Lecture 2: Geometry of Riemannian manifolds

2.0. A Remark on the Connection 

\( (U, g) \)

\[ \frac{d}{ds} |_{s=0} L_s = \int_0^1 g_{\sigma(t)} (\nabla_{\gamma(t)} X(t), \gamma'(t)) \, dt. \]

\[ \nabla_{\gamma} X(t) = \frac{dX}{dt}(t) + \Gamma_{\sigma(t)} (X(t), \gamma'(t)) \]

\[ \Gamma_{\sigma(t)} (X(t), \gamma'(t)) = \frac{1}{2} G^{-1}_{\sigma} DG_x (X(t), \gamma'(t)). \]

\[ \Gamma_x (u, v) = \frac{1}{2} G^{-1}_x DG_{-u} (v) \]

- Covariant derivative

\[ \Gamma_x (u, v) \in T_x U \text{ if } u \in T_x U, \forall \in T_x U. \]

\[ \nabla_{\gamma} X = \frac{dX}{dt} + \Gamma_{\sigma(t)} (\gamma'(t), X(t)) \]

- \( g \)-parallel

- If \( \Gamma \) is symmetric, then \( \nabla \) is a Levi-Civita connection.
Definition of curvatures

- **Riemann curvature tensor**
  \[ R(x,y,z,w) = g(\mathcal{R}(x,y)z,w) \]

- **Sectional curvature**
  \[ \text{Sec}(x,y) = \frac{\mathcal{R}(x,y,y,x)}{||x||^2 ||y||^2 - g(x,y)^2} \]

- **Ricci tensor**
  \[ \text{Ric}(x,y) = \sum_{j=1}^{n} \mathcal{R}(E_j, x, y, E_j) \] for \( \{E_j\}_{j=1}^{n} \) orthonormal basis

- **Scalar curvature**
  \[ Sc = \text{tr}_g(\text{Ric}) = \sum_{j=1}^{n} \text{Ric}(E_j, E_j) \]
2.1 Some applications of Jacobi fields

Let $\gamma: [0,1] \rightarrow U$ be a geodesic with $\gamma(0) = p$, $\gamma'(0) = v$. Then $\gamma$ is a geodesic.

Let $U(t)$, $W(t)$ be the Jacobi fields along $\gamma$ with $U(0) = W(0) = 0$, $U'(0) = u$, $W'(0) = v$.

Lemma. Let $\gamma: [0,1] \rightarrow U$ be a geodesic. Then for all $t \in T_{\gamma(0)}U$, there exists a unique Jacobi field $J(t)$ along $\gamma$ with $J(0) = 0$, $J'(0) = \xi$.

Proof. \[ J''(t) + R(J(t), \gamma'(t))\gamma'(t) = 0 \]

- Compute the Taylor expansion of $\langle U(t), W(t) \rangle$ at $t = 0$.

\[ \langle U(0), W(0) \rangle = 0, \quad \langle U, W \rangle'(0) = 0, \]
\[ \langle U, W \rangle'' = 2 \langle U'(0), W'(0) \rangle = 2 \langle u, v \rangle, \]
\[ \langle U, W \rangle''' = 0, \quad \langle U, W \rangle^{(4)}(0) = -8R(\gamma(0), u, v). \]
- Taylor expansion of $g_{ij}$

\[ V_i(t, s) = (t x_1, \ldots, t(x_i + s), \ldots, t x_n) \]

\[ \gamma(t) = t \cdot x \]

\[ J_i(t) = \frac{\partial}{\partial s} \bigg|_{s=0} V_i(t, s) = (0, \ldots, t, \ldots, 0) \]

\[ = t \partial_i \]

\[ g_{ij}(t) = \langle \partial_i, \partial_j \rangle = t^{-2} \langle J_i(t), J_j(t) \rangle \]
\[
\begin{align*}
\delta_{ij} - \frac{R(X) e_i, e_j, X)}{3} t^2 + o(t^3) \\
= \delta_{ij} - \frac{1}{3} R^k_{ijl} X^k X^l + o(t^3) \\
X^k = t \cdot x_k, \quad X^l = t \cdot x_l \\
\sqrt{\det(g_{ij})} \\
\text{Vol}(B_r(p)) = \int_{B_r(p)} \text{dvol}_g \\
\text{dvol}_g = \sqrt{\det(g_{ij})} \, dx_1 \cdots dx_n \\
B_r(p) = \{ y \in U | \, d_g(y, p) < r \} \\
S_r(p) = \partial B_r(p).
\end{align*}
\]
- Example \( S^2: x^2 + y^2 + z^2 = 1 \)

\[
\forall p \in S^2, \\
g = dr^2 + \sin^2(r) \, d\theta^2, \quad r \in [0, \pi], \\
\theta \in [0, 2\pi]
\]

Geodesic \( \gamma = (\cos(t), \sin(t), 0) \).

Jacobi field along \( \gamma \)

\[
\overline{J(t)} = (0, 0, C_1 \cos t + C_2 \sin t)
\]

\underline{ODE Theory in Riemannian Geometry}

- Distance structure (Geodesic) → Jacobi field
- Curvature tensor
- Global Geometry/Topology
2.2. A Quick Introduction to Riemannian Manifolds

**Definition** A differentiable $n$-manifold is a Hausdorff and second countable topological space $M^n$, together with a $C^\infty$-atlas $\{ (U_a, \phi_a) \}$ on $M^n$ s.t.

1. $M^n = \bigcup U_a$
2. $\phi_a : U_a \to \mathbb{R}^n$ homeomorphism
3. the transition map $\phi_a \circ \phi_b^{-1} : \phi_b(U_a \cap U_b) \to \phi_a(U_a \cap U_b)$ is $C^\infty$

**Example** $S^1 \subset \subset S^2, \ T^2$

**Definition** Let $M^n$ be a differentiable manifold. A tangent vector at $p$ is a linear operator $\mathbf{v} : C^\infty(M^n) \to \mathbb{R}$ called a
derivation at $p$ which satisfies
\[ \nabla (fg) = f(p) \nabla (g) + g(p) \cdot \nabla (f), \]
\[ \forall f, g \in C^\infty (M^n). \]

**Definition** A vector field $X$ on $M^n$ is
\[ C^\infty \text{-map} \] that assigns every $p \in M^n$
to a tangent vector $X(p) \in T_p M^n$. The space of vector fields on $M^n$:
\[ \mathcal{X} (M^n). \]

**Definition (tensor)**
\[ T : T_p M^n \times \ldots \times T_p M^n \to \mathbb{R} \]
is a $(0,c)$ tensor $T$ if $T$ is a
multilinear operator.
Definition (Riemannian manifold) \( M^n \): differentiable manifold

\[ g : TM^n \times TM^n \to \mathbb{R} \]

Symmetric \((0,2)\)-tensor field and \( g(u, v) \geq 0 \) "=" iff \( v = 0 \).

Definition (Completeness) A Riemannian manifold \((M^n, g)\) is said to be complete if \( \forall p \in M^n, \ Exp_p : T_p M^n \to M^n \) is well-defined.

Example

- \( \{x^2 + y^2 < 1\} \) is incomplete

- \( \mathbb{R}^2 \setminus \{0\} \) is incomplete

- Closed Riemannian manifolds are always complete.
Theorem (Hopf - Rinow)

The following statements are equivalent:

1. $(M^n, g)$ is geodesically complete
2. $(M^n, dg)$ is a complete metric space
3. Every bounded closed subset in $M^n$ is compact.
4. Every geodesic $\gamma: [0, a) \rightarrow M^n$ can be extended to a continuous path $\tilde{\gamma}: [0, a] \rightarrow M^n$.

If one of the above holds, then for any minimal geodesic connecting $p$ and $q$, (NOT UNIQUE)

Example

\[ \begin{array}{c}
N \\
\hline
1 \\
\end{array} \]
Example: $\mathbb{R}^2$, $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$

Base elements $(1,0), (0,1)$

Q: Find a geodesic $\gamma$ s.t. $\gamma(1) = \gamma^2$.

- Definition (Isometry)

A map $\Phi: (M^7, g) \rightarrow (N^k, h)$ is called an isometric embedding if

$$d_g(\Phi(p), \Phi(q)) = d_g(p, q)$$

for all $p, q \in M^7$.

An isometric embedding $\Phi$ is called an isometry if $\Phi$ is surjective.
Lemma. If \( \varphi : (M^n, g) \to (N^k, h) \) is an isometry. Then

1. \( M^n, N^k \) are diffeomorphic. \((n = k)\)
2. \( \varphi \) preserves the Riemannian metric,
   \( h_{\varphi(p)}(D\varphi(u), D\varphi(v)) = g_p(u, v) \)
   \( p \in M^n, \ u, v \in T_p M^n \)
3. \[
\begin{aligned}
   T_p M^n \xrightarrow{D\varphi} T_{\varphi(p)} N^n \\
   \sim \quad \sim \\
   \exp_p \downarrow \quad \downarrow \exp_{\varphi(p)} \\
   M^n \xrightarrow{\varphi} N^n \\
   \varphi \circ \exp_p = \exp_{\varphi(p)} \cdot D\varphi_p
\end{aligned}
\]
Example: Let $\Psi$ be an isometry on $(\mathbb{R}^n, g_0)$. Then $\exists A \in O(n)$ st.

$$\Psi(x) = A \cdot x + \Psi(0), \quad \forall x \in \mathbb{R}^n.$$ 

$\text{Isom}(\mathbb{R}^n) = O(n) \times \mathbb{R}^n$.

**Proof.** We just assume $\Psi(0) = 0$.

Let us work with $\Psi = \Psi_0(D\Psi_0)^{-1}$.

$D\Psi_0 = \text{Id}$. For any $u \in \mathbb{R}^n$, $\Psi(tu)$ is a geodesic.

$\Psi(0) = 0$, $D\Psi_0 = \text{Id}$.

$\Psi(tu) = tu$. $t=1$, $\Psi(1) = u$.

$\Rightarrow \Psi = \text{Id}$. $\Psi = D\Psi_0 \in O(n)$.
\[ \text{Ex: } U \subseteq \mathbb{R}^n. \]

\[ \varphi: U \rightarrow \varphi(U) \text{ isometry.} \]

\[ \exists \psi \in \text{Isom}(\mathbb{R}^n) \text{ s.t. } \varphi = \psi|_U. \]