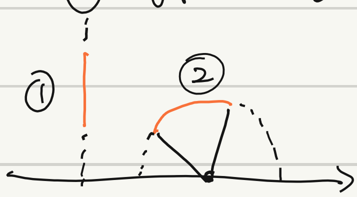
$\ell_g(\gamma) < \ell_g(\eta)$



$\ell_E(\gamma) < \ell_E(\eta)$.

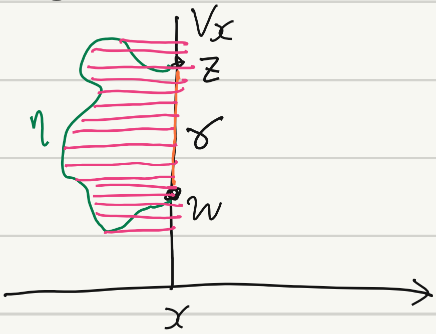
Rmk: In $\mathbb{H}$, "locally minimize" $\Leftrightarrow$ "global minimize".

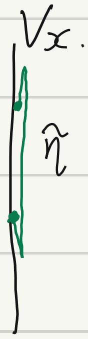
Prop: A geod connecting w and z in $\mathbb{H}$ is
① either a vertical segment.
② or a circular arc in circle with center in $\mathbb{R}$. (Euclidean)



Proof: ① $\mathrm{Re}(w) = \mathrm{Re}(z)$
② $\mathrm{Re}(w) \neq \mathrm{Re}(z)$.

① $\mathrm{Re}(w) = \mathrm{Re}(z)$.





$l_{\mathbb{H}}(\eta) > l_{\mathbb{H}}(\eta') \geq l_{\mathbb{H}}(\gamma)$.

$\eta(t) = (x(t), y(t))$ $\quad \underline{\dot{\eta}(t)} = (\dot{x}(t), \dot{y}(t))$

$\hat{\eta}(t) = (x, y(t))$ $\quad \underline{\dot{\hat{\eta}}(t)} = (0, \dot{y}(t))$

$$l_{\mathbb{H}}(\eta) = \int_a^b \|\dot{\eta}(t)\|_{\mathbb{H}}\, dt = \int_a^b \frac{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{y(t)}\, dt.$$

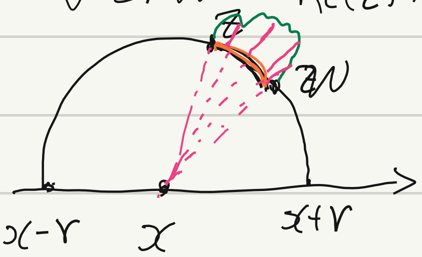
$$\geq \int_a^b \frac{\sqrt{\dot{y}(t)^2}}{y(t)}\, dt. = l_{\mathbb{H}}(\hat{\eta}).$$

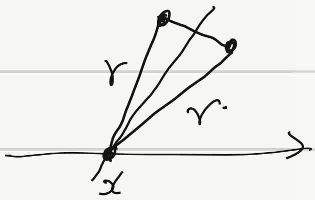
"=" $\Longleftrightarrow$ $\eta = \hat{\eta}$.

$\geq l_{\mathbb{H}}(\gamma)$

$l_{\mathbb{H}}(\hat{\eta}) = l_{\mathbb{H}}(\gamma) \Longleftrightarrow \hat{\eta} = \gamma$.

② $\forall z, w$ $\quad \mathrm{Re}(z) \neq \mathrm{Re}(w)$, $\exists C(x, r)$ s.t.

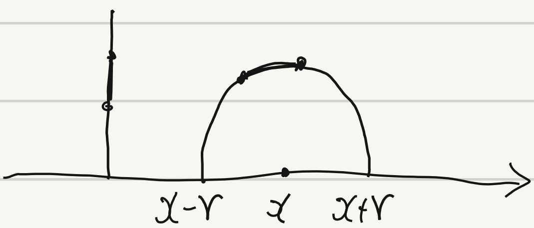




$\underline{\underline{x = 0}}$

$$\|\dot{\gamma}(t)\|_{\mathbb{H}} = \frac{\sqrt{\dot{r}(t)^2 + r(t)^2\dot{\theta}(t)^2}}{\underline{r(t)}\sin\theta(t)} = \frac{\sqrt{r(t)^{-2}\dot{r}(t)^2 + \dot{\theta}(t)^2}}{\sin\theta(t)}$$

■



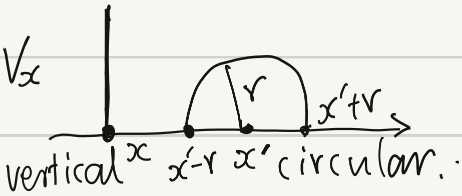
$C(x, r)$

$\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$

Rmk: $\underline{\underline{\mathcal{G}(\mathbb{H})}} = \hat{\mathbb{R}} \times \hat{\mathbb{R}} \setminus \{(x, x) \mid x \in \hat{\mathbb{R}}\}$

$\mathbb{Z}_2$.

Def: A complete geodesic is a path $\gamma : \mathbb{R} \to \mathbb{H}$ s.t.
∞. $\forall t < t' \in \mathbb{R}$, $\gamma|_{[t, t']}$ is a geodesic (segment) of length $|t - t'|$.



End points ① V_x; x, ∞

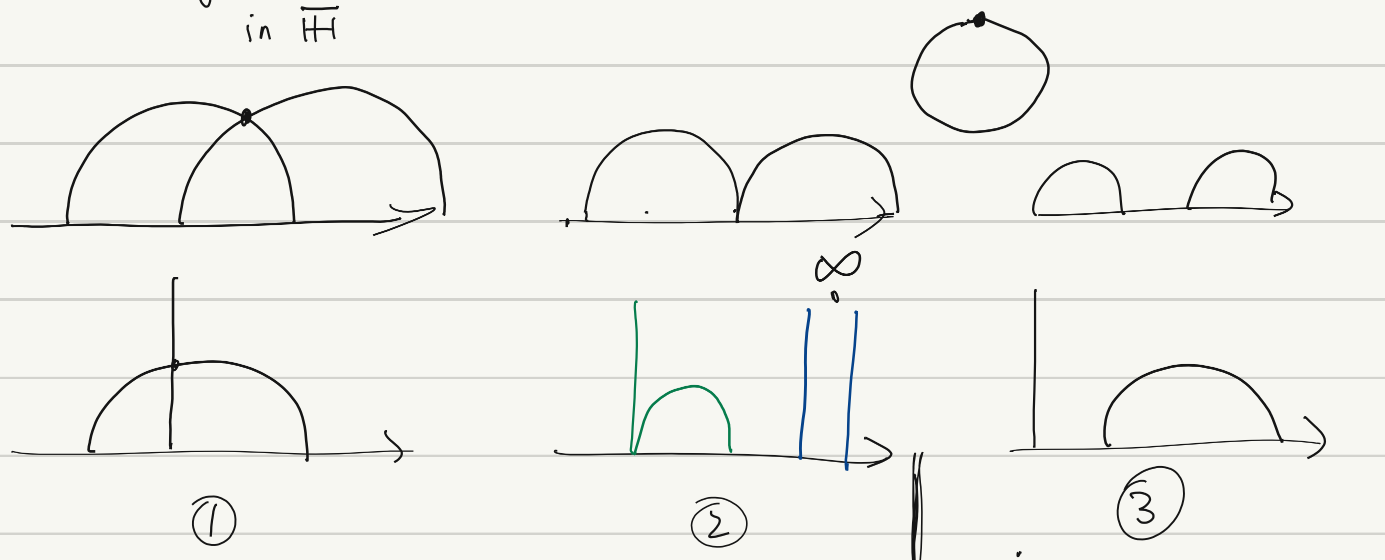
② $C(x', r)$; $x'-r$, $x'+r$

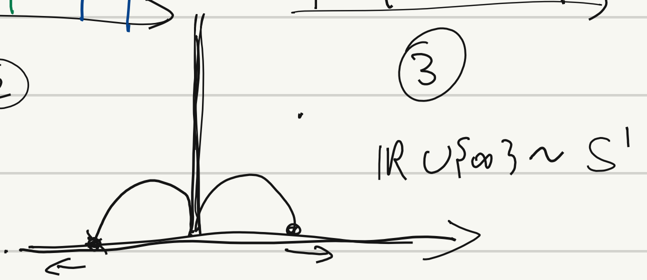
Relative positions between 2 geodesics. $\gamma \neq \eta$

① intersecting : $\gamma \cap \eta \neq \emptyset$ $\exists!$ intersection point.

② parallel : $\gamma \cap \eta = \emptyset$, share 1 endpoint $\in \hat{\mathbb{R}} = \{\mathbb{R} \cup \{\infty\}\}$

③ disjoint : $\gamma \cap \eta = \emptyset$, no common endpoint.
in $\overline{\mathbb{H}}$

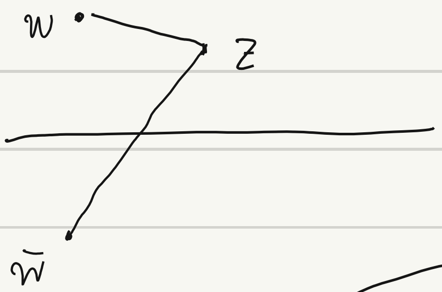




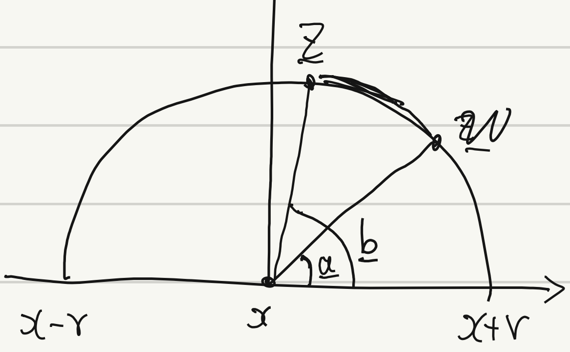
5. Distance formula :

Prop : For $w, z \in \mathbb{H}$,

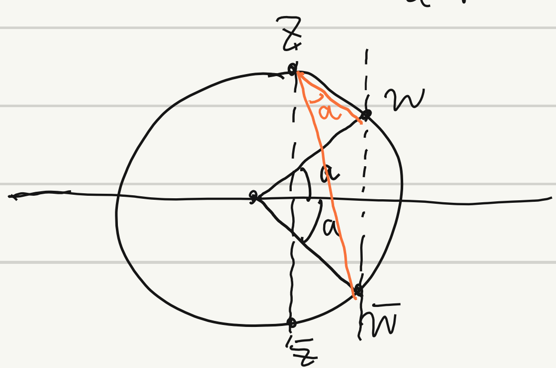
$$d_{\mathbb{H}}(w, z) = \log \frac{|\bar{w} - z| + |w - z|}{|\bar{w} - z| - |w - z|}$$

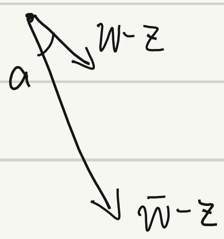


Proof : First consider circular geod. then show it holds for vertical geod.



$$d_{\mathbb{H}}(w,z) = \log \frac{\sin b}{\cos b + 1} - \log \frac{\sin a}{\cos a + 1}$$





$$w - z = (\bar{w} - z) \cdot e^{ia} \cdot \left| \frac{w - z}{\bar{w} - z} \right|$$

∎

Rmk: $d_{\mathbb{H}}(w,z) = \log \dfrac{|\bar{w}-z| + |w-z|}{|\bar{w}-z| - |w-z|}$ $\quad |\bar{w}-z| \neq 0$

$$= \log\left(\frac{2}{1 - \left|\frac{w-z}{\bar{w}-z}\right|} - 1 \right)$$

Cor: $d_{\mathbb{H}}(w,z) = d_{\mathbb{H}}(w',z')$
iff $\left|\dfrac{\bar{w}'-z'}{w'-z'}\right| = \left|\dfrac{\bar{w}-z}{w-z}\right|$.

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh d_{\mathbb{H}}(w,z) = 1 + \frac{|w-z|^2}{\operatorname{Im} w \cdot \operatorname{Im} z}$$

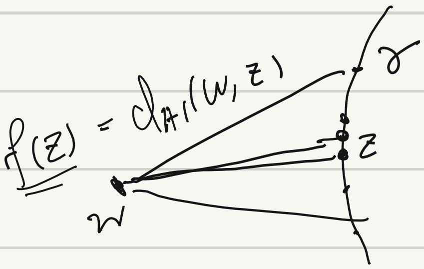
$$\sinh\left(\frac{d_{\mathbb{H}}(w,z)}{2}\right) = \frac{|w-z|}{2(\operatorname{Im} w \cdot \operatorname{Im} z)^{1/2}}$$

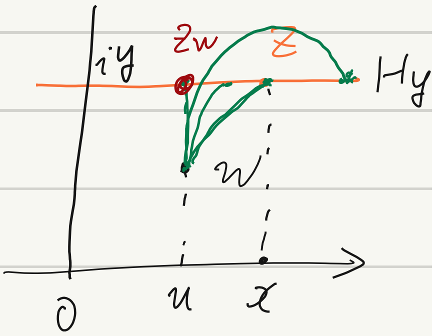
$$\cosh\left(\frac{d_{\mathbb{H}}(w,z)}{2}\right) = \frac{|\bar{w}-z|}{2(\operatorname{Im} w \cdot \operatorname{Im} z)^{1/2}}$$

$$\tanh \frac{d_{\mathbb{H}}(w,z)}{2} = \frac{|w-z|}{|\bar{w}-z|}.$$

6. Convexity of distance function:

① $w \in \mathbb{H}$, $z \in H_y$.
$w = u + iv$, $z = x + iy$



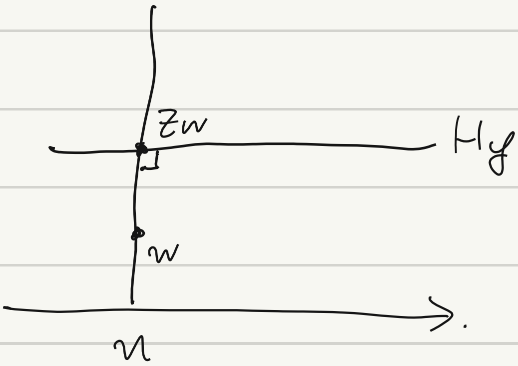


$$f(x) = d_{\mathbb{H}}(w,z)$$

$$\cosh f(x) = 1 + \frac{|w-z|^2}{\operatorname{Im} w \operatorname{Im} z}$$

$$= 1 + \frac{(u-x)^2 + (v-y)^2}{vy}$$

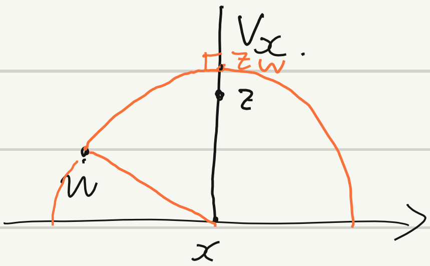
$$\underbrace{\sinh f(x)}_{>0} \cdot f'(x) = \underbrace{\left(\frac{2}{vy}\right)}_{>0} (x-u). \qquad f'(x) = 0 \text{ iff } x = u.$$



Ob: $\exists!\, z_w \in H_y$ s.t.
$d_{\mathbb{H}}(w, H_y) = d_{\mathbb{H}}(w, z_w)$.

z_w is the orthorgonal proj of w to H_y

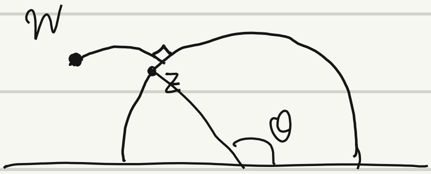
② $w \in \mathbb{H}$ $\quad z \in V_x$

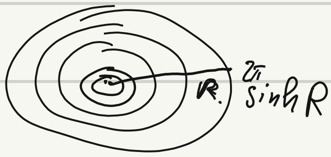


Prop. $\exists!\, z_w \in V_x$ s.t.
$d_{\mathbb{H}}(w, V_x) = d_{\mathbb{H}}(w, z_w)$
z_w is the orth proj of w to V_x

$f(y) = d_{\mathbb{H}}(w, z)$
$$\cosh f(y) = 1 + \frac{(u-x)^2 + (v-y)^2}{vy}$$
$$(\sinh f(y))\, f'(y) = \frac{1}{v}\left(1 - \frac{(u-x)^2 + v^2}{y^2}\right)$$

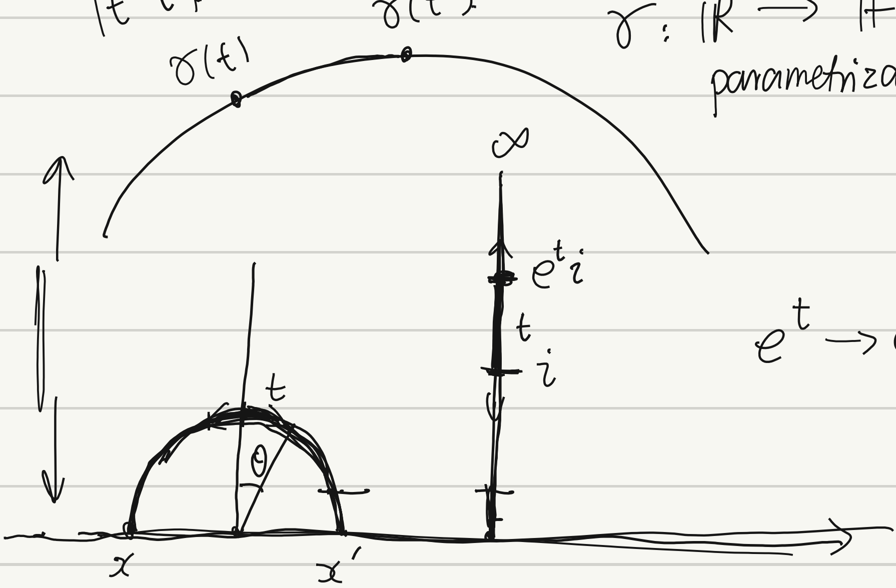
③

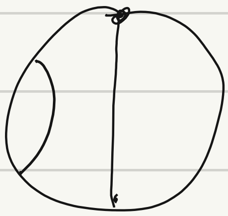




$|t-t'| = \gamma|_{[t,t']}$ is a geod.

$\gamma: \mathbb{R} \to \mathbb{H}$.
parametrizati





$e^t \to 0 \quad (t \to -\infty)$