V Discrete subgroups of Isom($\mathbb{H}$)

1. Topology on Isom($\mathbb{H}$) & $\text{Isom}^+(\mathbb{H})$

$\text{Isom}(\mathbb{H}) \cong \underline{P}GL(2,\mathbb{R})$

$\text{Isom}^+(\mathbb{H}) \cong \underline{P}SL(2,\mathbb{R})$

Consider $\mathbb{M} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R}) \;\middle|\; ad-bc = \pm 1 \right\}$

$SL_2(\mathbb{R}) = \{ \text{—} \quad \text{—} \quad \text{—} \quad = 1 \}$

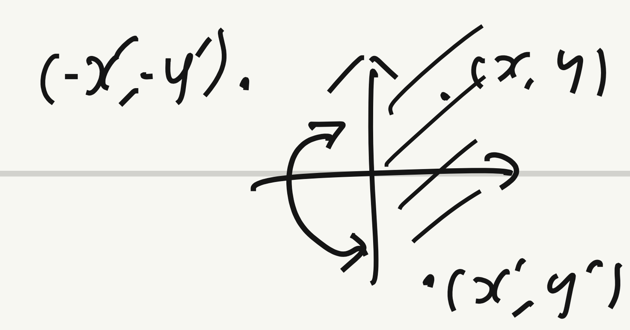
$\| \ \|_2 : M_{2\times2}(\mathbb{R}) \to \mathbb{R} \qquad M_{2\times2}(\mathbb{R}) \cong \mathbb{R}^4$

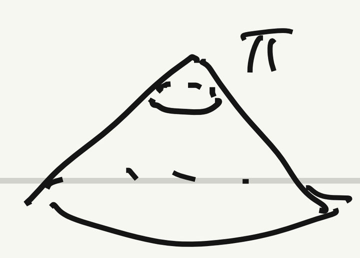
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \sqrt{a^2+b^2+c^2+d^2}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \qquad d_2(A,B) := \|A-B\|_2 = \sqrt{(a-p)^2+(b-q)^2+(c-r)^2+(d-s)^2}$

Problem $\quad \text{Isom}(\mathbb{H}) \quad \text{Isom}^+(\mathbb{H})$

- $A \in \mathbb{M}, \quad f_A = f_{-A}$
- $\|A-(-A)\|_2 = 2\|A\|$





Hence: $\forall A, B \in \mathbb{M}$

$$d(f_A, f_B) = \min\{\|A-B\|_2, \|A+B\|_2\}$$

induces a metric on Isom($\mathbb{H}$). (similarly for $\text{Isom}^+(\mathbb{H})$)

Rmk: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \quad \begin{pmatrix} -p & -q \\ -r & -s \end{pmatrix}$

Def: $\Gamma < \text{Isom}\,\mathbb{H}$ is discrete if $\nexists\, \overbrace{(f_n)_{n\in\mathbb{N}}}^{\text{non constant.}} \in \Gamma$, s.t. $\lim_{n\to\infty} f_n = f \in \Gamma$.

$\uparrow$ subgroup.

no accumulation point.

Prop: Γ is discrete iff $id \in \Gamma$ is not an accumulation point.

Proof: $f_n \to f, \ n\to\infty \qquad \begin{bmatrix} a+\Delta a & b+\Delta b \\ c+\Delta c & d+\Delta d \end{bmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Delta a\ \Delta b \ldots \Delta d \to 0$

$f_n \circ f^{-1} \to id, \ n \to \infty \qquad \begin{bmatrix} a+\Delta a & b+\Delta b \\ c+\Delta c & d+\Delta d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1+d\Delta a - c\Delta b & -b\Delta a + a\Delta b \\ d\Delta c - c\Delta d & 1-b\Delta c + a\Delta d \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Delta a\ \Delta b\ \Delta c\ \Delta d \to 0.$

Ex:
- $\phi_{\lambda^2}(z) = \lambda^2 \cdot z \quad \lambda^2 > 1 \quad A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \qquad A^n = \begin{bmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{bmatrix} \quad \Gamma = \langle A \rangle = \{\ldots, A^{-2}, A^{-1}, \underset{id.}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, A, A^2, \ldots, A^n \ldots\}$

  $d(\phi_{\lambda^2}^n, id) = \sqrt{(\lambda^n-1)^2 + (\lambda^{-n}-1)^2} \geq \max\{|\lambda^n-1|, |\lambda^{-n}-1|\}$

- $T_t(z) = z+t \quad A = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad A^n = \begin{bmatrix} 1 & nt \\ 0 & 1 \end{bmatrix} \quad \Gamma = \langle A \rangle$

  $t > 0$

  $d(T_t^n, id) = |n|\, t$

• (non-ex) ρ_θ $\theta = \alpha\pi$ $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ irrational rotation.

$$A^n = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

$$d(\rho_\theta^n, id) = \sqrt{2(\cos n\theta - 1)^2 + 2\sin^2 n\theta}$$
$$= \sqrt{4 - 4\cos n\theta}.$$

① $\exists\ \infty$ elts in $\langle f_A \rangle = \Gamma$

② $\exists j_n$, $j_n\theta \to 0 \mod 2\pi$, $j_n \to \infty$ $\quad d(\rho_\theta^{j_n}, id) \to 0$ $\quad j_n \to \infty$

α rational $\# \langle A \rangle < \infty$.

2. Properly discontinuously action (Geometric point of view)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad d_{\mathbb{H}}(f_A(i), i) = \log \dfrac{\left|\frac{-ai+b}{-ci+d} - i\right| + \left|\frac{ai+b}{ci+d} - i\right|}{\left|\frac{-ai+b}{-ci+d} - i\right| - \left|\frac{ai+b}{ci+d} - i\right|}$

$\det A = 1$

$d_{\mathbb{H}}(f_A(i), i) \to \infty$ as $\|A\| \to \infty$.

$$= \log \frac{|(b-c) + (a+d)i| + |(b+c) + (a-d)i|}{|(b-c) + (a+d)i| - |(b+c) + (a-d)i|}$$

If $\|A\|$ is not big so is $d_{\mathbb{H}}(f_A(i), i)$.

$$= \log \frac{\sqrt{(b-c)^2 + (a+d)^2} + \sqrt{(b+c)^2 + (a-d)^2}}{\sqrt{(b-c)^2 + (a+d)^2} - \sqrt{(b+c)^2 + (a-d)^2}}$$

$$= \log \frac{2(a^2+b^2+c^2+d^2) + 2\sqrt{\Delta}}{4(ad-bc)} \qquad \Delta > 0.$$

$$= \log \frac{\|A\|^2 + M}{2} \qquad M > 0.$$

$\Gamma < \mathrm{Isom}(\mathbb{H})$

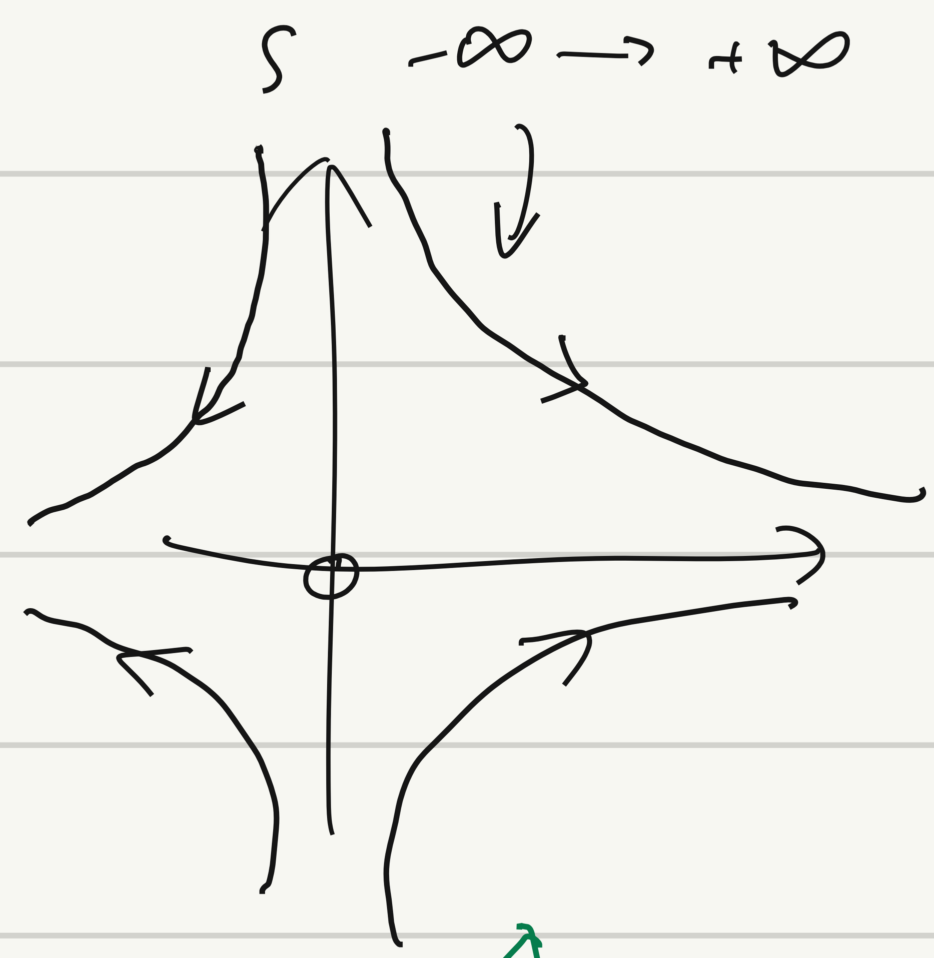
Def: Γ acts properly discontinuously on $\mathbb{H}$ if $\forall K \subset \mathbb{H}$ compact.

$\# \{ f \in \Gamma \mid f(K) \cap K \neq \phi \} < \infty$.

Let $z \in \mathbb{H}$.

Def: Orbit: $[z]_\Gamma := \{ f(z) \mid f \in \Gamma \} \subseteq \mathbb{H}$.

Rmk: (Thurston) $\mathbb{R}^2 \setminus \{(0,0)\}$



$\mathbb{R} \ni s : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$

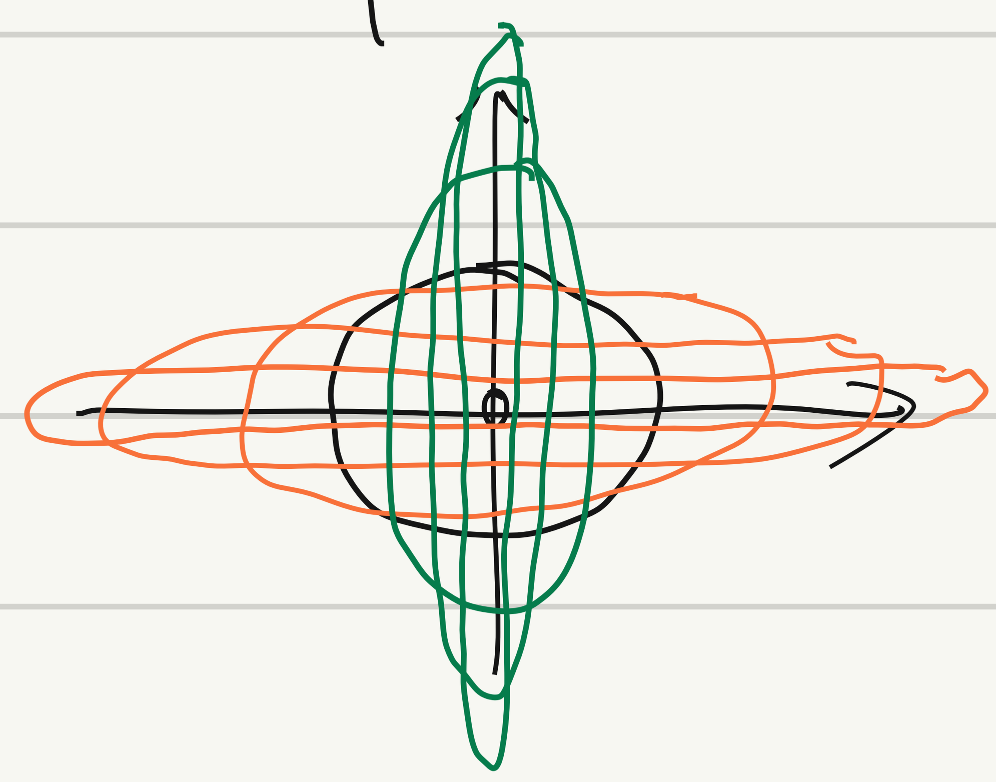
$(x, y) \longmapsto (2^s \cdot x, 2^{-s} y)$

$\mathbb{Z}$ discrete in $\mathbb{R}$.

$[(x,y)]_{\mathbb{Z}}$ discrete in $\mathbb{R}^2 \setminus \{(0,0)\}$

" $\{(2^n x, 2^{-n} y) \mid n \in \mathbb{Z}\}$.

$\mathbb{Z}$-action is not properly discontinuously.



For our case,

Prop: Γ is discrete in $\mathrm{Isom}(\mathbb{H})$ iff Γ-action is prop. discont.

Proof: "$\Rightarrow$" • if $\nexists f_n \to id$, $n \to \infty$, $\Rightarrow \forall M > 0$, $\#\{f \in \Gamma \mid d(f, id) < M\} < \infty$

• $d(f_n, id) \to \infty$, $n \to \infty$ iff $\|A_n\| \to \infty$, $n \to \infty$.

$\Rightarrow \forall M$, $\#\{f_A \in \Gamma \mid \|A\| < M\} < \infty$

$\Rightarrow$ (*) $\forall R > 0$, $K = \bar{D}(i, R)$ $\#\{f \in \Gamma \mid f(K) \cap K \neq \phi\} < \infty$

(**) $\forall K \subset \mathbb{H}$ compact, $\exists R$ s.t. $K \subseteq \bar{D}(i, R)$.

if $\exists f$ s.t. $f(K) \cap K \neq \emptyset$, then $f(\bar{D}(i,R)) \cap \bar{D}(i,R) \neq \phi$.

(*) + (**) $\Rightarrow$ Γ prop. discont.

$A_n \to Id.$ $n \to \infty$

"$\Leftarrow$" if Γ is not discrete, $\exists (f_n)_{n \in \mathbb{N}} \subseteq \Gamma$ s.t. $f_n \to id$, $n \to \infty$

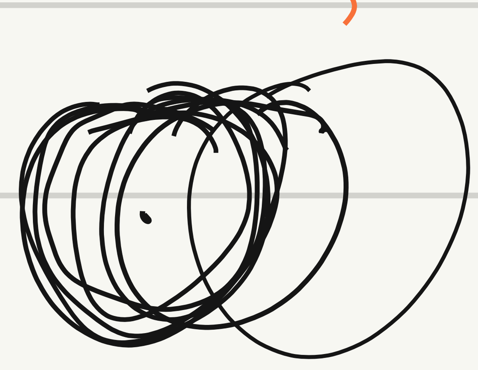
Consider.

$$d_{\mathbb{H}}(f_A(i), i) = \log \frac{|\overset{\nearrow 2a}{i(a+d)} + \overset{\nearrow 0}{(b-c)}| + |\overset{\nearrow 0}{i(a-d)} + \overset{\nearrow 0}{(b+c)}|}{|\underset{\searrow 2a}{i(a+d)} + \underset{\searrow 0}{(b-c)}| - |\underset{\searrow 0}{i(a-d)} + \underset{\searrow 0}{(b+c)}|}$$

$A \to Id.$ $a - d \to 0$ $\to 0$, as $n \to \infty$,

$b \to 0$

$c \to 0$.

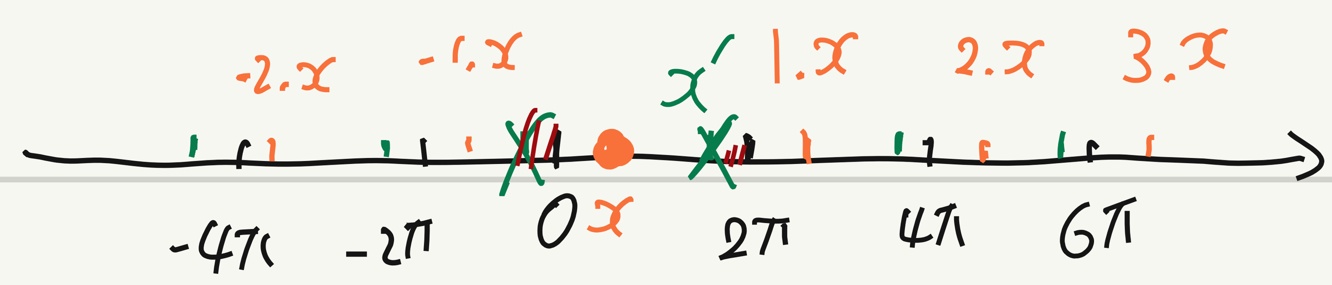


$\Rightarrow$ not prop. discont.

3. Quotient space

Baby example:

$n.x = x + 2\pi n.$



$\circlearrowleft \mathbb{Z}$ $\quad n: \mathbb{R} \to \mathbb{R}$, $\quad x \mapsto x + 2\pi \cdot n$

$\mathbb{R}/\mathbb{Z} = \{ [x]_{\mathbb{Z}} \mid x \in \mathbb{R} \} \cong S^1$ (isometric)

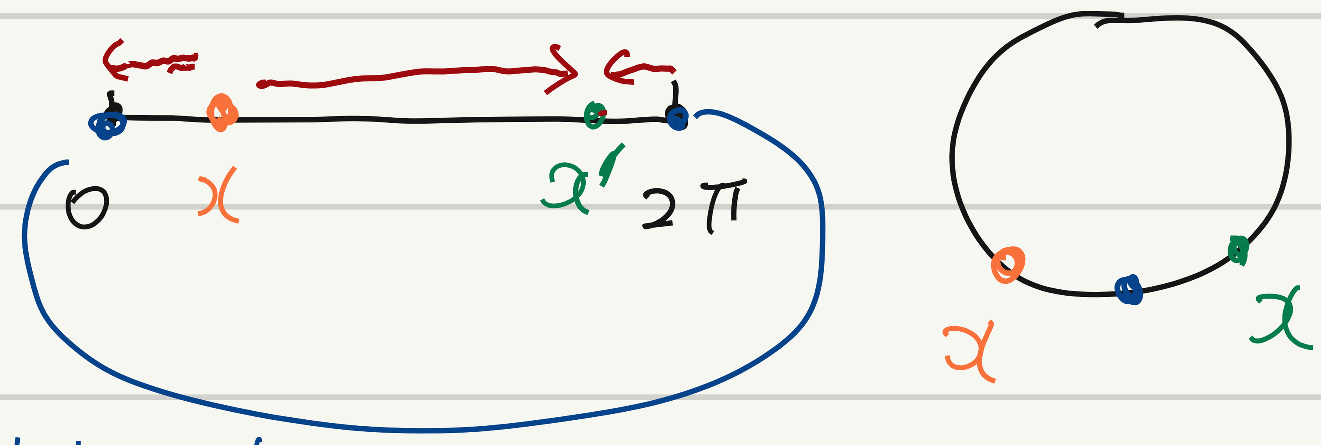
$\forall n \quad x \sim n.x.$

$\operatorname{dist}([x]_{\mathbb{Z}}, [x']_{\mathbb{Z}}) = \inf \{ |y - y'| \mid y \in [x]_{\mathbb{Z}}, y' \in [x']_{\mathbb{Z}} \}$

$|m.x - n.x'| = |x - (n-m).x'|$ $\quad$ is actually a minimum.

$|x + 2\pi m - (x' + 2\pi n)| = |x - (x' + 2\pi(n-m))|$

$K = [x - \varepsilon, x + \varepsilon]$



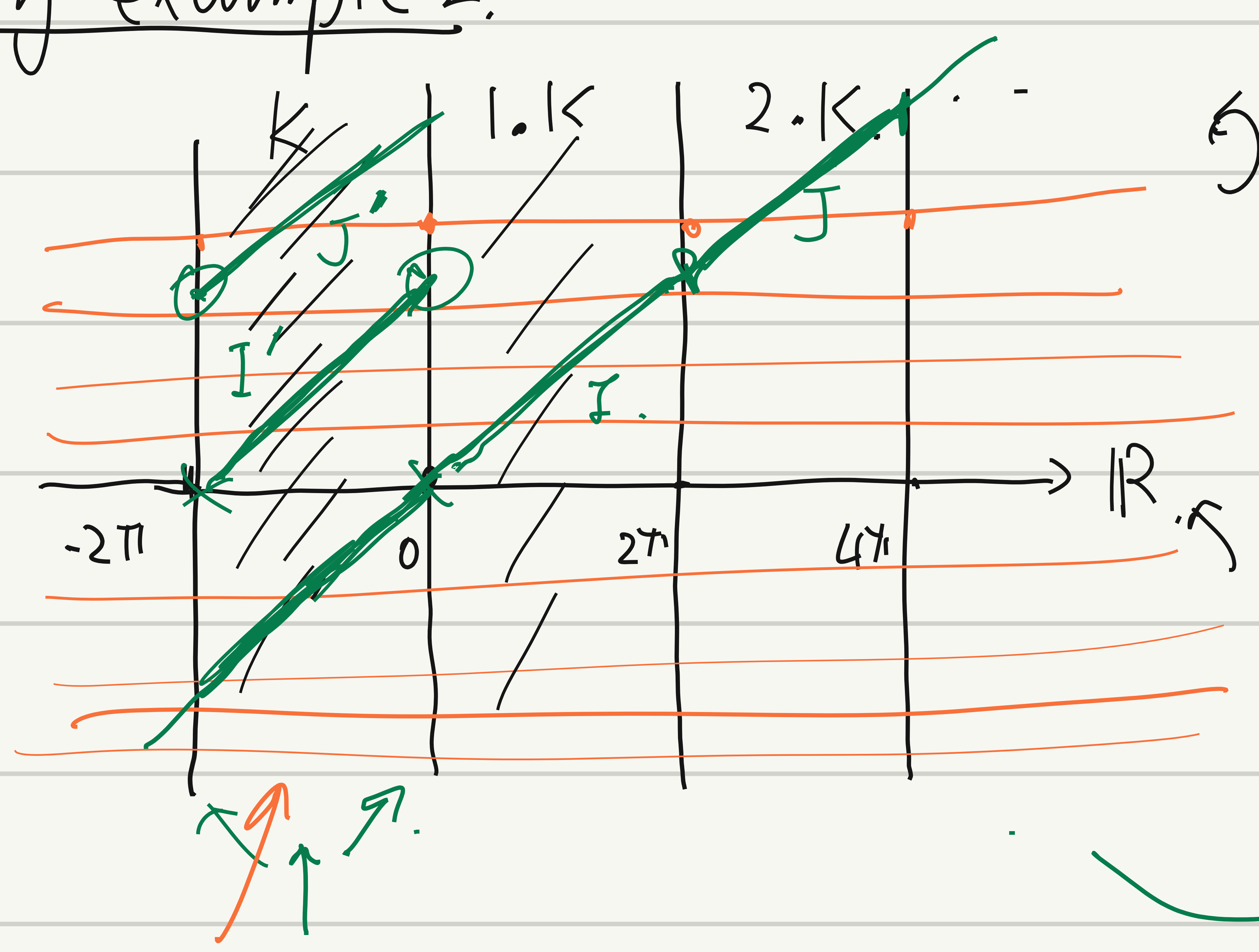
$\cong S^1$

Same orbit. glue together.

Prop. discont. action $\Rightarrow \exists$ finitely n. $n.K \cap K \neq \phi$,

$\varepsilon \searrow \varepsilon'$ s.t. $\forall n \quad nK' \cap K' = \phi$.

Baby example 2:

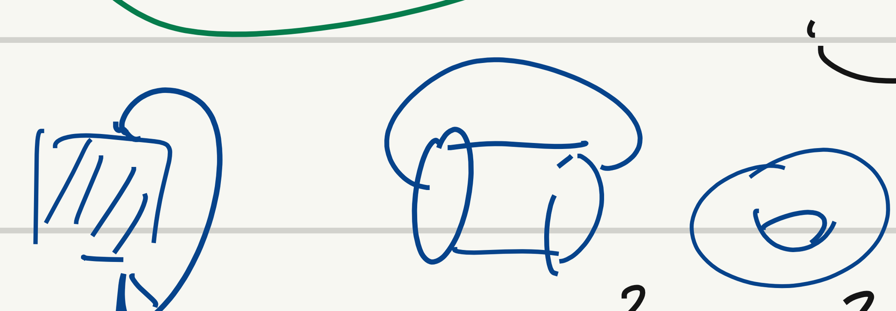


$\circlearrowleft \mathbb{Z}$ $\quad n: \mathbb{R}^2 \to \mathbb{R}^2$

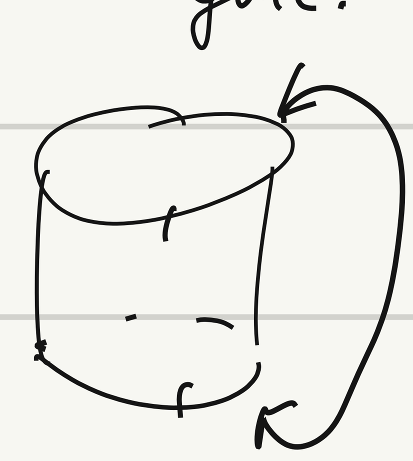
$(x, y) \mapsto (x + 2\pi n, y)$

$\mathbb{R}^2/\mathbb{Z} \cong$

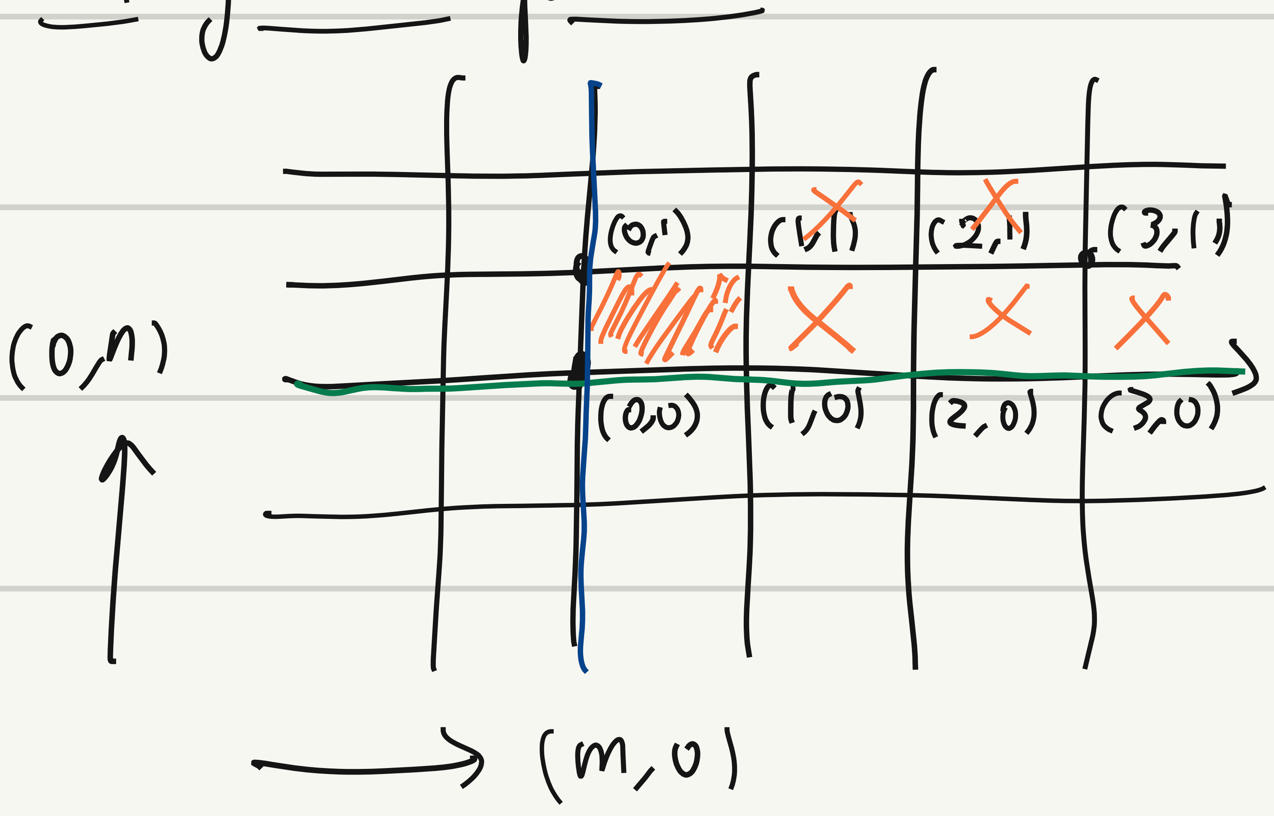
isometric.



glue.



Baby example 3

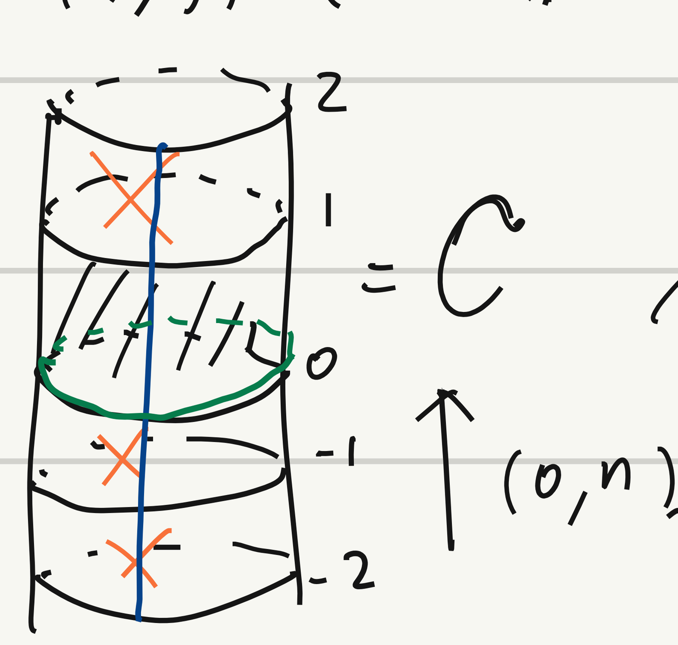


$\circlearrowleft \mathbb{Z}^2$

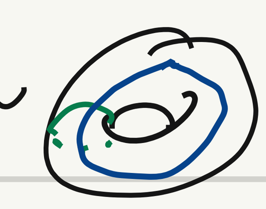
$(m, n): \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \mapsto (x, y) + (m, n).$

$\mathbb{R}^2 / \{(m, 0) \mid m \in \mathbb{Z}\} \cong$



$= C$ $\quad C / \{(0, n) \mid n \in \mathbb{Z}\} \cong$



4. Discrete groups of Isom(H)

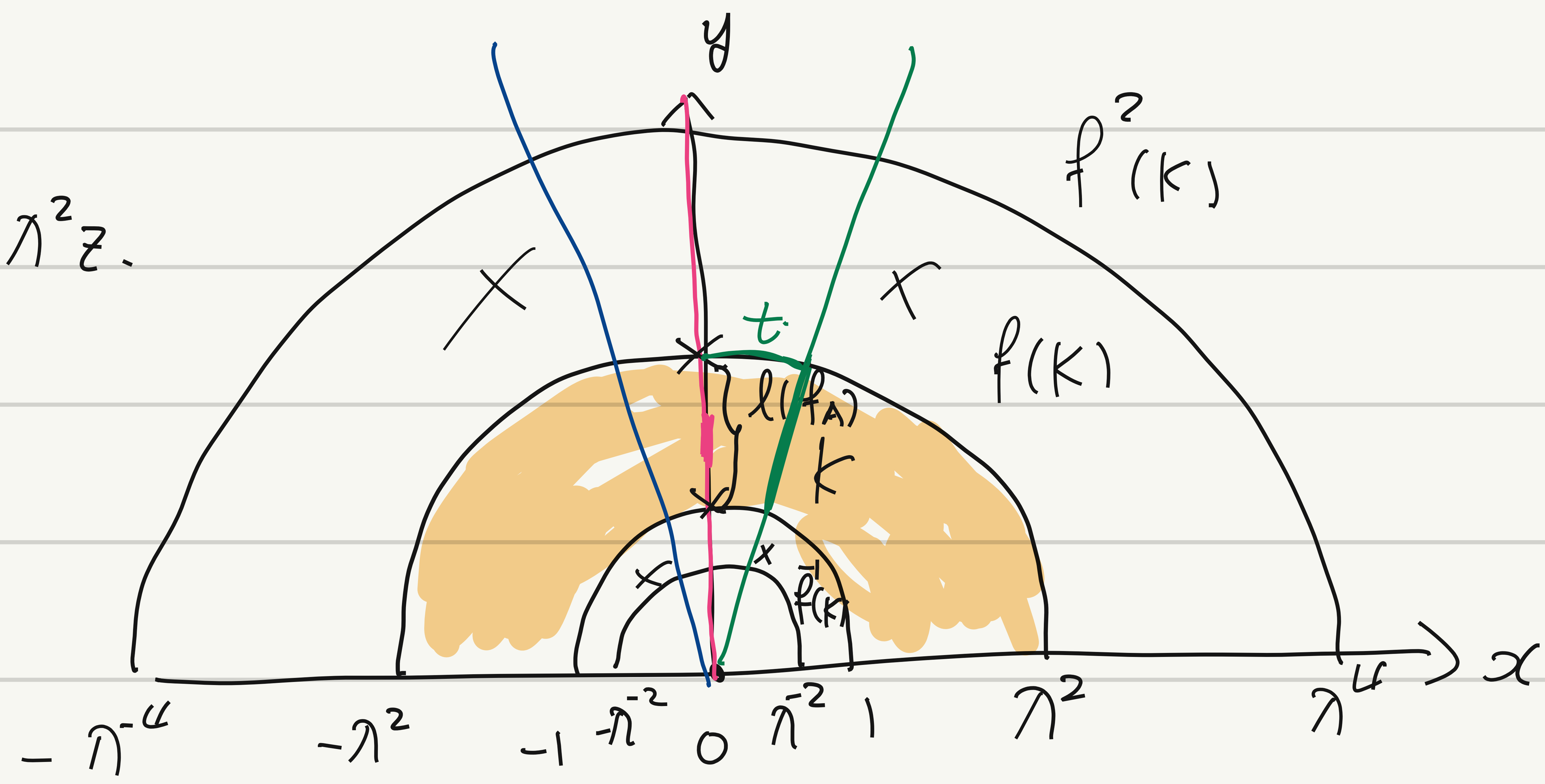
① Cyclic group

$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ $\quad f(z) = \lambda^2 z.$

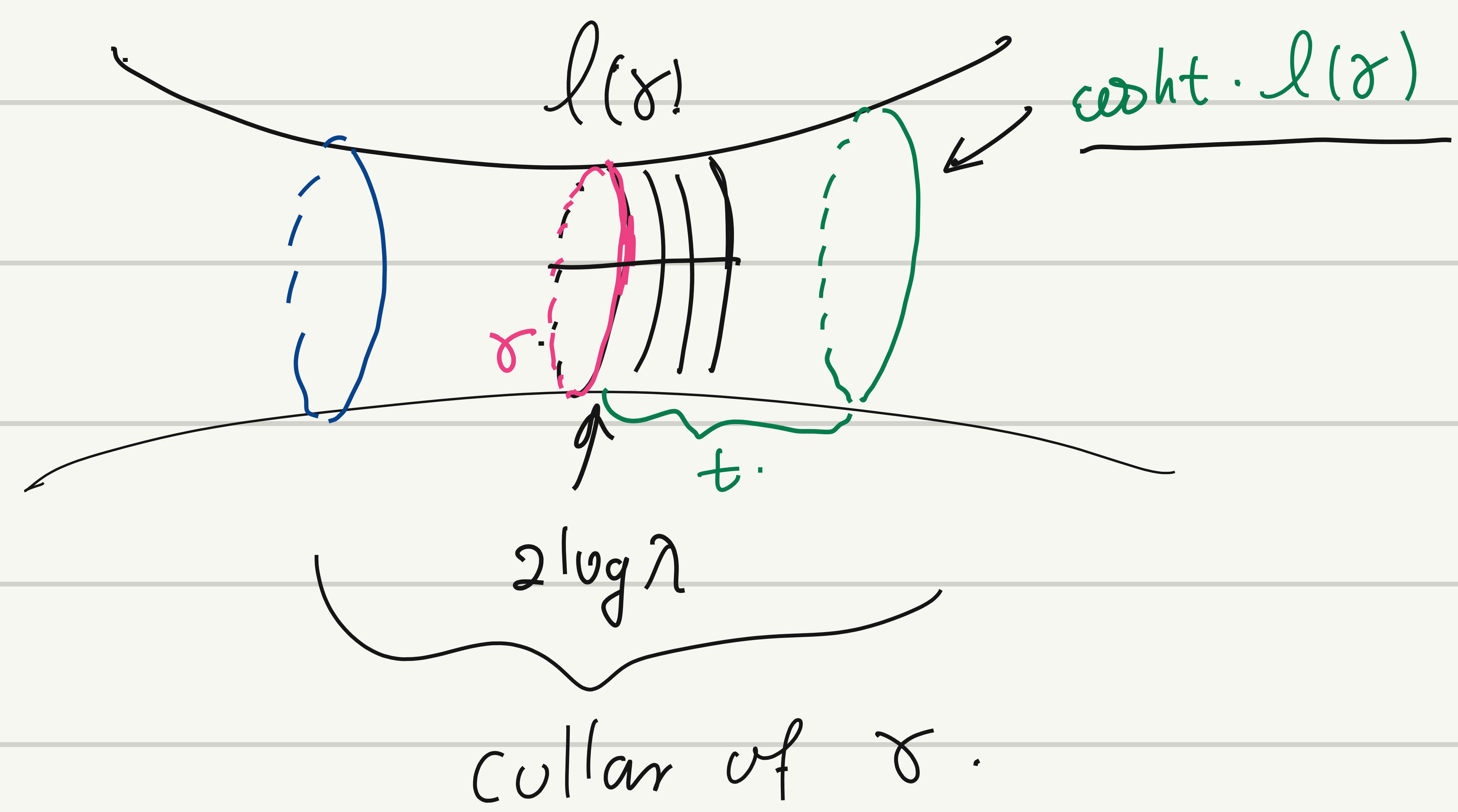
$\lambda > 1$

$\Gamma = \langle A \rangle \cong \mathbb{Z}$

$\ell(f_A) = 2 \log \lambda$



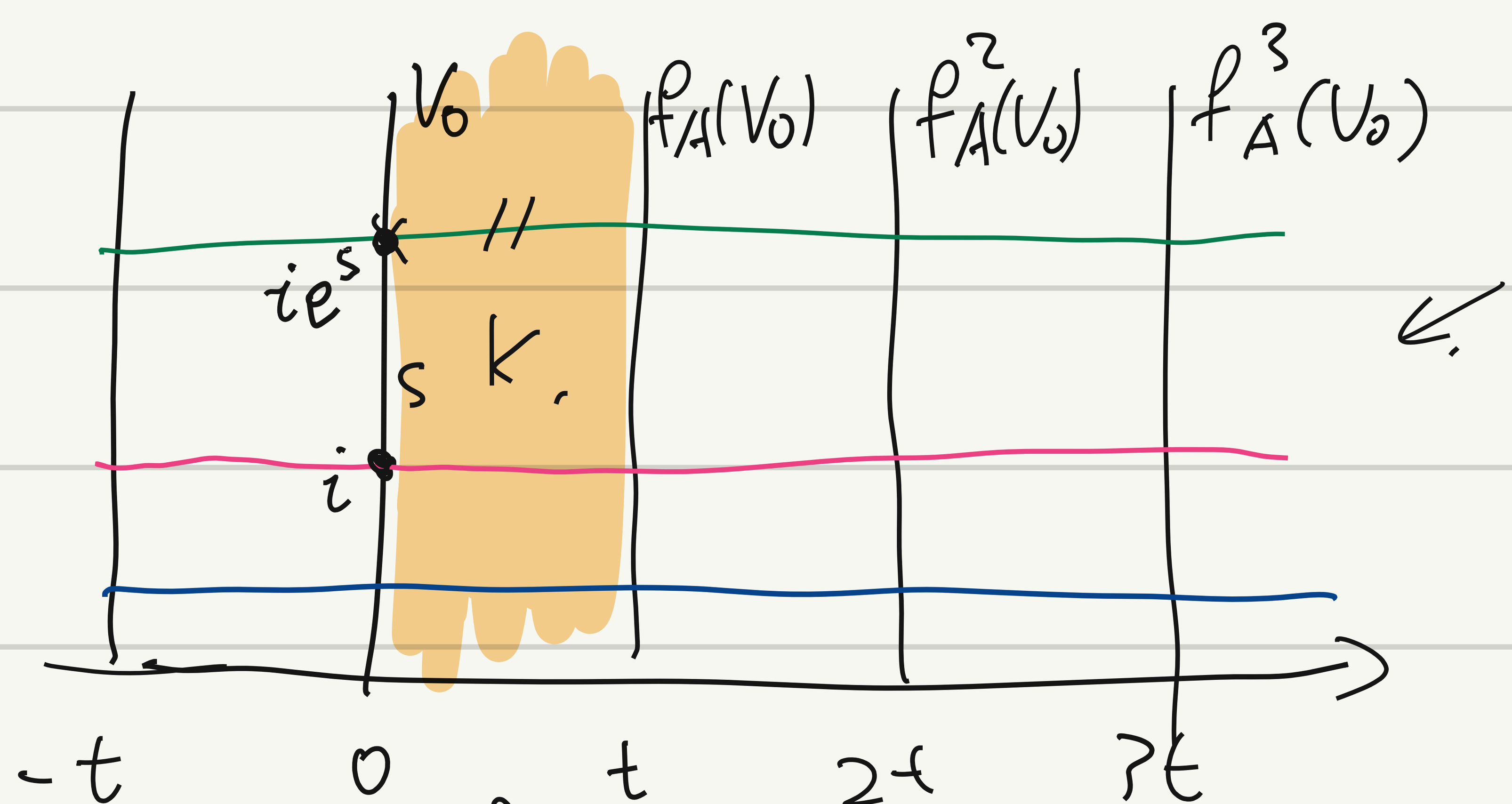
∃! closed geod.



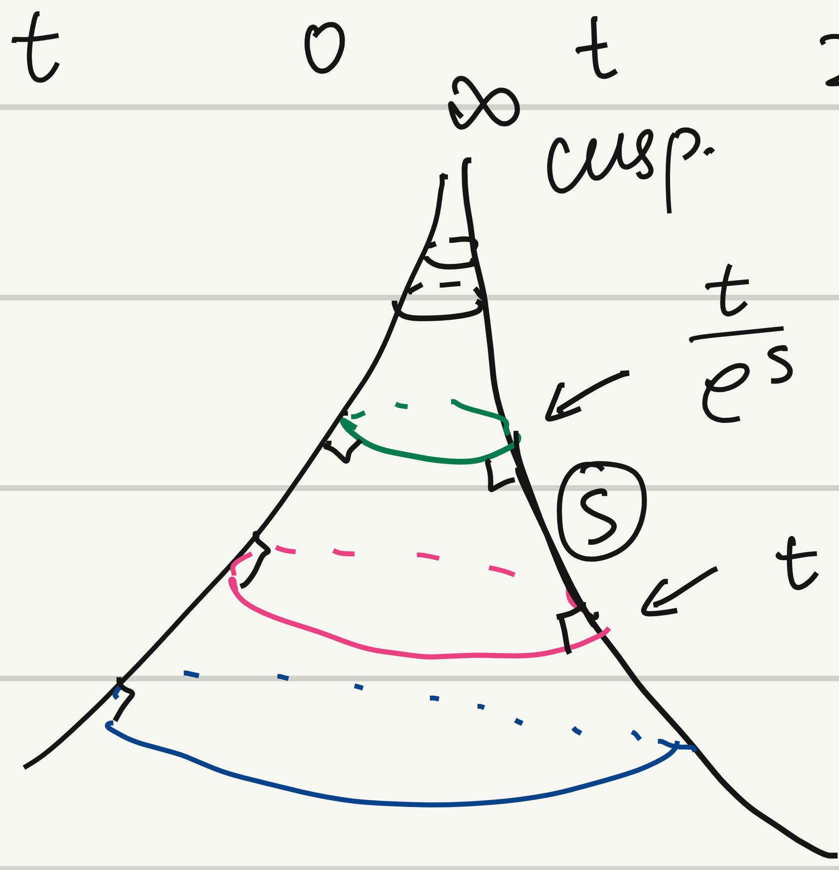
$A = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ $\quad f_A(z) = z + t$

$t > 0.$

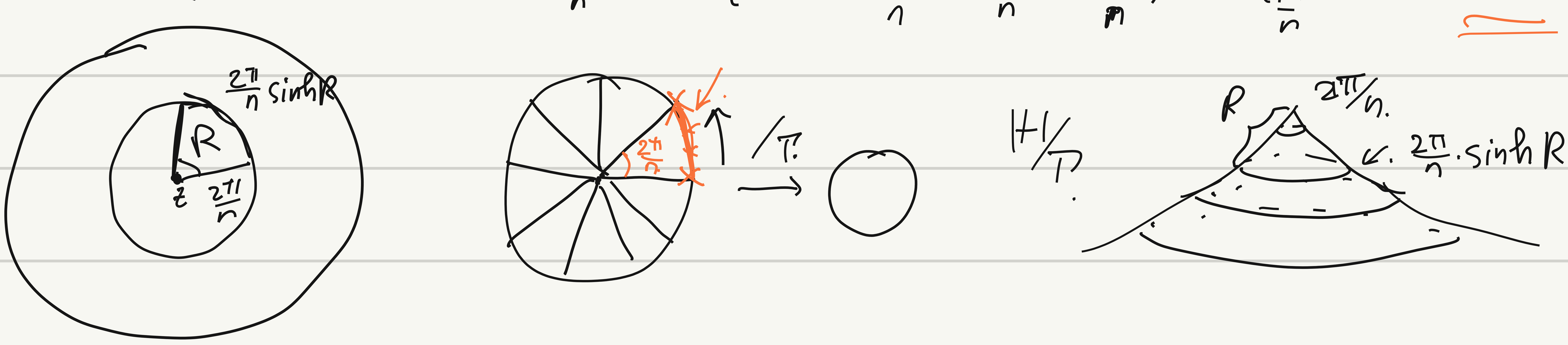
$\Gamma = \langle A \rangle \cong \mathbb{Z}$



∄ closed geod.

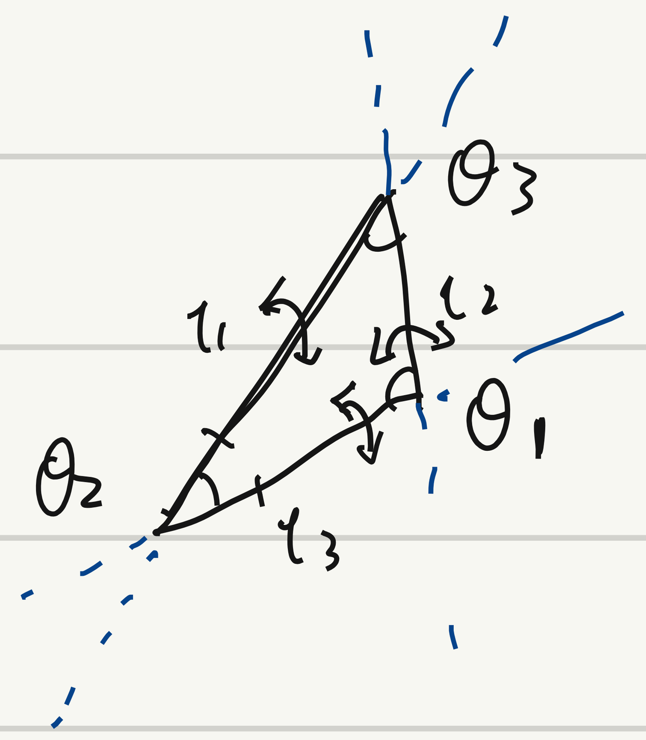


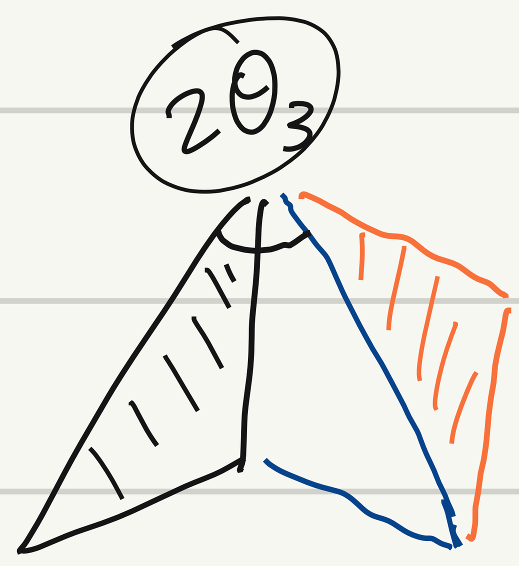
$A = \begin{bmatrix} \cos\frac{\pi}{n} & \sin\frac{\pi}{n} \\ -\sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{bmatrix}$ $\quad \Gamma = \langle \rho_{\frac{\pi}{n}} \rangle = \{ id = \rho_{\frac{\pi}{n}}^0, \rho_{\frac{\pi}{n}}, \rho_{\frac{\pi}{n}}^2, \dots, \rho_{\frac{\pi}{n}}^{n-1} \} \cong \mathbb{Z}/n\mathbb{Z}.$



② $\Delta(p,q,r)$ $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, $p,q,r \in \mathbb{N}_{>1} \cup \{\infty\}$

$\mathbb{R}^2$





$T = \langle \iota_1, \iota_2, \iota_3 \rangle$ $\iota_1 \circ \iota_2$ ↻ $2\theta_3$

Want T discrete. $\iota_2 \circ \iota_3$ ↻ $2\theta_1$

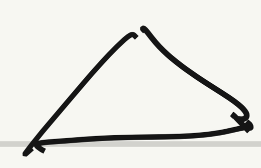
$2\theta_1 = \frac{2\pi}{p}$ $p,q,r \in \mathbb{N}_{>1}$ $\iota_3 \circ \iota_1$ ↻ $2\theta_2$

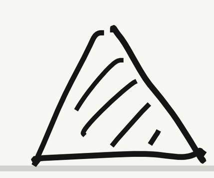
$2\theta_2 = \frac{2\pi}{q}$

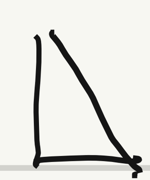
$2\theta_3 = \frac{2\pi}{r}$

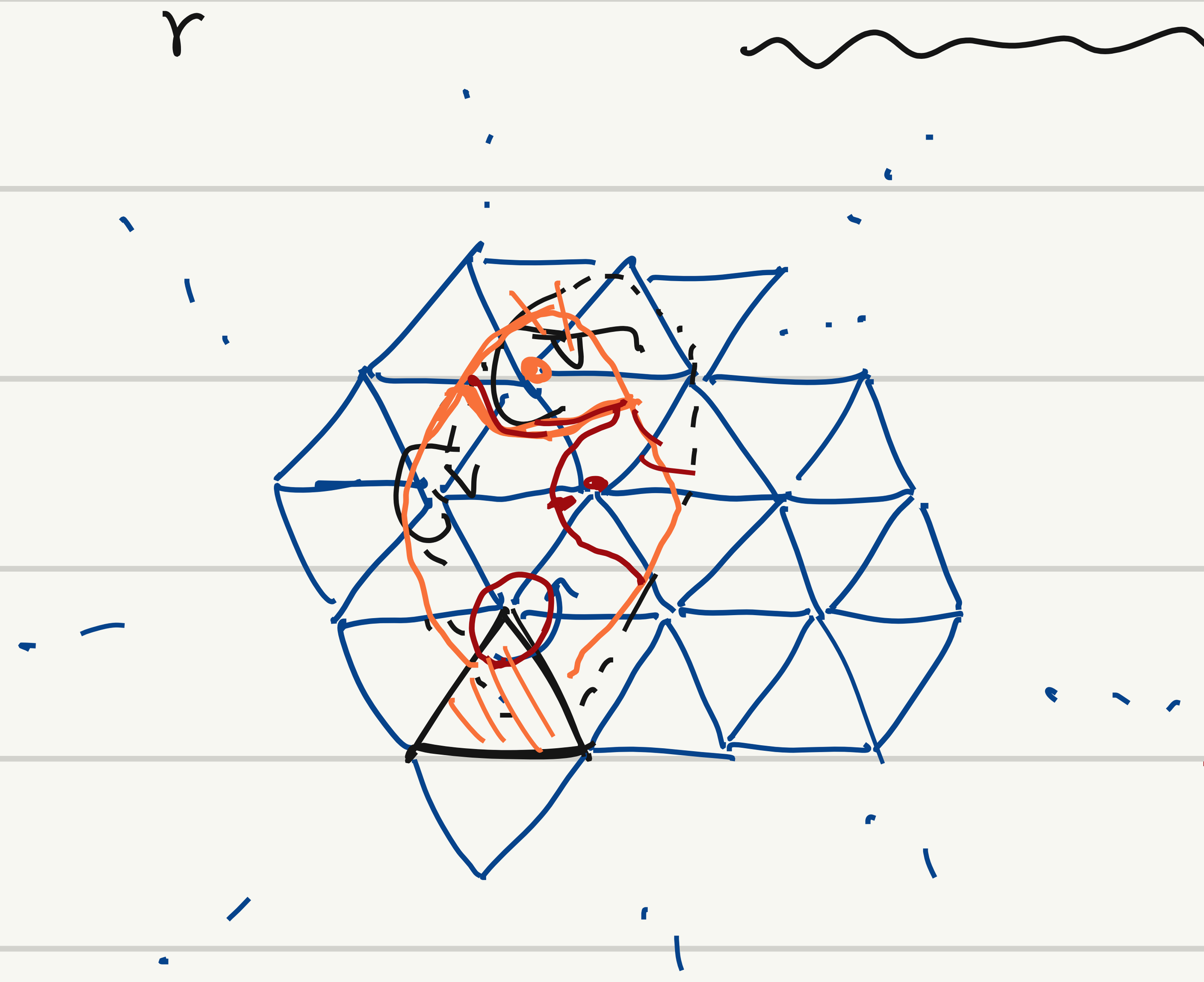
$\theta_1+\theta_2+\theta_3=\pi \Rightarrow \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$

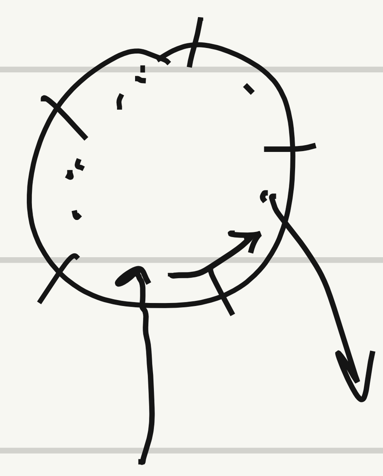
$\mathbb{R}^2$ $(2,4,4)$ $(3,3,3)$ $(2,3,6)$

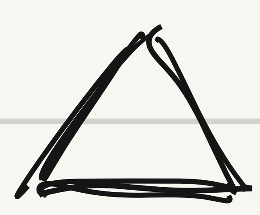


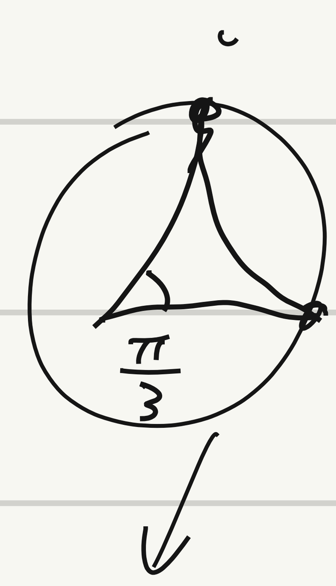






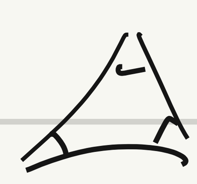


$\mathbb{R}^2/\Delta(3,3,3) =$ 

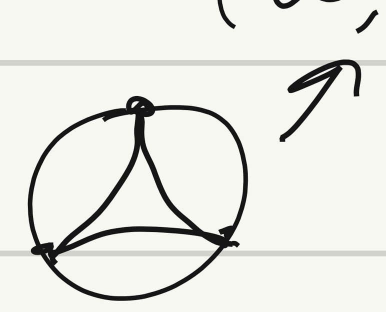


$\mathbb{H}$ $\theta_1+\theta_2+\theta_3<\pi \Rightarrow \frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$

$(p,q,r) = (2,3,8), (6,6,6), (5,6,7), (2,3,\infty), (3,\infty,\infty)$

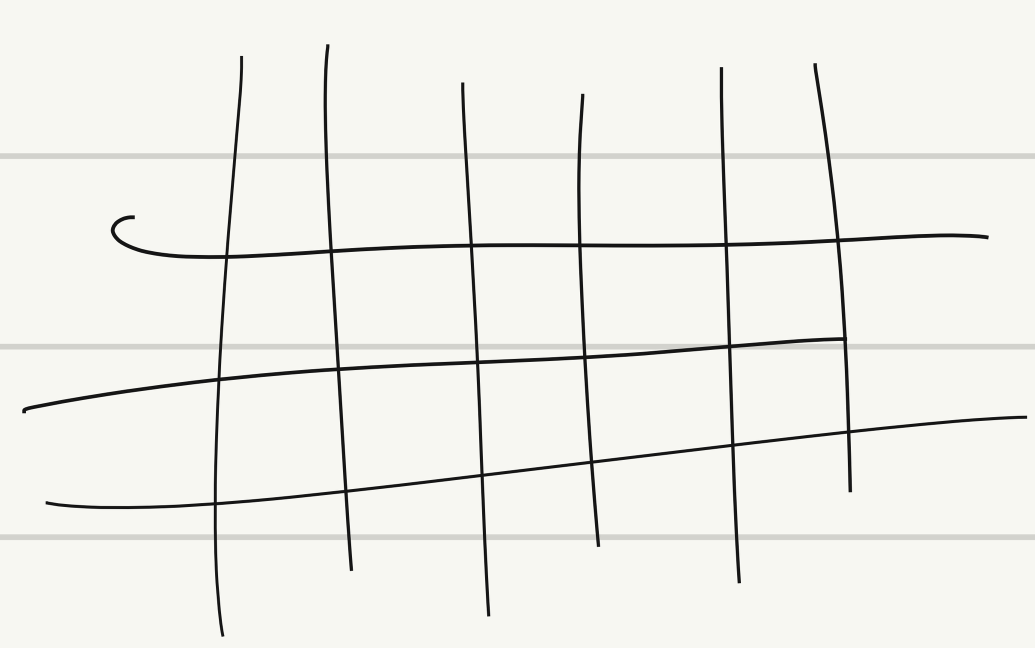
$\mathbb{H}/T \cong$ 

(∞,∞,∞) ...



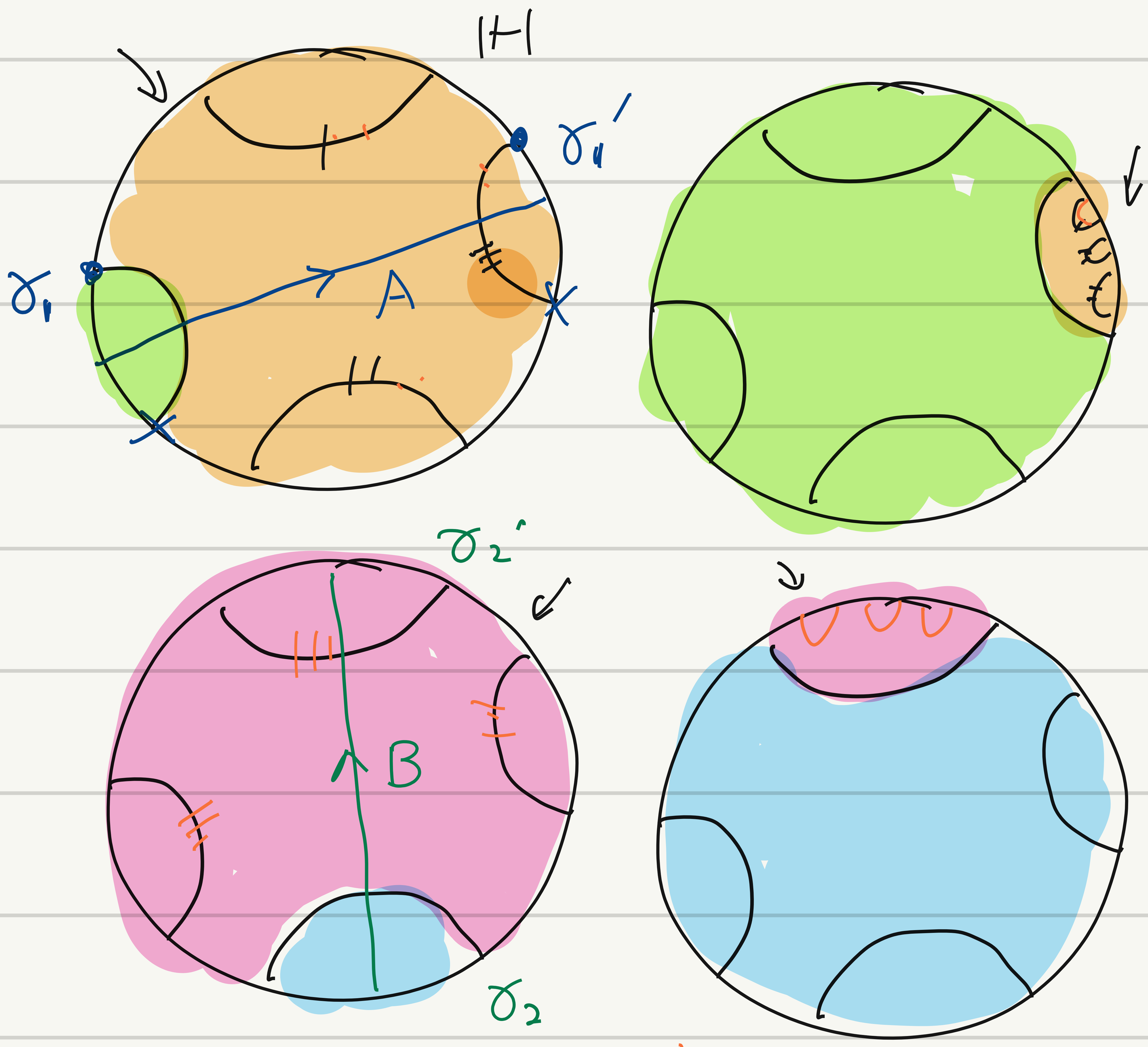
$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$

$\mathbb{S}$ $(p,q,r) = (2,2,n)$, $(2,3,3)$ ...



$f \in \mathrm{Isom}^+(\mathbb{H}) \simeq PSL(2,\mathbb{R})$.

③ Schottky group :



$\Gamma = \langle A, B \rangle \cong F_2$

free group of 2 letters.

$\langle a,b \rangle = \{ e, w(a,b) \mid \text{word of } a,b \}$.

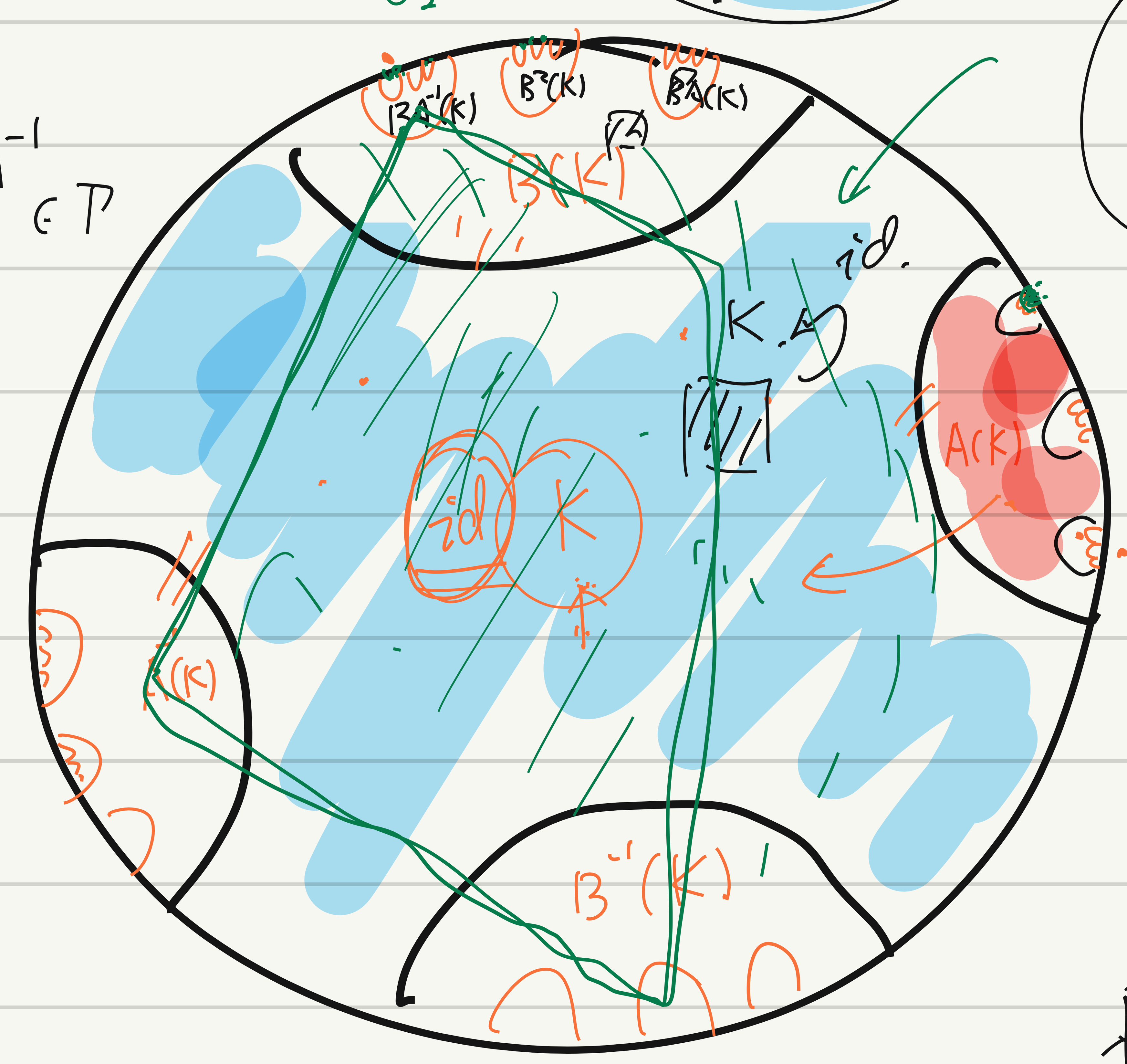
$w(a,b) = a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k}$

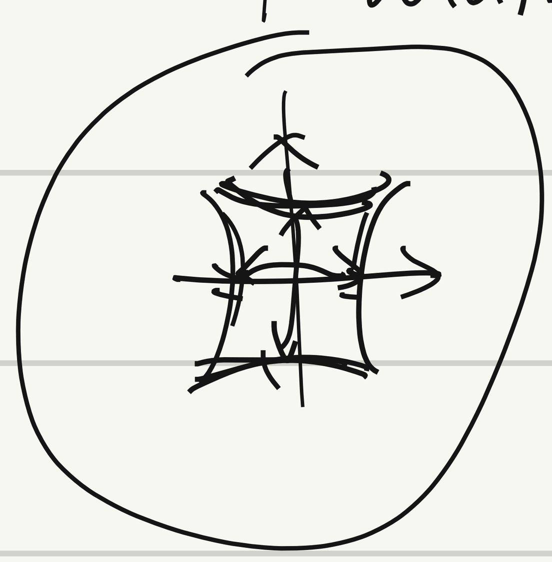
word.

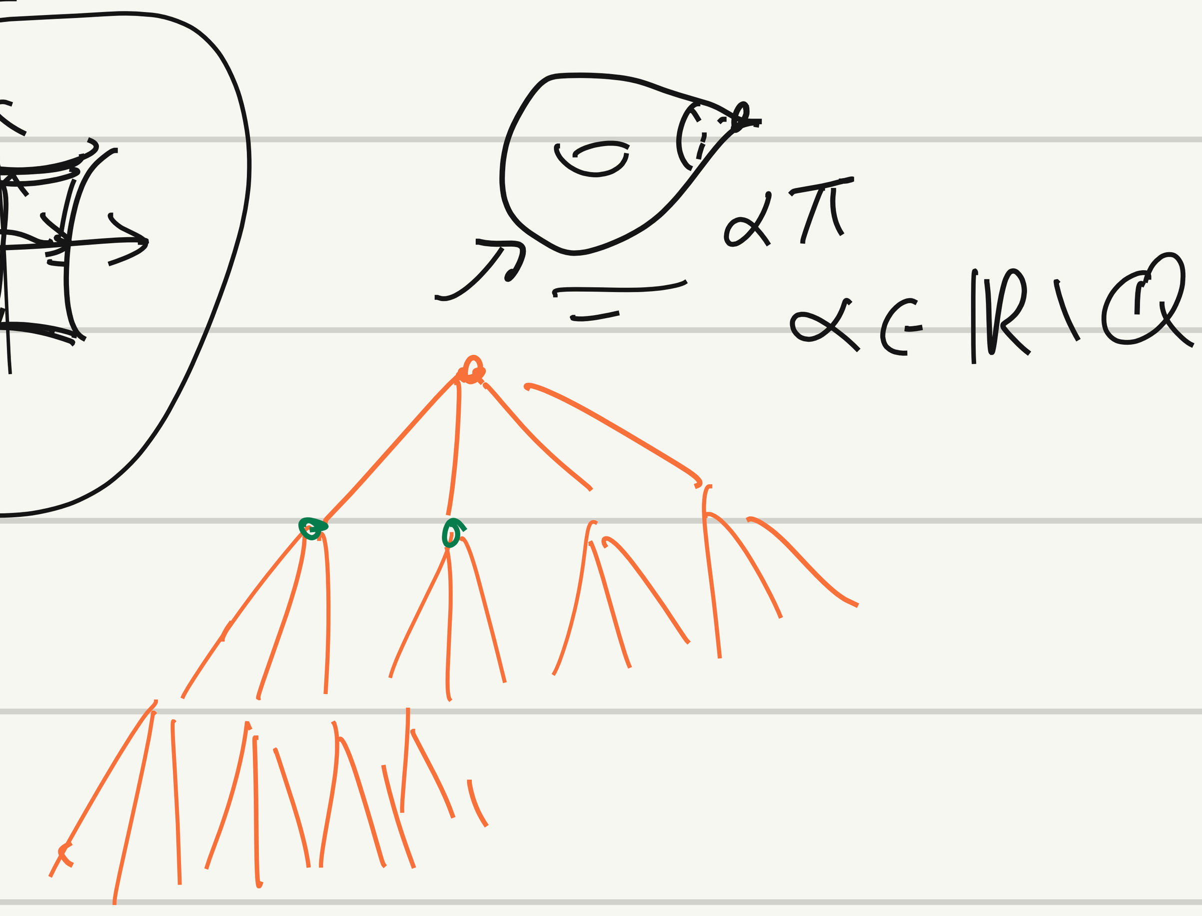
$w'(a,b)\, w(a,b)$

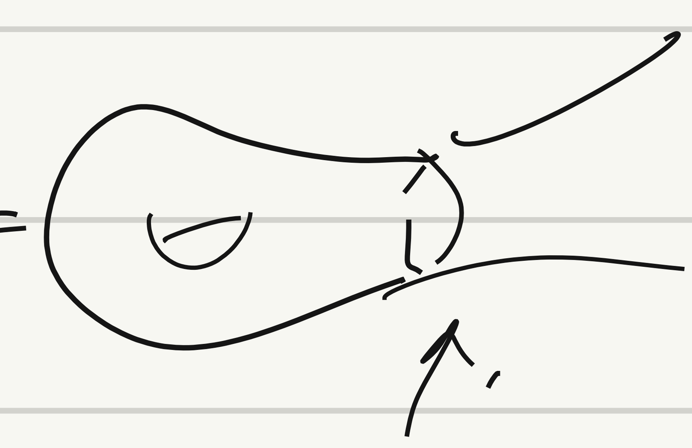
$\cdot\, w(a,b)\, e = w(a,b)$!

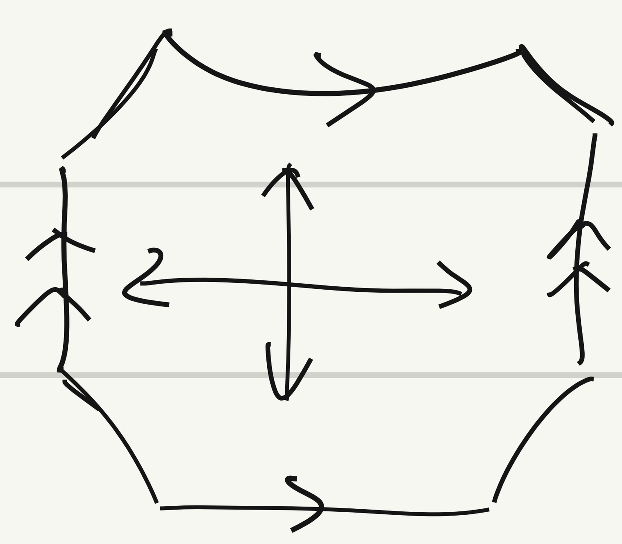
BA^{-1}, $B^2 \in \Gamma$



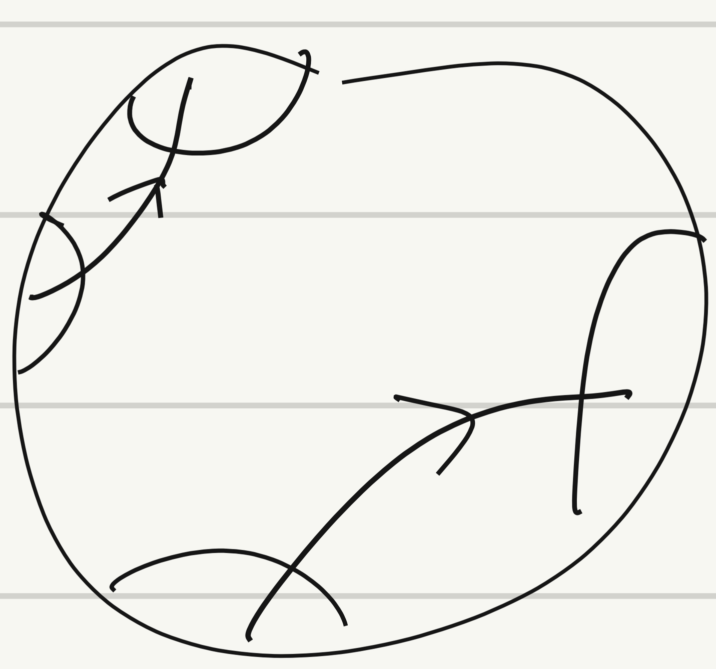


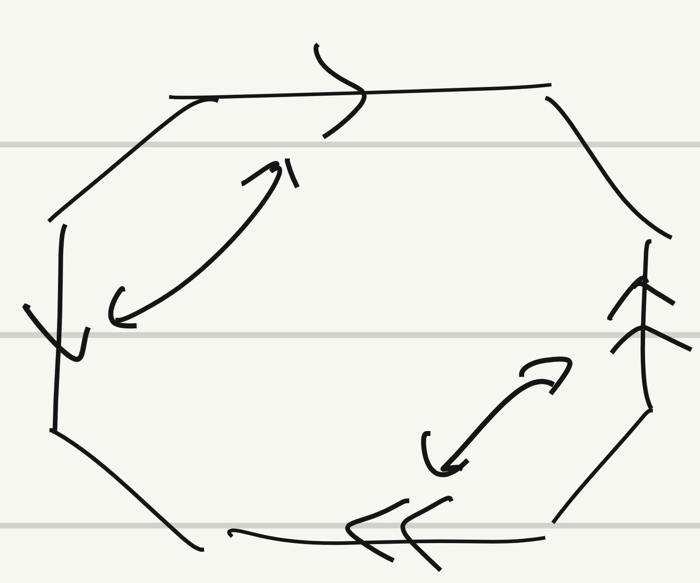


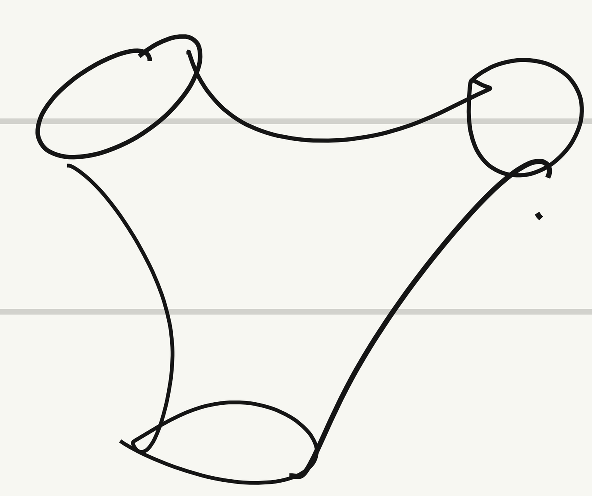
$\mathbb{H}/\Gamma =$ 



Second case Γ





$\mathbb{H}/\Gamma =$ 

Γ be a discrete group of $\mathrm{Isom}^+(\mathbb{H})$.

Def: A (open) domain $D \subset \mathbb{H}$ is called a fundamental domain of Γ if

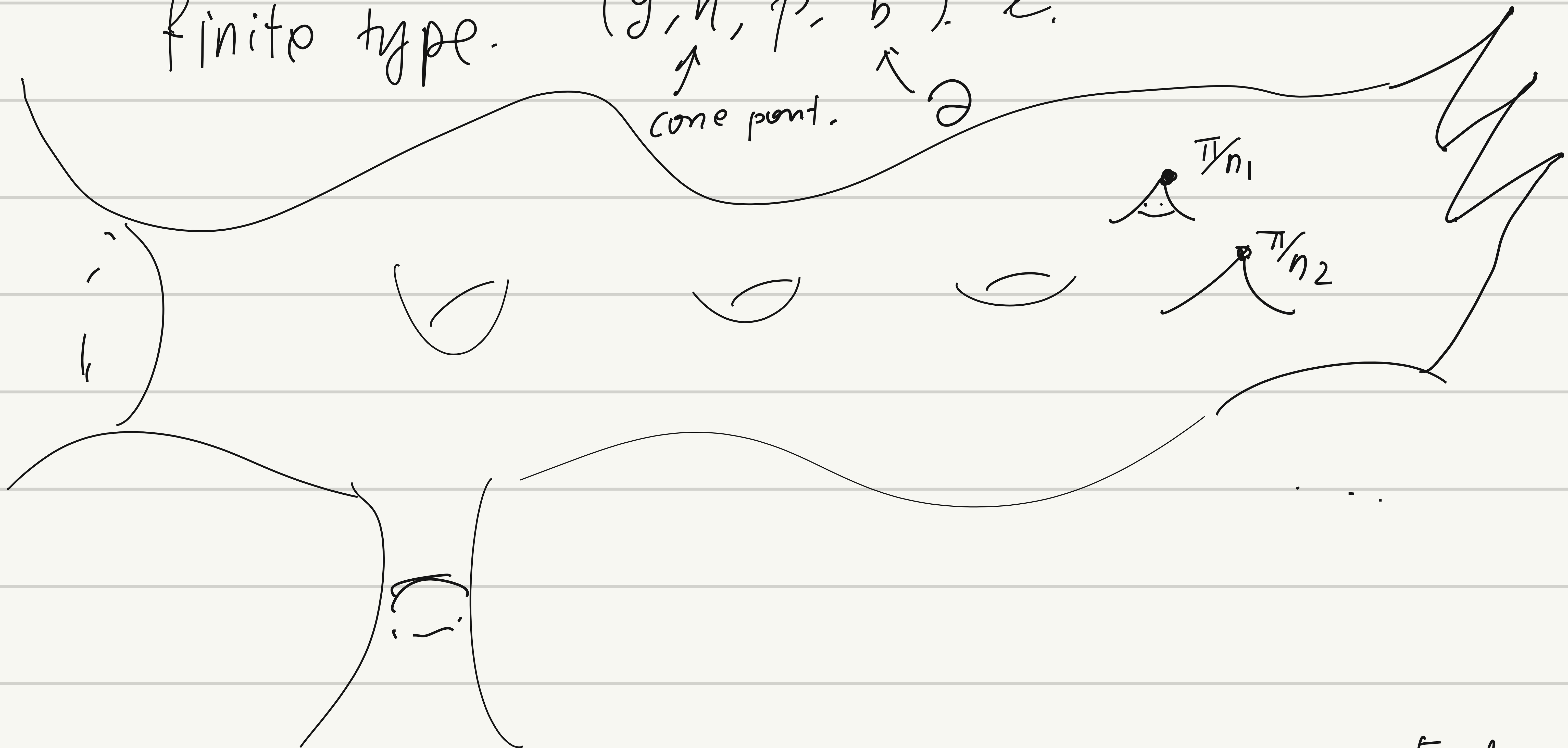
① $\forall f \in \Gamma \setminus \{id\}$, $f(D) \cap D = \emptyset$

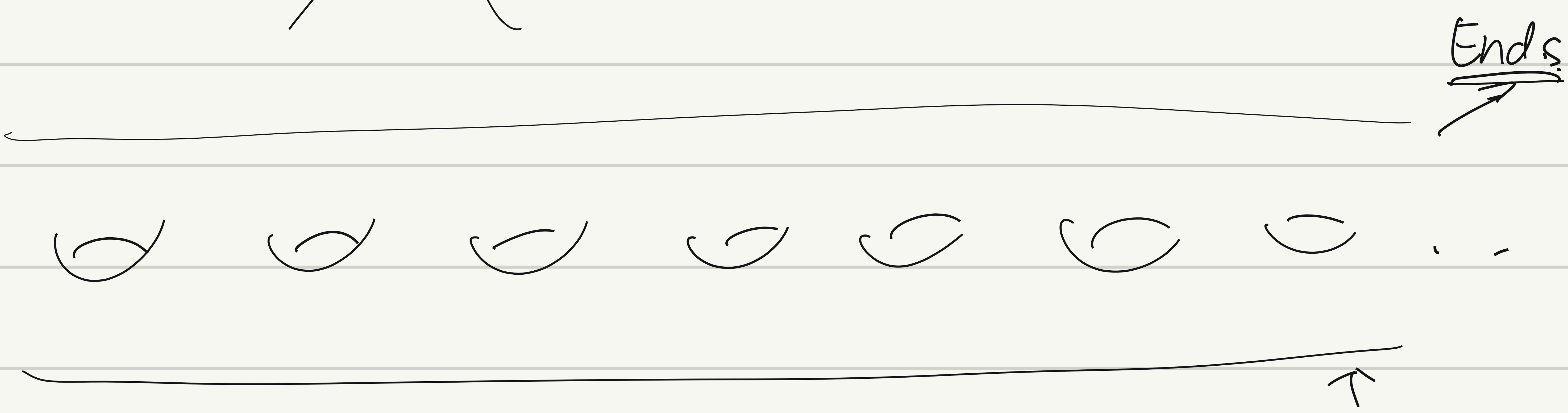
② $\bigcup_{f \in \Gamma} \overline{f(D)} = \mathbb{H}$.

genus / cusp

finite type. (g, n, p, b)

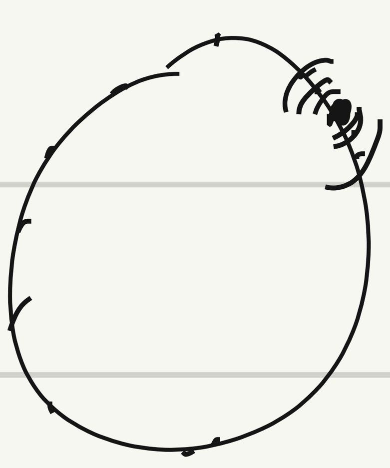
cone point. / ∂

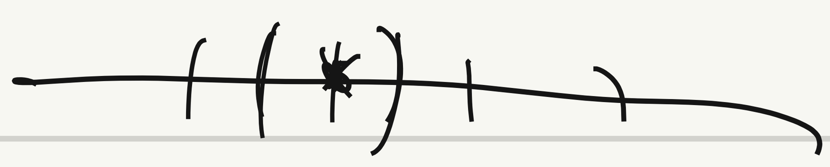




infinite typer surface.

+ types of ends.





$\{n\sqrt{2} \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$.

$$\begin{bmatrix} \cos n\sqrt{2}\pi & \sin n\sqrt{2}\pi \\ -\sin n\sqrt{2}\pi & \cos n\sqrt{2}\pi \end{bmatrix} \begin{bmatrix} m & m \\ m & n \end{bmatrix} = \begin{matrix} m+n & m+n \\ m+n & m+n \end{matrix}$$

$\Phi: (G_1, \circ_1) \longrightarrow (G_2, \circ_2)$

$\phi(id_1) = id_2$

$\phi(f_1 \circ_1 f_1') = \phi(f_1) \circ_2 \phi(f_1')$

$\frac{m}{q}$

$\frac{k}{q+1}$