K gK

Lemma 6.21 (Svarc-Milnor Lemma). *Suppose G acts properly and co-compactly on a proper length space (X, d). Then*

$X/_G$ is compact.

(1) *G is finitely generated by a set S.*

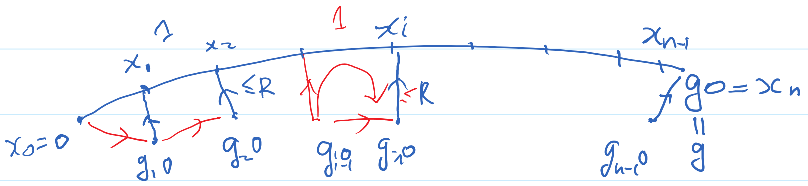
(2) *Fix a basepoint $o \in X$. Then the map*

$$(G, d_S) \to (Go, d), \ g \to go,$$

is a G-equivariant quasi-isometric map. (quasi- isometry)

Proof: Let $K \subseteq X$ compact s.t. $G \cdot K = X$. Set $R = \mathrm{Diam}(K) < \infty$

Fix $o \in K$.



Then $S \triangleq \{ s \in G : d(o, so) \le 2R+1 \}$ generates G. $\# S < \infty$

$g = 1 \cdot g = 1 \cdot g_1 (g_1^{-1} g_2)(g_2^{-1} g_3) \cdots (g_{n-1}^{-1} g_n)$. (word length n)

Ψ S

Cayley graph of G w.r.t. S:

$g_1 \overset{s}{\longrightarrow} g_2$ iff $g_2 = g_1 \cdot s$ $s \in S$

s^{-1}

$g_1 = g_2 s^{-1}$

Now prove that

$\Phi: \ g \longmapsto go$ is Q.I.E. map

Define $\lambda = \max_{s \in S} d(o, so)$

- $d_X(go, ho) \le \lambda \, d_S(g, h) = n$ (very general)

- $d_S(g, h) \le n \le d_X(go, ho) + 1$ ✓

WLOG. $g = 1$, $h = g$ (上图)

#

Corollary 6.23. *Let G be a finitely generated group. Then any finite index subgroup is finitely generated and quasi-isometric to G.*

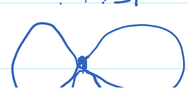
$H < G \overset{\text{free}}{\curvearrowright} \mathrm{Cay}(G, S)$

$hg_1 \longrightarrow hg_2 = g_1 s$

$\mathrm{Cay}(G, S)/_G$

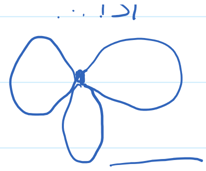
$|S|$

$G = Hg_1 \cup \cdots \cup Hg_m = H\{g_1 \cdots g_m\}$



Exercise 6.24. *Let*

$G = Hg_1 \cup \cdots \cup Hg_m = H\{g_1, \cdots, g_m\}$



Exercise 6.24. *Let*

$$1 \to N \to G \xrightarrow{\pi} \Gamma \to 1$$

be a group extension, where G is finitely generated. Assume that $\boxed{N \text{ is finite}}$*. Then G is quasi-isometric to Γ.*

$G \curvearrowright Cay(\Gamma, S)$: proper

$G_x \cong N$

Examples 6.25. (1) If $n \neq m$, then $\mathbb{R}^n$ is not quasi-isometric to $\mathbb{R}^m$.

(2) All free groups of finite rank at least two are quasi-isometric.

$\forall\ \mathbb{F}_n$ is of finite index in $\mathbb{F}_2 \Rightarrow \mathbb{F}_n \overset{Q.I}{\cong} \mathbb{F}_m$

Pf of (1):

$\mathbb{Z}^n \overset{geom}{\curvearrowright} \mathbb{R}^n \quad\quad \mathbb{Z}^m \overset{geom}{\curvearrowright} \mathbb{R}^m$

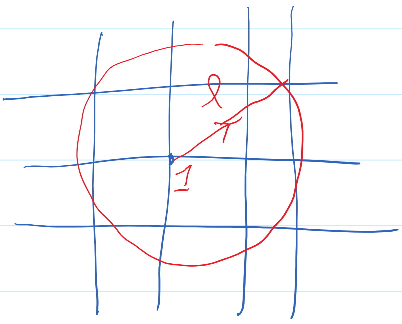
$\Downarrow \quad\quad \Downarrow$

$\mathbb{Z}^n \overset{Q.I}{\cong} \mathbb{R}^n \quad\quad \mathbb{Z}^m \overset{Q.I}{\cong} \mathbb{R}^m$

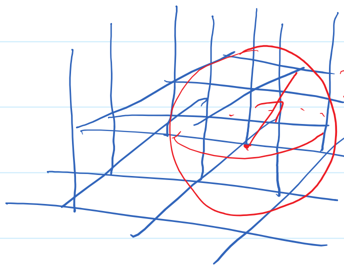
If $\mathbb{R}^n \overset{Q.I}{\cong} \mathbb{R}^m \quad (n \neq m)$

then $\mathbb{Z}^n \overset{Q.I}{\cong} \mathbb{Z}^m$

$n=2$



$\not\simeq$



$B_\ell(\mathbb{Z}^m) \asymp \ell^m$

$\ell \longmapsto \# B_\ell(\mathbb{Z}^n) = \#\{g \in \mathbb{Z}^n : d(1,g) \le \ell\}$ growth function

$\asymp \ell^n$

biLip.

群的增长分类

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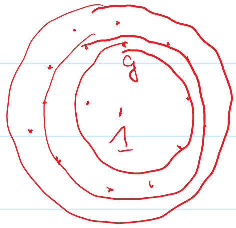
$(G, S) \longmapsto (Cay(G, S), d_S)$

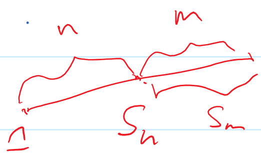


Growth function:

$$\phi(n) := \#B(n) = \#\{g \in G : d_S(1, g) \le n\}$$

$$= \sum_{i=0}^{n} \#S(i) = \{g : d_S(1, g) = i\}$$

Ex: $\#S(n+m) \le \#S(n) \cdot \#S(m)$.



Let $\phi, \varphi : \mathbb{N} \to \mathbb{N}$ be two monotonically non-discreasing functions. We say that ϕ *dominates* φ if there exists $C \ge 1$ such that

$n^2 \le n^3$

$$\varphi(n) \le C\phi(Cn)$$

$\varphi(n) \le C\phi(Cn + C) + C$

for $n > 0$. Denote $\varphi \prec \phi$.

Two functions ϕ, φ are *equivalent* if they dominate each other. Note that $a_0 + a_1 n + ... + a_i n^i$ is equivalent to n^i. All exponential functions like λ^n for $\lambda > 1$ are equivalent to the the standard one e^n. But e^n dominates any polynomial function.

Fact: Equivalent Types of Growth functions are Quasi-Isometric Invariant.

Ex. Finitely generated Abelian groups: $\phi_{\mathbb{Z}^n}(\ell) \sim n^{\ell}$

Free groups of finite rank:

$$\phi_{\mathbb{F}_n}(\ell) \sim e^{\ell}$$

Definition 5.22. Let G be a finitely generated group. Let $\phi(n)$ be the growth function of G.

(1) (Polynomial growth) G has *polynomial* growth if there exists $d \in \mathbb{N}$ such that $\phi(n) \prec n^d$.

(2) (Exponential growth) G has *exponential* growth if $e^n \prec \phi(n)$. $e^n \sim \phi(n)$

(3) (Intermediate growth) G has *intermediate* growth if G does not belong to the polynomial and exponential growth types.

Remark. Many classes of groups have either polynomial or exponential growth. For example, there are no groups of intermediate growth in linear groups (Tits

Remark. Many classes of groups have either polynomial or exponential growth. For example, there are no groups of intermediate growth in linear groups (Tits alternative), in solvable groups (Milnor, Wolf)... However, Grigorchuk constructed the first group of intermediate growth in 1983, answering a long-standing open question of Milnor about whether there exist groups of intermediate growth.

$\forall G \to \Gamma \qquad \phi_G(\ell) \gtrsim \phi_\Gamma(\ell)$

Theorem 5.16 (J. Wolf, 1968). *A finitely generated nilpotent group has a polynomial growth function.* $\langle [g, g_0] : g \in G,\ g_0 \in G \rangle$

$G = G_0, \quad \triangleq [G, G_0] = G_1, \quad G_n \triangleq [G\ G_{n-1}] = 1$ for $n \geq 0$.

In 1981, Gromov proves the converse of Theorem 5.16. Given a property P, we say a group G has *virtually* property P if G contains a finite index subgroup which has property P.

See Terrence Tao for an elementary proof of Thm 5.17.

Theorem 5.17 (Gromov, 1981). *Let G be a finitely generated group of polynomial growth, then G is virtually nilpotent: G contains a finite index subgroup which is nilpotent.*

Gromov - Hausdorff distance Asymptotic Cone

Exercise 5.18. *Find a non-abelian group but which is virtually abelian, and a non-nilpotent group but which is virtually nilpotent.*

Proof of Thm 5.16:

Nilpotent group $G = G_0, \quad G_1 = [G, G_0] \cdots \quad G_m = [G, G_{m-1}] = 1$ for some $m \geq 0$. nil. degree

$\subseteq$ Solvable group $\quad H \leq G, \quad [H, H_0] \leq G_1, \quad H_m = 1$

Facts: [Lemma 5.14, 5.15]

1). Subgroups of nilpotent groups are Nilpotent.

$(*) \quad [G\ G] \to G \to G/[G, G]$ Abelian

2). If G is nilpotent of degree m, then $[G, G]$ is of degree $\leq \boxed{m-1}$

Solvable: $\quad \mathbb{Z}^2 \to G \to \mathbb{Z}^2$

3). If G is generated by S then $[G\ G]$ is generated by $= \langle [g, f] : g, f \in G \rangle$

$\#T < \infty \quad T \triangleq \{ \underbrace{[[[\cdot, \cdot];], \cdots, \cdot]}_{\leq (m-1)\text{-fold Commutator}} : \cdot \in S \}$ $\quad ([\cdot;], \cdot) \cdots$ 2-fold, $[\ ,\ ]$ is 1-fold commutator

$g f g^{-1} f^{-1} \triangleq (s_1 s_2 \cdot s_m) \cdots s_1' \cdots s_\ell' \cdot (s_m^{-1} \cdots s_1^{-1})(s_1'^{-1} \cdots s_1'^{-1})$

Goal: $\Phi_G(n)$ is of polynomial type.

Use induction on Nilpotent degree m:

$$[G\ G] \longrightarrow G \longrightarrow G/[G\ G]$$

$\le m-1$ m 1

✓   ✓

Obs: Any word over S of length l in G

$W = s_1 s_2 \cdots s_l$: $s_i \in S \triangleq \{a_i^{\pm 1} : 1 \le i \le n\}$

Can be written as

$$\underbrace{(a_1^{l_1} \cdot a_2^{l_2} \cdots a_n^{l_n})}_{\in \mathbb{Z}^n} \cdot (t_1 \cdot t_2 \cdots t_k)$$

where $\cdot\ \sum l_i = l$

$\cdot\ t_i \in T$ $\# T < \infty$

$\cdot\ \boxed{k < c \cdot l^m}$ nil. degree

$$\Longrightarrow \phi_G(l) \lesssim l^n \cdot l^m \lesssim l^{n+m}$$

$G = G_0$ $G_1 = [G\ G]$, $G_2 = [G\ G_1] = 1$

$\ \ \ \|$

$\langle a, b \rangle$

$w = ab\underline{a}b\,\underline{a^{-1}}\,\underline{b^{-1}}$ $ba = ab\,[b^{-1}, a^{-1}]$

$\langle a, b \rangle$

$w = abab\, a^{-1} b^{-1}$

$ba = ab\,\underline{[b^{-1}, a^{-1}]}$

$= aab[b^{-1} a^{-1}]\, b a^{-1} b^{-1}$

$= a^2 b\, [b^{-1} a^{-1}]\, b a^{-1} b^{-1}$

$= a^2 b\, \underline{[b^{-1} a^{-1}], a^{-1}}\, b\, [b^{-1} a]\, b^{-1}$

$= a^2 b a^{-1}\, \underline{[b^{-1} a^{-1}]}\, b\, \underline{[b^{-1} a]}\, b^{-1}$

双曲空间定义

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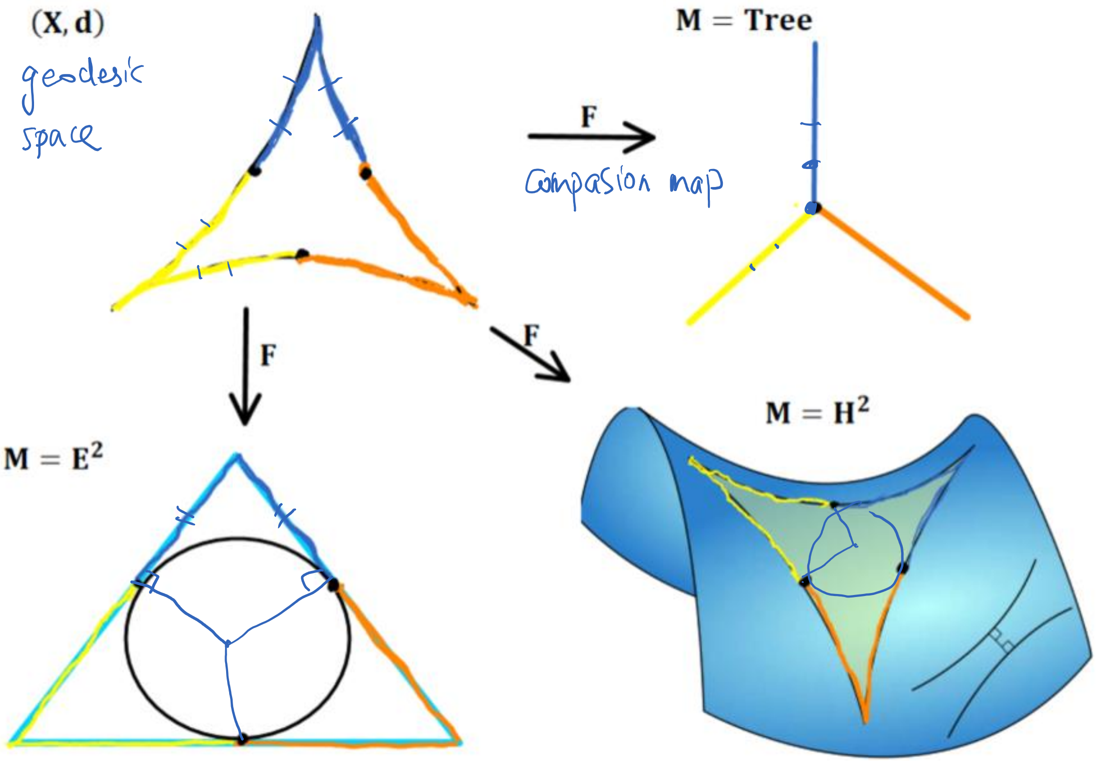
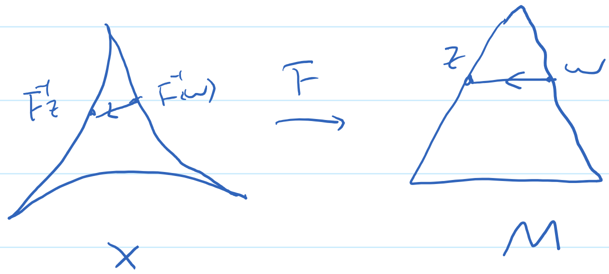
三角形比较:



Def: (X, d) is δ-thinner than (M, d) $\begin{cases} (\text{Tree}, d) \\ (E^2, d) \\ (H^2, d) \\ \dots \end{cases}$

if any triangle comparsion map

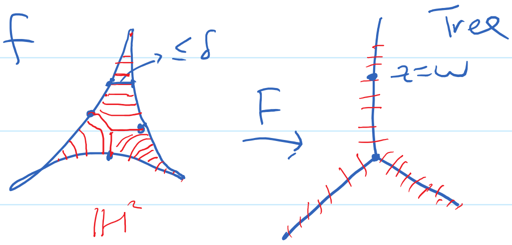
$F: \Delta_X \to \Delta_M$



is δ-thin:

$$\forall z, w \in \Delta_M \quad d_X(F^{-1}(z), F^{-1}(w)) \leq d_M(z, w) + \delta.$$

Def: • (X, d) is called δ-hyperbolic if it is δ-thinner than tree.

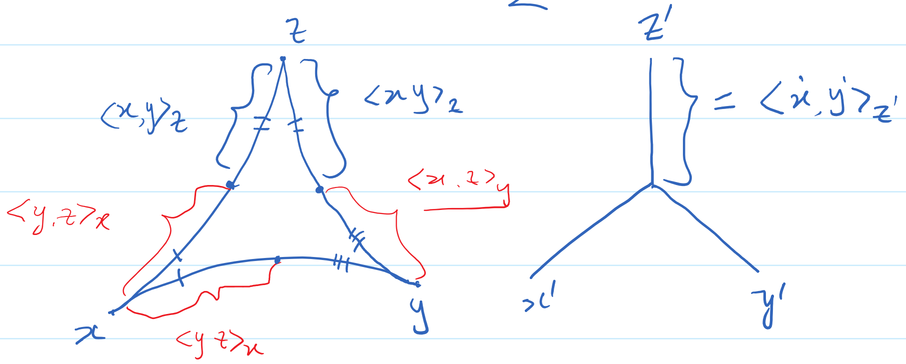


• (X, d) is called CAT(k) for $k \leq 0$

- (X,d) is called CAT(k) for $k \le 0$ if it is 0-thin than $M_k := \begin{cases} \mathbb{H}^2 & k=-1 \\ \mathbb{E}^2 & k=0 \end{cases}$

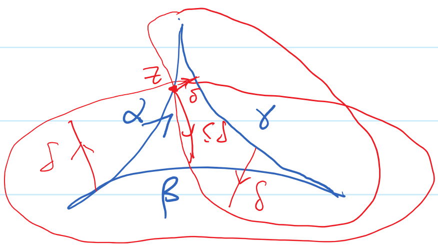
Def: Gromov Product:

$$\langle x . y \rangle_z \triangleq \frac{d(x,z)+d(y,z)-d(x,y)}{2}$$



Def. (X,d) has δ-slim triangle property if every geodesic triangle is δ-slim:

|| Every side is contained in δ-nbhd of the other two sides.

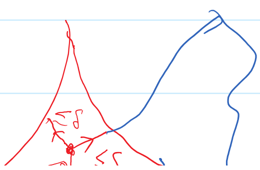


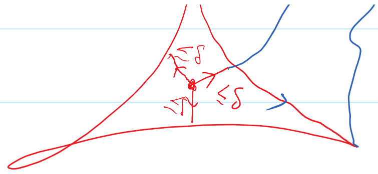
$\alpha \subseteq N_\delta(\beta) \cup N_\delta(\gamma)$

$\Rightarrow \exists z \in (\alpha \cap N_\delta(\beta)) \cap (\alpha \cap N_\delta(\gamma)) \neq \phi$

Def: (X,d) has δ-thin triangle property if every geodesic triangle has δ-centers

|| ∃ a point (called δ-center) is δ-close to each side





Lemma 7.4: Assume (X,d) has δ-thin $\triangle$.
(Gromov product)

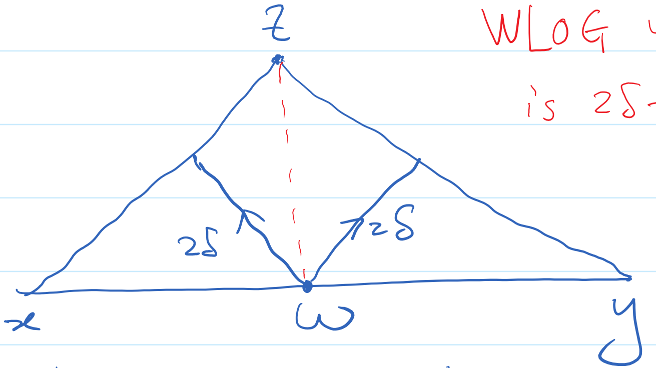
then $\forall x, y, z \in X$

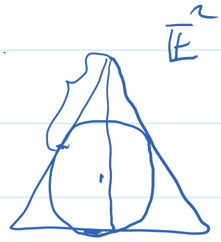
holds for any metric space

$$d(z,[x,y]) - 2\delta \le \langle x, y\rangle_z \le d(z,[x,y])$$

Proof:

WLOG $w \in [x,y]$ is 2δ-center





$$2\langle x, y\rangle_z = d(x,z) + d(y,z) - d(x,y)$$

$$\ge \cancel{d(x,w)} + d(w,z) + 4\delta + \cancel{d(y,w)} + d(w,z) + 4\delta - \cancel{d(x,y)}$$

$$\ge 2d(w,z) + 4\delta \ge 2d(z,[x,y]) + 4\delta \qquad ▨$$

A path p is called $\boxed{t\text{-taut}}$ for some $t \ge 0$ if $\mathrm{Len}(p) \le d(p_-, p_+) + \boxed{t}$.

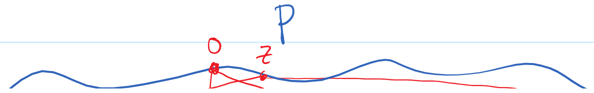
$\Rightarrow$ $[0, \mathrm{Len}\, p] \to X$ is $(1, t)$-quasi-isometric embedding

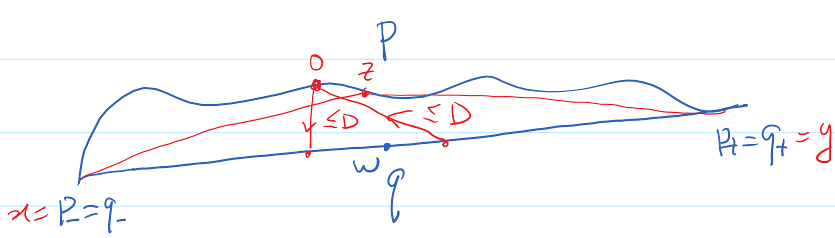
Fact: Any subpath of a t-taut path is t-taut.

Lemma 7.7. *Let p be a t-taut path for $t \ge 0$, and q be a geodesic with same endpoints as p. Then there exists $D = D(t, \delta)$ such that*

$$p \subset N_D(q), q \subset N_D(p).$$

(assuming X has δ-thin triangle property)





proof: 1) $z \in p \subseteq N_D(q)$

$$\langle x \cdot y \rangle_z = \frac{d(x,z) + d(y,z) - d(x,y)}{2} \leq \frac{\mathrm{len}(p) - d(x,y)}{2}$$

$$\leq \frac{t}{2}$$

$$\Rightarrow \underline{d(z, q)} \leq \frac{t}{2} + 2\delta \triangleq D(t, \delta)$$

2) $q \subseteq N_D(p)$: $w \in q$

[Any geodesic with same endpts has $\underline{2\delta\text{-Hausd}}$ distance]

p, q

$p \subseteq N_\delta(q)$ $q \subseteq N_{2\delta}(p)$

Use connected arguments: $q = [x\, w] \cup [w, y]$

$$p \subseteq N_D(q) = N_D([x\, w]) \cup N_D([w, y])$$

Connected

$$\underset{p}{\Rightarrow} (p \cap N_D([x, w])) \cap (p \cap N_D([w, y])) \neq \phi$$

$$\Rightarrow \exists\, o \in p \text{ s.t. } d(o, [x, w]) \leq D$$

$$d(o, [w, y]) \leq D$$

$$\Rightarrow d(w, o) \leq 3D.$$

$$\Rightarrow w \in q \subseteq N_{3D}(p) \qquad \blacksquare$$

双曲空间定义等价性

2021年7月20日 22:43

Thm: TFAE:

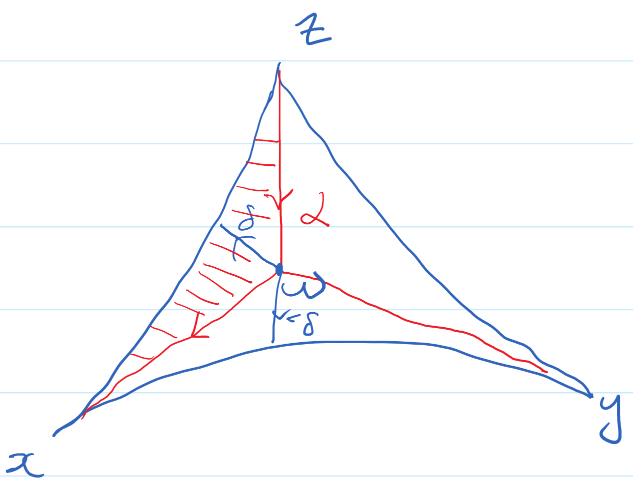
(1). (X, d) is δ_1-hyperbolic

(2). (X, d) has δ_2-slim triangle property.

(3). (X, d) has δ_3-thin triangle property.

Proof: (1) => (2) => (3): ✓

(3) => 2):



- Every center w produces three δ-2δ-taut paths:

$$\underline{\mathrm{Len}}\, \alpha \leq d(\alpha_-, \alpha_+) + 2\delta.$$

where

$$\underline{\alpha = [z, w] \cdot [w, x]}.$$

$\Rightarrow [x, z] \subseteq N_{2D}([x, y]) \cup N_{2D}([y, x])$

$\delta \triangleq 2D$

(2) => (1): Exercise.

Fact: δ-Center is uniformly close to congruent points