Introduction to hyperbolic surfaces

Exercises III

For $x \in \mathbb{R}$, $y > 0$ and $r > 0$, we use H_y for the horizontal line passing iy, V_x for the vertical geodesic with end point x and ∞, and $C(x, r)$ for the circular geodesic with Euclidean center x and Euclidean radius r.

1. Let $x \in (0, 1)$. Let γ_x denote the circular geodesic with end points x and $1/x$.
   a) (Easy) Compute the formula for the reflection ι_x of $\mathbb{H}$ along γ_x.
   b) (Easy) Show that
   $$\lim_{x \to 0+} \iota_x = \iota_0,$$
   where ι_0 is the reflection along V_0, i.e. for any $z \in \mathbb{H}$, we have
   $$\lim_{x \to 0+} \iota_x(z) = \iota_0(z).$$
   c) (Normal) Compute the distance $d(x)$ between γ_x and V_0.
   d) (Normal) Let $d_0 > 0$ be a constant. Find the hyperbolic isometry f such that
      - the axis of f is $C(0, 1)$;
      - the translation distance $l(f)$ of f is d_0;
      - the translation direction is from -1 to 1.

   (Hint: Write x as a function of d. Use ι_0 as one of the two reflections)

2. Consider the parabolic isometry ϕ_t.
   a) (Easy) Find $x \in \mathbb{R}$ such that $V_x = \phi_t(V_0)$.
   b) (Easy) Compute the length l_y of the segment in H_y between V and V_0.
   c) (Easy) Show
   $$\lim_{y \to +\infty} l_y = 0,$$
   and use it to conclude that the translation distance $l(\phi_t)$ of ϕ_t is 0.
   d) (Easy) Show that $l(\phi_t)$ is not realizable, i.e. there is no $z \in \mathbb{H}$ such that
   $$l(\phi_t) = \mathrm{d}_{\mathbb{H}}(z, \phi_t(z)).$$

3. (Easy) Let $z = x + iy \in \mathbb{H}$. Find the elliptic isometry whose fixed point is z with rotation angle π. (Hint: Find two geodesics intersecting each other at z with intersection angle $\pi/2$.)