Lecture 2: Geometry of Riemannian manifolds

- 2.0. A Remark on the Connection

$(\mathcal{U}, g)$

$$\frac{d}{ds}\Big|_{s=0} L_s = \int_0^l g_{\gamma(t)}(\nabla_{\gamma'} X(t), \gamma'(t))\, dt.$$

- $\hat{\nabla}_{\gamma'} X(t) \equiv \frac{dX}{dt}(t) + \boxed{\Gamma_{\gamma(t)}(\underline{X(t)}, \underline{\gamma'(t)})}$

$$\Gamma_{\gamma(t)}(X(t), \gamma'(t)) \equiv \frac{1}{2} G^{-1} DG_{X(t)}(\gamma'(t)).$$

$$\Gamma_x(\underline{u}, \underline{v}) = \frac{1}{2} G_x^{-1} DG_u(v)$$

- Covariant derivative

- $\Gamma_x(u, v) \in T_x\mathcal{U}$ if $u \in T_x\mathcal{U}$, $v \in T_x\mathcal{U}$.

$$\nabla_{\gamma'} X = \frac{dX}{dt} + \underline{\Gamma_{\gamma(t)}(\underline{\gamma'(t)}, \underline{X(t)})}$$

- $\underline{g\text{-parallel}}$

- If Γ is symmetric, then $\hat{\nabla}$ is a Levi-Civita connection.

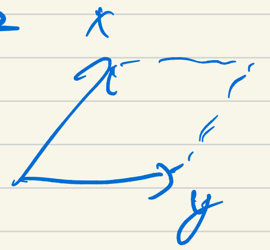
Definition of curvatures

- Riemann curvature tensor

$$R(X,Y,Z,W) = g(R(X,Y)Z, W)$$

- Sectional curvature

$$\sec(X,Y) = \frac{R(X,Y,Y,X)}{\|X\|^2\|Y\|^2 - g(X,Y)^2}$$



- Ricci tensor

$$\mathrm{Ric}(X,Y) = \sum_{j=1}^{n} R(E_j, X, Y, E_j)$$

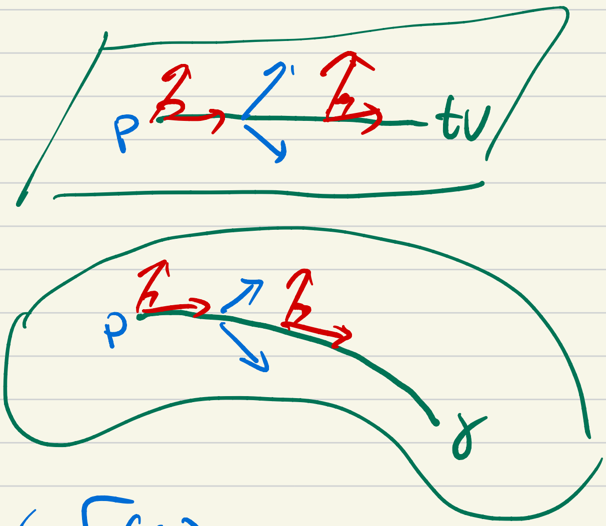
$\{E_j\}_{j=1}^{n}$ Orthonormal Basis

- Scalar curvature

$$Sc = \mathrm{tr}_g(\mathrm{Ric}) = \sum_{j=1}^{n} \mathrm{Ric}(E_j, E_j)$$

2.1 Some applications of Jacobi fields

$\gamma : [0,1] \to \underline{U}$ geodesic
with $\gamma(0) = p$,
$\gamma'(0) = v$.



Jacobi fields $U(t)$, $W(t)$:

$$U(0) = W(0) = 0, \quad U'(0) = u, \quad W'(0) = w.$$

Lemma. Let $\gamma : [0,1] \to U$ be a geodesic. Then $\forall\, \xi \in T_{\gamma(0)} U$, $\exists$ a unique Jacobi field $J(t)$ along γ with $\underline{J(0) = 0}$, $\underline{J'(0) = \xi}$.

<u>Pf.</u> $\quad J''(t) + R(J(t), \gamma'(t))\gamma'(t) = 0 \quad \square$

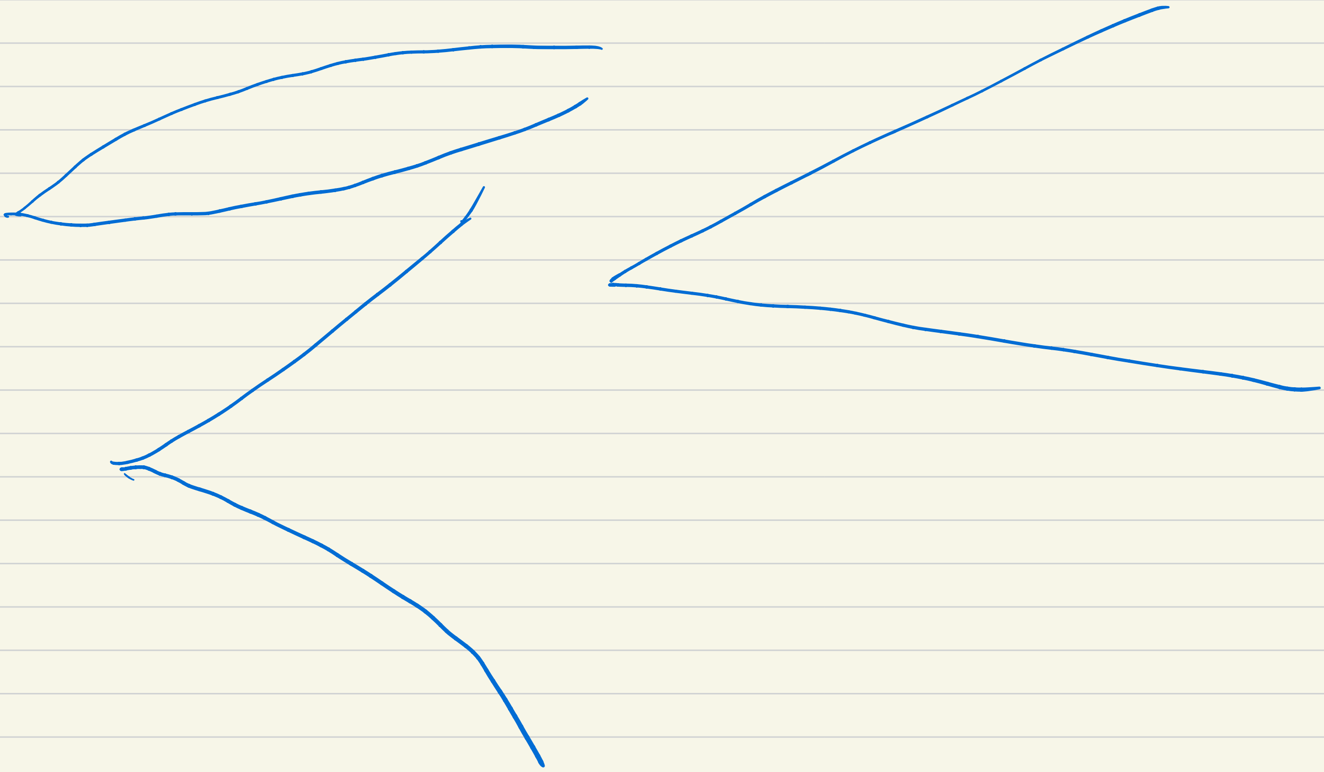
- Compute the Taylor expansion of $\underline{\langle U(t), W(t) \rangle}$ at $t = 0$.

$$\langle U(0), W(0) \rangle = 0, \quad \langle U, W \rangle'(0) = 0,$$

$$\langle U, W \rangle'' = 2\langle U'(0), W'(0) \rangle = \underline{2\langle u, w \rangle},$$

$$\langle U, W \rangle''' = 0, \quad \langle U, W \rangle^{(4)}(0) = -8R(\underline{v}, u, w, v)$$

• $\|u(t)\|^2 = t^2 - \frac{1}{3}\underline{\mathrm{Sec}(v,u)}\,t^4 + O(t^5)$



• Taylor expansion of g_{ij}

$\{x_1, \dots, x_n\}$ geodesic normal coordinates at p

$$V_i(t,s) = (t x_1, \dots, t(x_i + \underline{s}), \dots, t x_n)$$

$$\gamma(t) = t \cdot x$$

$$J_i(t) = \frac{\partial}{\partial s}\Big|_{s=0} V_i(t,s) = (0, \dots, \underset{\substack{\uparrow \\ i\text{th}}}{t}, \dots, 0)$$

$$= t\,\partial_i$$

$$g_{ij}(t) = \langle \partial_i, \partial_j \rangle = t^{-2} \langle J_i(t), J_j(t) \rangle$$

$$= \delta_{ij} - \frac{R(X, e_i, e_j, X)}{3} t^2 + o(|t|^3)$$

$$= \delta_{ij} - \frac{1}{3} R_{kijl} X^k X^l + o(|t|^3)$$

$$\underline{X^k = t \cdot X_k , \; X^l = t \cdot X_l}$$

$$\sqrt{\det(g_{ij})}$$

$$\mathrm{Vol}(B_r(p)) = \int_{B_r(p)} \underline{d\mathrm{vol}_g}$$

$$d\mathrm{vol}_g = \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_n$$

$$B_r(p) = \{ y \in \mathcal{U} \mid d_g(y, p) < r \}$$

$$S_r(p) = \partial B_r(p).$$

- Example S^2: $x^2+y^2+z^2=1$

$\forall p \in S^2$,

$$g = dr^2 + \sin^2(r)\, d\theta^2, \quad r \in [0,\pi], \quad \theta \in [0,2\pi]$$

Geodesic $\gamma = (\cos t, \sin t, 0)$.

Jacobi field along γ

$$J(t) = (0,\, 0,\, C_1 \cos t + C_2 \sin t).$$

ODE Theory in Riemannian Geometry

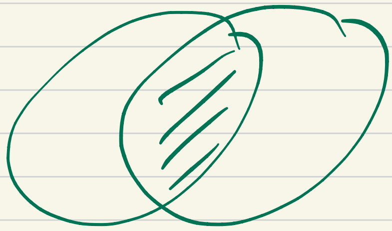
Distance structure (Geodesic) $\longrightarrow$ Jacobi field

Curvature tensor

Global Geometry / Topology

2.2. A Quick Introduction to Riemanian manifolds

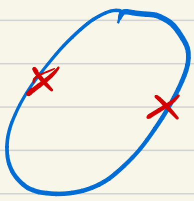
<u>Definition</u> A differentiable n-<u>manifold</u> is a
(Mannisfaltigkeit)
<u>Hausdorff</u> and <u>second countable</u> topological space
<u>M^n</u>, together with a C^∞-atlas $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}$
on M^n s.t.



(1) $M^n \equiv \bigcup \mathcal{U}_\alpha$

(2) $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n$ <u>homeomorphism</u>

(3) the transition map
$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$
is C^∞.

<u>Ex</u> S^1  S^2, T^2

<u>Definition</u> Let M^n be a differentiable manifold.

A tangent vector at p is a linear operator
$v : C^\infty(M^n) \to \mathbb{R}$ called a

derivation at p which satisfies

$$v(fg) = f(p)\, v(g) + g(p) \cdot v(f),$$

$$\forall f, g \in C^\infty(M^n).$$

<u>Definition</u> A vector field X on M^n is C^∞-map that assigns every $p \in M^n$ to a tangent vector $X(p) \in T_pM^n$.

The space of vector fields on M^n:

$$\mathfrak{X}(M^n).$$

<u>Definition</u> (tensor)

$$T : \underbrace{T_pM^n \times \ldots \times T_pM^n}_{r} \to \mathbb{R}$$ is

a $(0,r)$ tensor if T is a multilinear operator.

Definition (Riemannian manifold)

M^n : differentiable manifold

$$g : TM^n \times TM^n \to \mathbb{R}$$

Symmetric $(0,2)$-tensor field and

$$g(v,v) \geq 0 \quad \text{"="} \text{ iff } v = 0.$$

* **Definition** (Completeness) A Riemannian manifold (M^n, g) is said to be complete if $\forall p \in M^n$, $\mathrm{Exp}_p : T_pM^n \to M^n$ is well-defined.

Example

* • $\{x^2+y^2<1\}$ is incomplete

* • $\mathbb{R}^2 \setminus \{0^2\}$ is incomplete

• Closed Riemannian manifolds are always complete.

Theorem (Hopf - Rinow)

The following statements are equivalent:

(1) (M^n, g) is geodesically complete

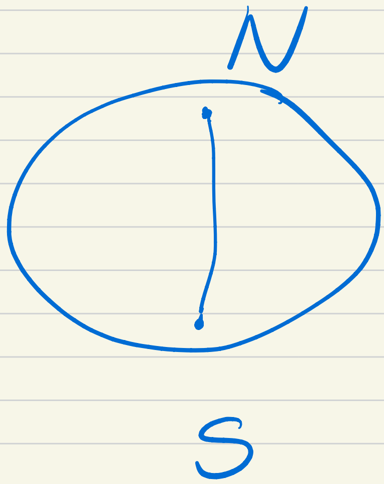
(2) (M^n, d_g) is a complete metric space

(3) Every bounded closed subset in M^n is compact.

(4) Every geodesic $\gamma : [0, a) \to M^n$ can be extended to a continuous path $\bar{\gamma} : [0, a] \to M^n$.

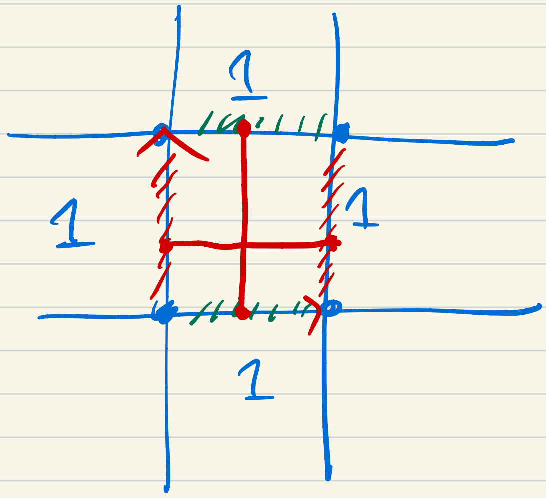
If one of the above holds, then $\forall p, q \in M^n$, $\exists$ a minimal geodesic connecting p and q. (NOT UNIQUE)

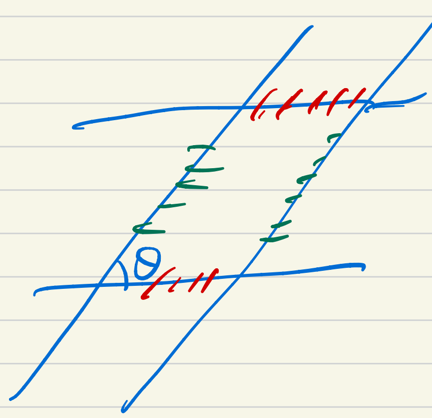
Example



Example $\quad \mathbb{R}^2$, $\underset{\sim}{\mathbb{T}^2} = \mathbb{R}^2/\mathbb{Z}^2$

Base elements $(1,0)$, $(0,1)$





Q: Find a geodesic γ s.t. $\overline{\gamma} = T^2$.

- Definition (Isometry)

A map $\varphi : (M^n, g) \to (N^k, h)$ is called an isometric embedding if

$$d_g(\varphi(p), \varphi(q)) = d_g(p,q)$$
$$\forall\ p, q \in M^n.$$

An isometric embedding φ is called an isometry if φ is surjective.

Lemma: If $\varphi : (M^n, g) \to (N^k, h)$ is an isometry. Then

(1) M^n, N^k diffeomorphic. $(n = k)$

(2) φ preserves the Riemannian metric,

$$h_{\varphi(p)}(D\varphi(u), D\varphi(v)) = g_p(u, v)$$

$p \in M^n$, $u, v \in T_pM^n$.

(3)

$$\begin{tikzcd} T_pM^n \arrow[r, "D\varphi"] \arrow[d, "\mathrm{Exp}_p"'] & T_{\varphi(p)}N^n \arrow[d, "\mathrm{Exp}_{\varphi(p)}"] \\ M^n \arrow[r, "\varphi"] & N^n \end{tikzcd}$$

$$\varphi \circ \mathrm{Exp}_p = \mathrm{Exp}_{\varphi(p)} \circ D\varphi_p$$

Example Let φ be an isometry on $(\mathbb{R}^n, g_0)$.

Then $\exists\, A \in O(n)$ s.t.

$$\varphi(x) = A\cdot x + \varphi(0), \quad \forall x \in \mathbb{R}^n.$$

$$\mathrm{Isom}(\mathbb{R}^n) \cong O(n) \ltimes \mathbb{R}^n.$$

Pf. We just assume $\varphi(0) = 0$.

Let us work with $\psi = \varphi \circ (D\varphi_0)^{-1}$.

$$D\psi_0 = \mathrm{Id}.$$

For any $v \in \mathbb{R}^n$. $\boxed{\psi(tv)}$ is a geodesic.

$$\psi(0) = 0, \quad D\psi_0 = \mathrm{Id}.$$

Uniqueness

$$\psi(tv) = tv. \qquad t=1, \ \boxed{\psi(v) = v}$$

$$\Rightarrow \boxed{\psi = \mathrm{Id}}.$$

$$\varphi = D\varphi_0 \in O(n)$$

□

Ex $\mathcal{U} \subseteq \mathbb{R}^n$.

$\varphi : \mathcal{U} \to \varphi(\mathcal{U})$ isometry.

$\exists\, \psi \in \mathrm{Isom}(\mathbb{R}^n)$ s.t. $\varphi = \psi|_{\mathcal{U}}$.