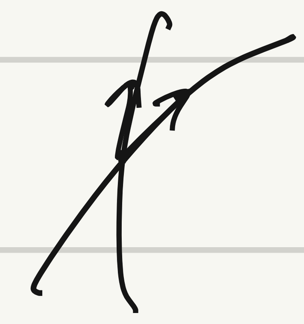
Last time:

- regular vs C^1 regular: $C^1 + \dot\gamma(t) \neq \vec{0}$



  we only care lengths of paths.

- Complete geodesics: ① locally geodesic

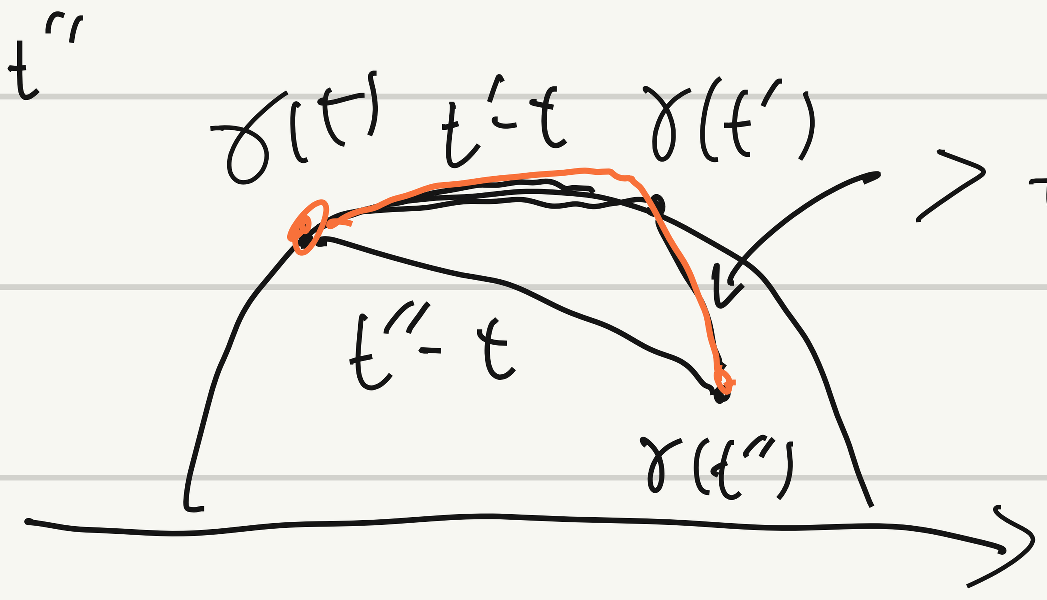
  ② infinite extensions on both ends.

Def: A complete geodesic in $\mathbb{H}$ is a path with

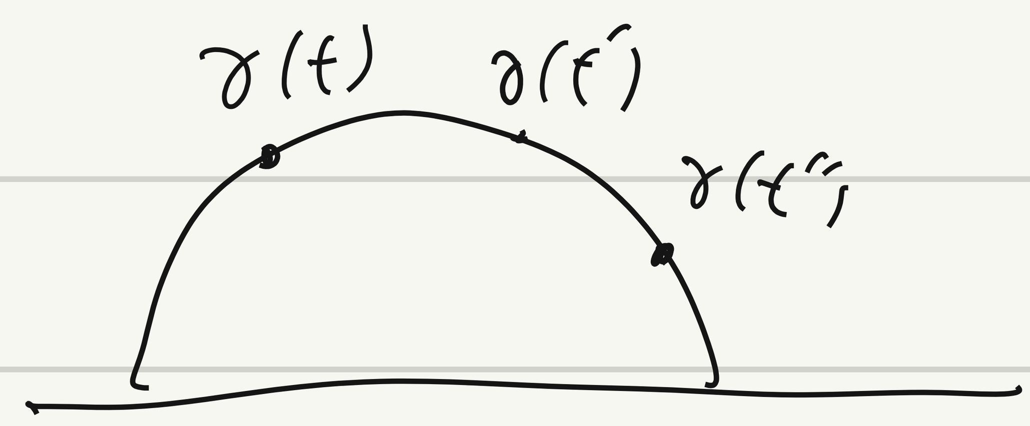
$$\gamma : \mathbb{R} \longrightarrow \mathbb{H}$$

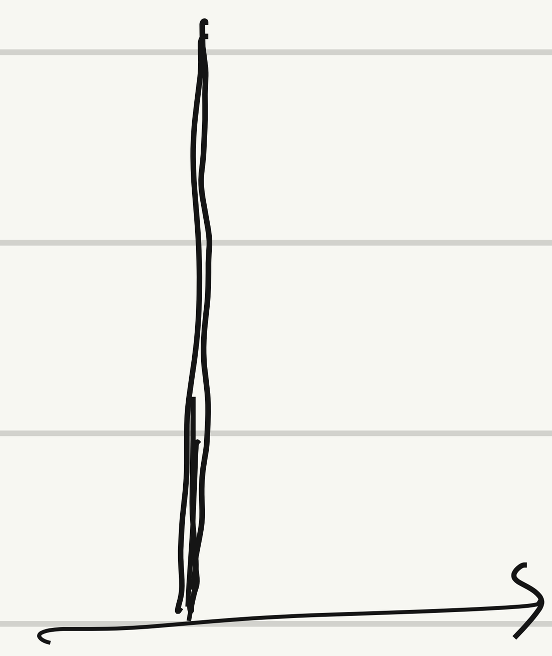
$$\forall t < t', \quad d_{\mathbb{H}}(\gamma(t), \gamma(t')) = |t - t'|$$

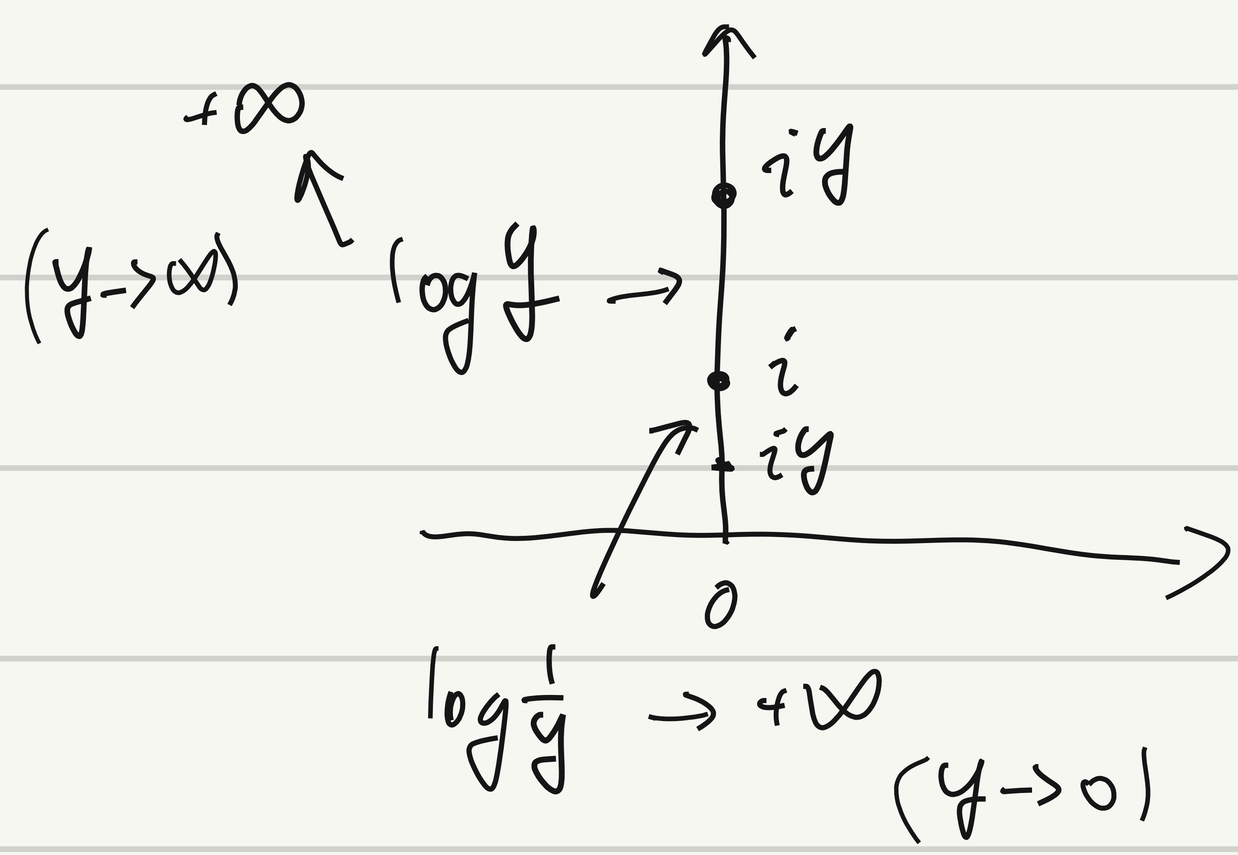
$t < t' < t''$



$> t'' - t'$?

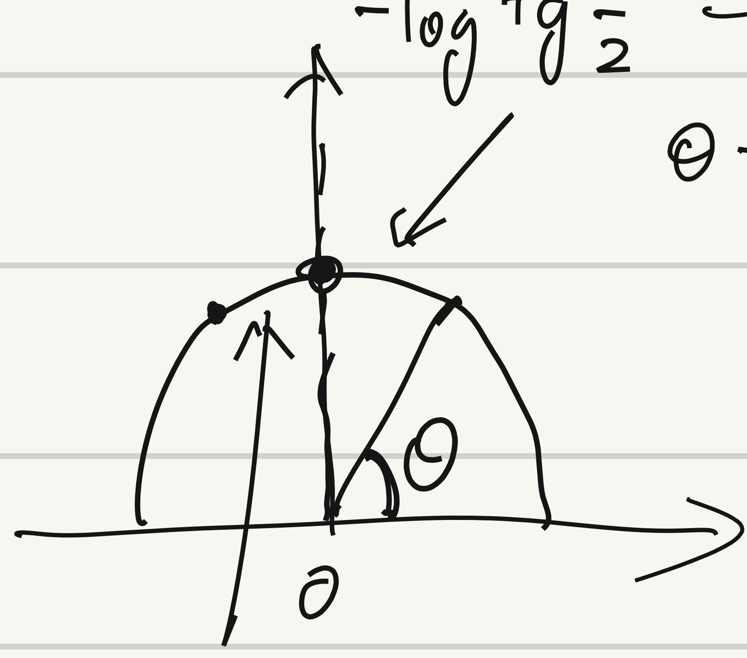






$-\log tg\frac{\theta}{2} \to \infty$

$\theta \to 0$



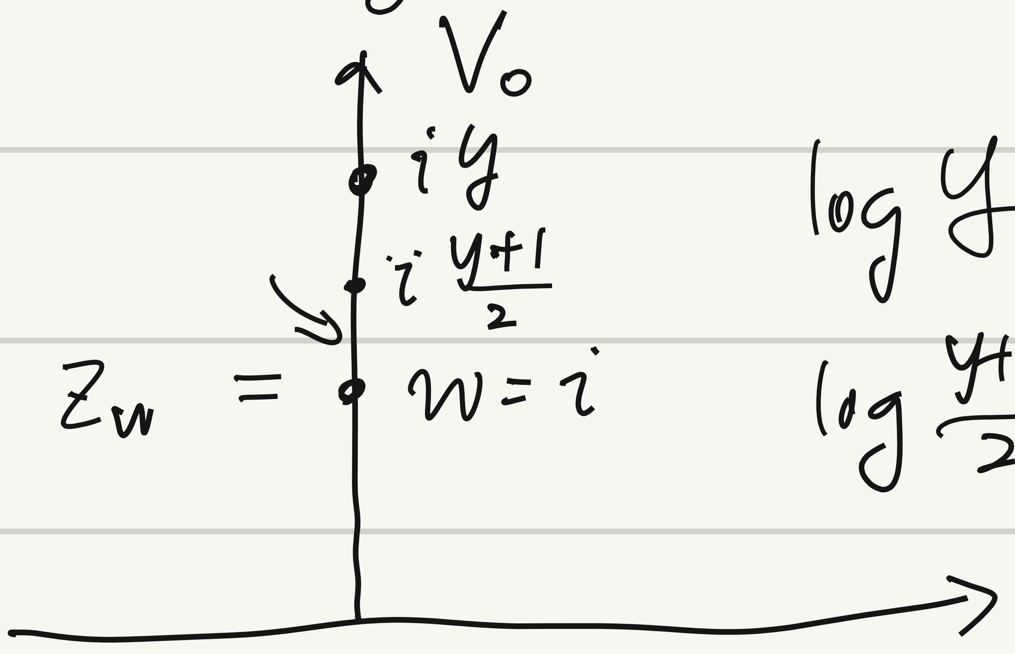
$\log tg\frac{\theta}{2} - 0 \to \infty$

$\theta \to \pi$

$$\frac{\sin\theta}{\cos\theta + 1} = tg\frac{\theta}{2}.$$

$\log tg\frac{\theta}{2} = 0 \quad \theta = \frac{\pi}{2}$

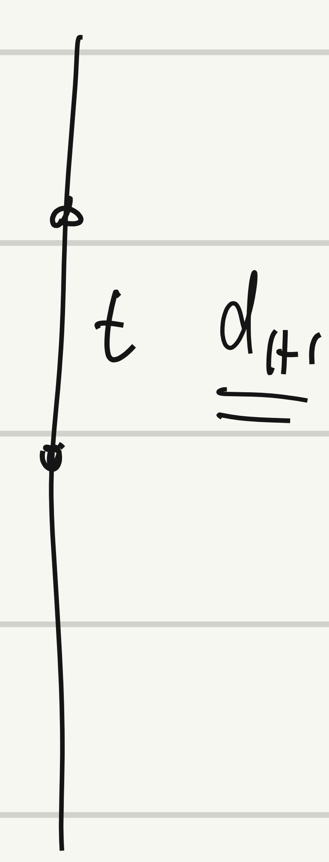
- Convexity.

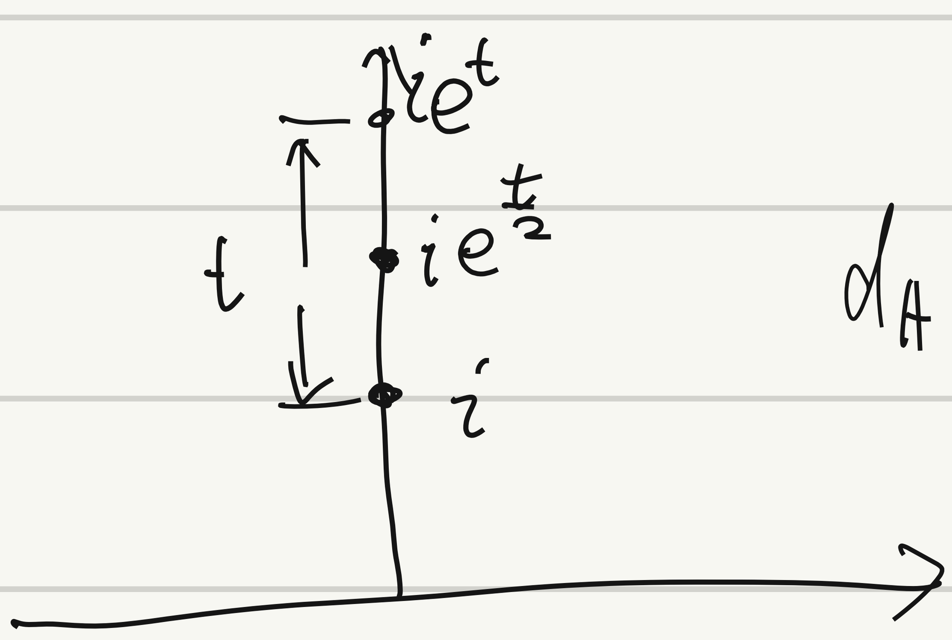


$\log y$.

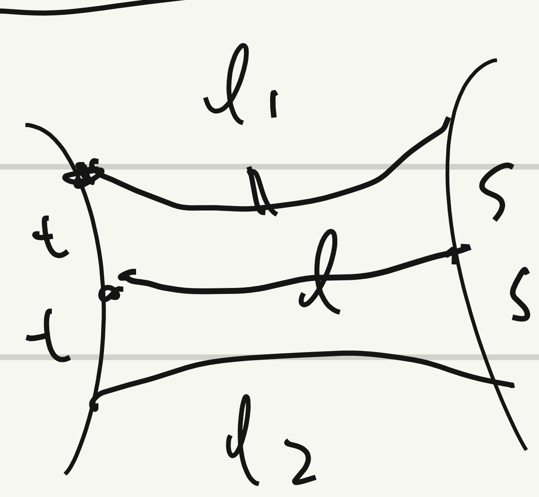
$\log\frac{y+1}{2} \sim \log y$

$\frac{1}{2}\log y$.



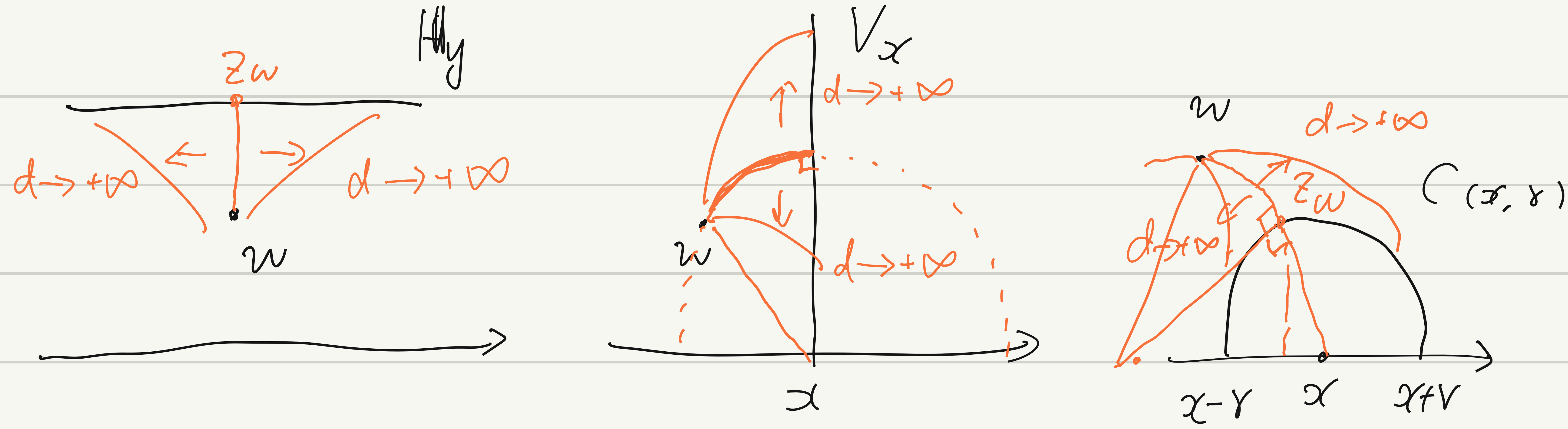


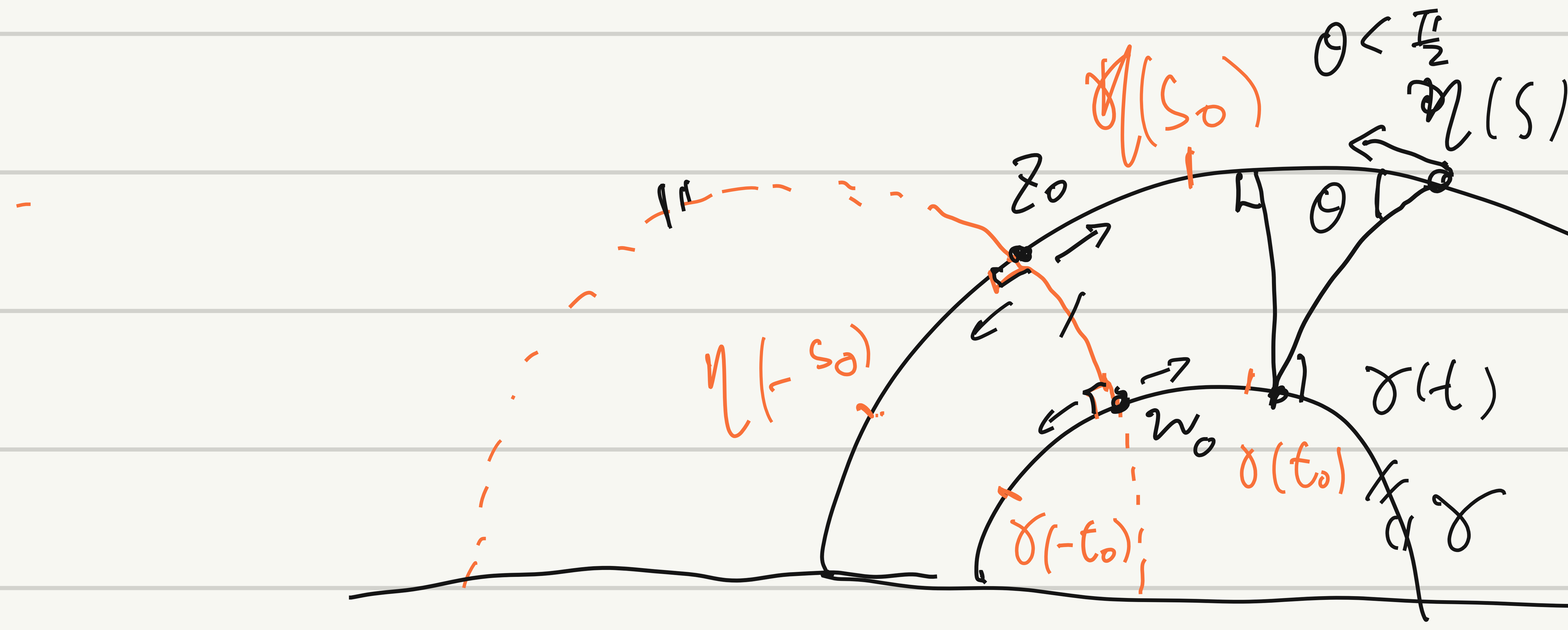
$$d_{\mathbb{H}}(ie^{\frac{t}{2}}, i) = \frac{1}{2} d_{\mathbb{H}}(ie^t, i)$$



$l < \frac{1}{2}l_1 + \frac{1}{2}l_2$

~~Convexity~~ $\exists!\, z_w$

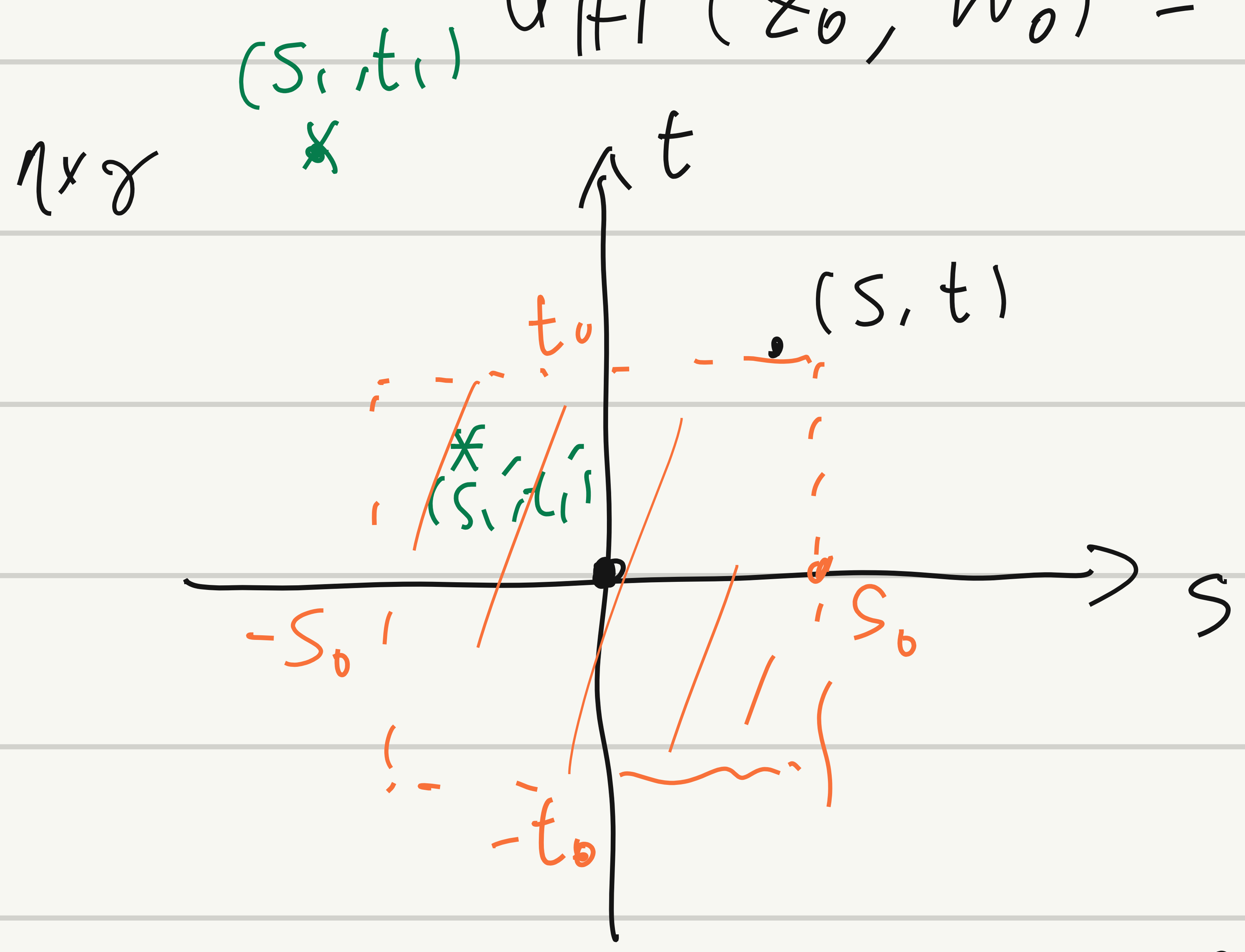




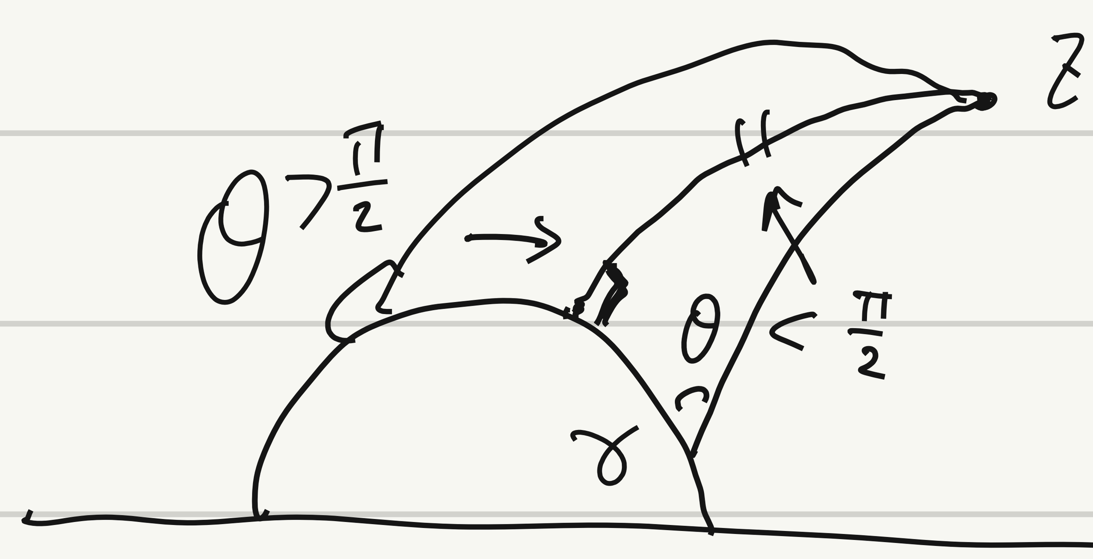
disjoint

$$d_{\mathbb{H}^1}(z_0, w_0) = d_{\mathbb{H}^1}(\eta, \gamma)$$

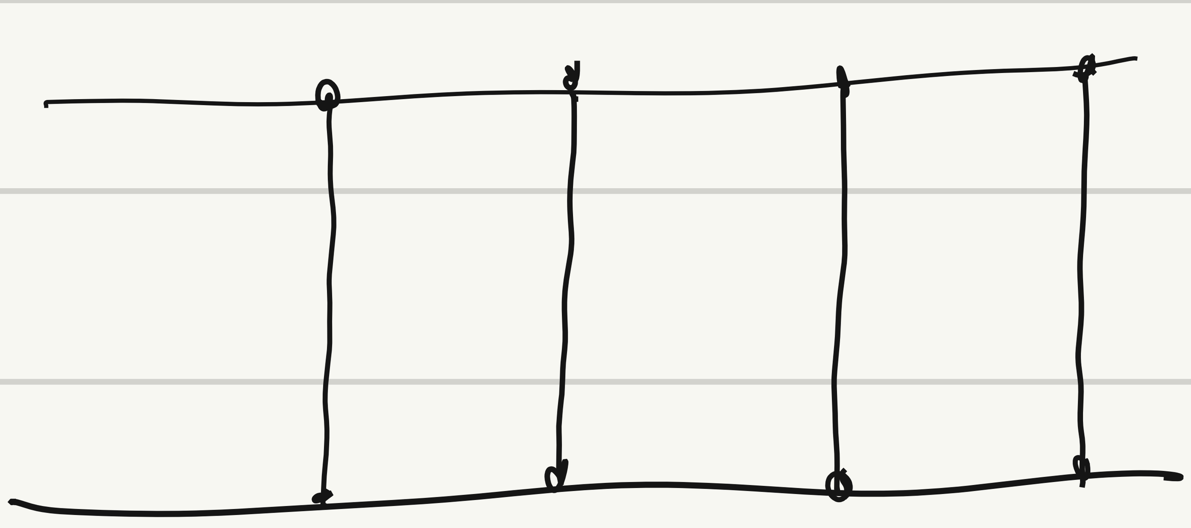
$$:= \inf\{d(z,w) \mid z \in \eta,\ w \in \gamma\}$$

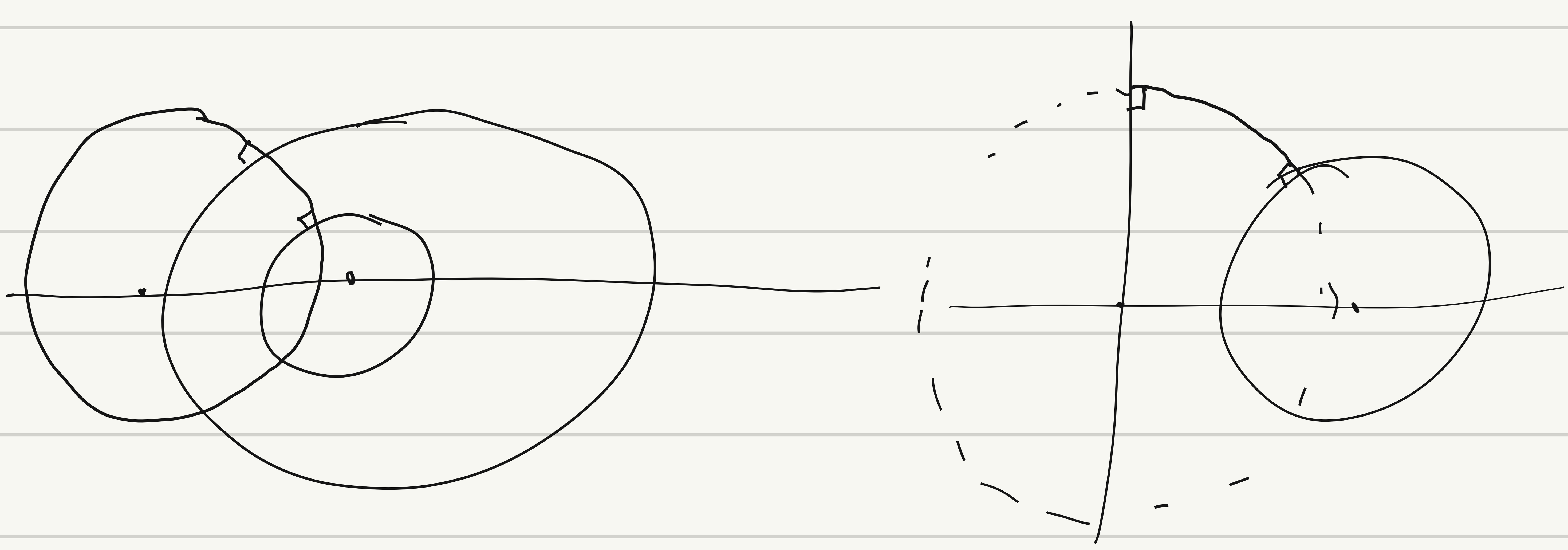


$$d_{\mathbb{H}^1}(\eta(s_1'), \gamma(t_1')) < d_{\mathbb{H}^1}(\eta(s_1), \gamma(t_1))$$



Rmk:





7 Circles in $\mathbb{H}$

Def: • A circle $C^{\mathbb{H}}(w, R)$ centered at $w \in \mathbb{H}$ with (hyperbolic) radius R is defined to be the subset:

$$C(w, R) := \{ z \in \mathbb{H} \mid d_{\mathbb{H}}(w, z) = R \}.$$

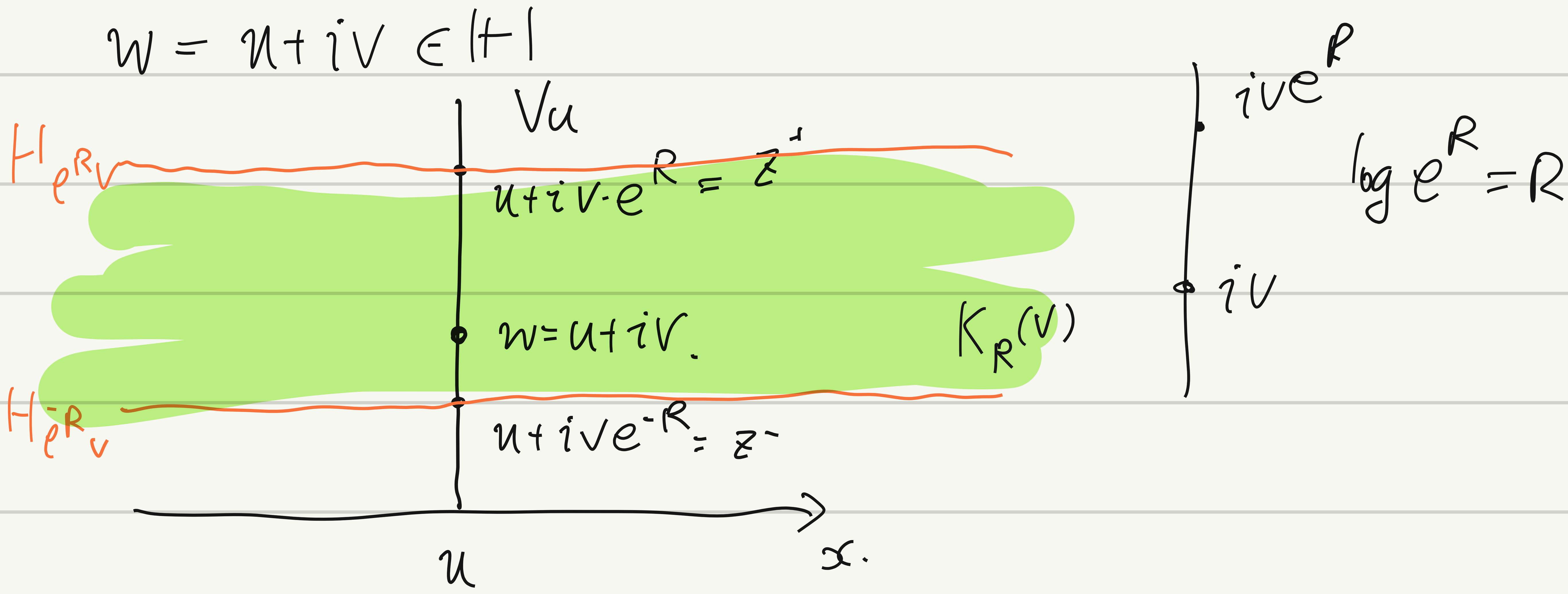
• Open disk

$$D(w, R) := \{ z \in \mathbb{H} \mid d_{\mathbb{H}}(w, z) < R \}$$

Closed disk

$$\overline{D}(w, R) := \{ z \in \mathbb{H} \mid d_{\mathbb{H}}(w, z) \leq R \}$$

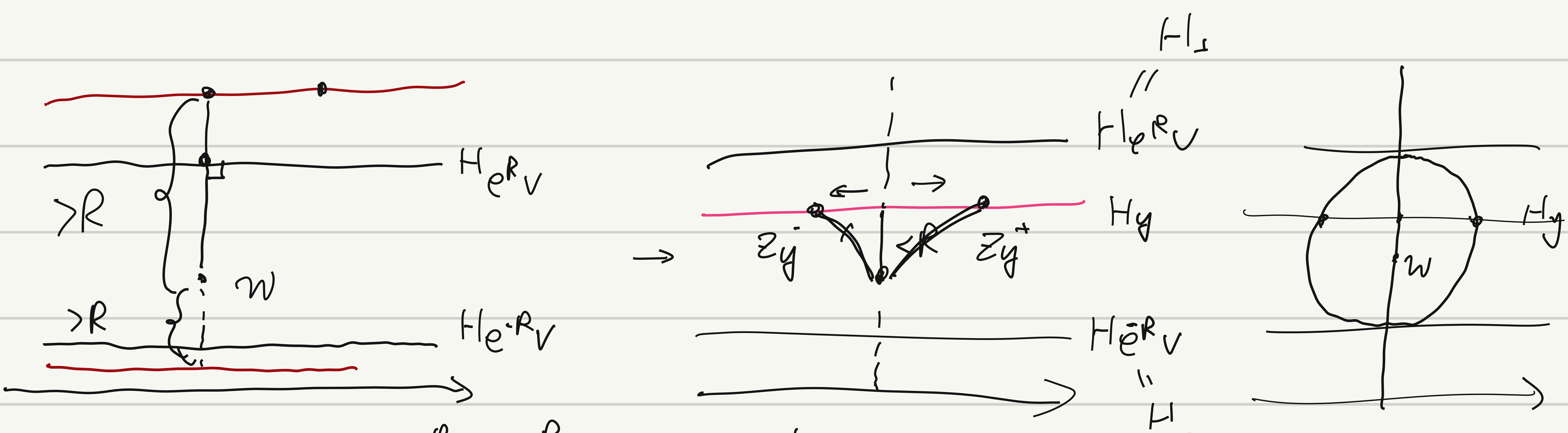
$w = u + iv \in \mathbb{H}$



Prop: $C(w, R)$ is non-empty.

$$K_R(v) := \{ z \in \mathbb{H} \mid \operatorname{Im} z \in [e^{-R} v, e^R v] \}$$

Prop: $C(w, R) \subseteq K_R(v)$



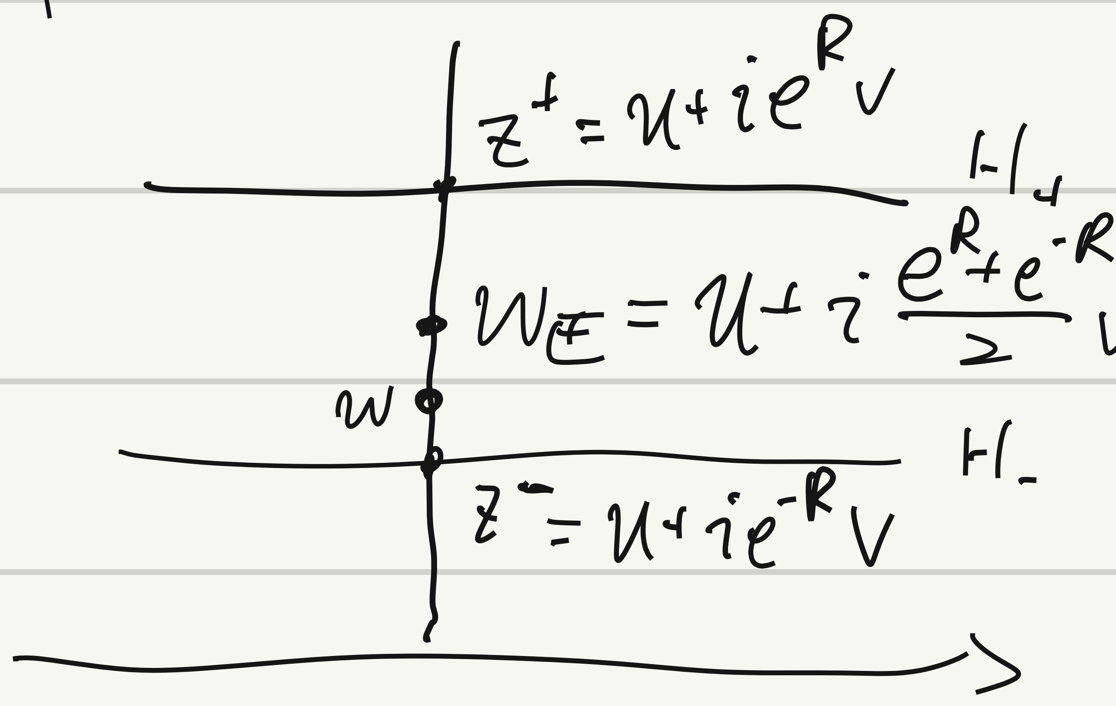
Prop: $\forall y \in (e^{-R} v, e^R v)$, $\exists z_y^+$ and $z_y^- \in H_y$

s.t. • $d_{\mathbb{H}}(w, z_y^+) = d_{\mathbb{H}}(w, z_y^-) = R$

• $\frac{1}{2} \operatorname{Re}(z_y^+ + z_y^-) = u$.

Prop: $C(w,R)$ is a circle w.r.t. Euclidean metric.

Proof: An Euclidean circle tangent to H_+ and H_-.



$$r = \frac{e^R v - e^{-R} v}{2} = \mathrm{sh}R \cdot v.$$

$$C_E(w_E, r) = \{ z_\theta = w_E + (v\,\mathrm{sh}R)\cdot e^{i\theta} \mid \theta \in [0, 2\pi] \}$$

$w_E = u + i\,\mathrm{ch}R\, v$, $r = \mathrm{sh}R\, v$.

$$= u + i\,\mathrm{ch}R\, v + (v\,\mathrm{sh}R) e^{i\theta}$$

Check: $f(\theta) = d_{H}(z_\theta, w)$ as a function on θ is constant function.

$\Updownarrow$

$\left| \dfrac{w - z_\theta}{\bar{w} - z_\theta} \right|$ is independant of θ.

∎

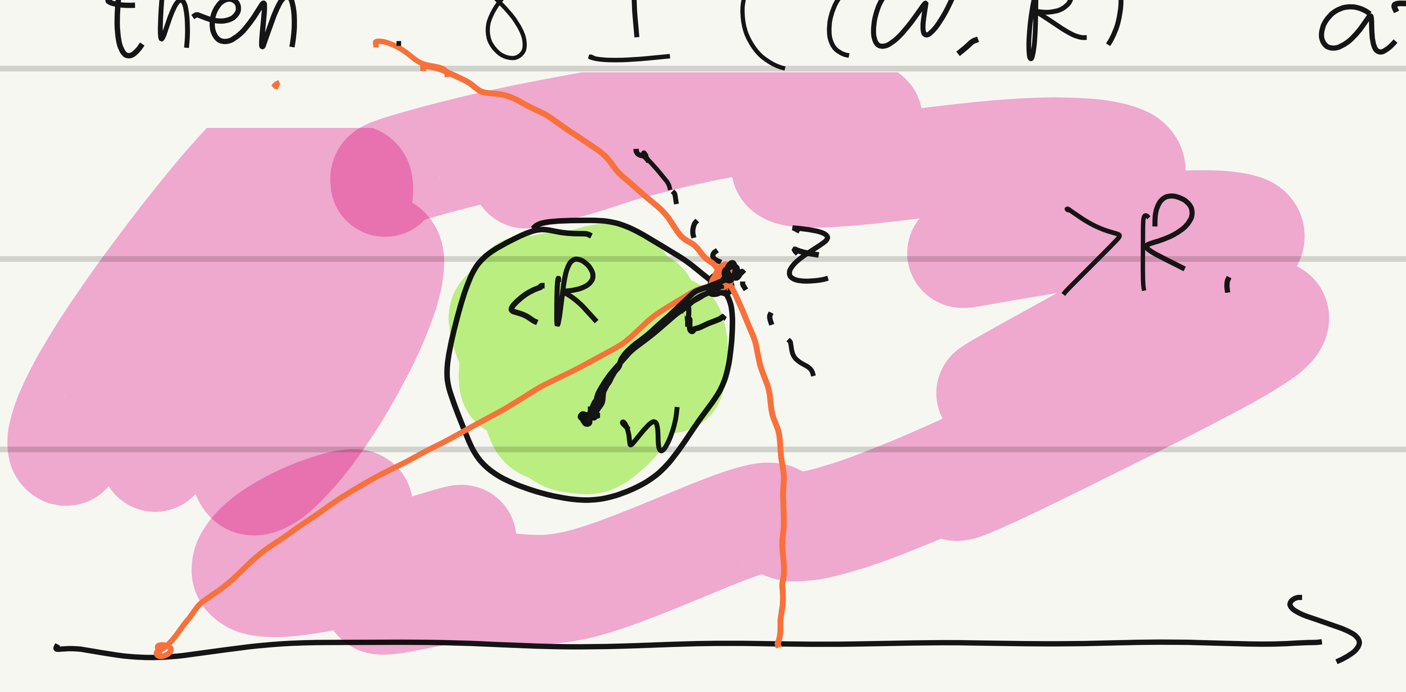
Coro: $D(w,R)$ is an open disk w.r.t. Euclidean metric.
$\bar{D}(w,R)$ is a closed disk.

Coro: The topology on H induced by d_H and that induced by d_E are equivalent.

Def:
- $\forall z \in C(w,R)$, the geod connecting w and z is called a *radius* of $C(w,R)$.
- $\forall z_1, z_2 \in C(w,R)$, if the geod connecting z_1 and z_2 passes w, then we call it a *diameter*.

Prop: Let γ be a radius of $C(w,R)$, then $\gamma \perp C(w,R)$ at $z \in \gamma \cap C(w,R)$.

Proof:



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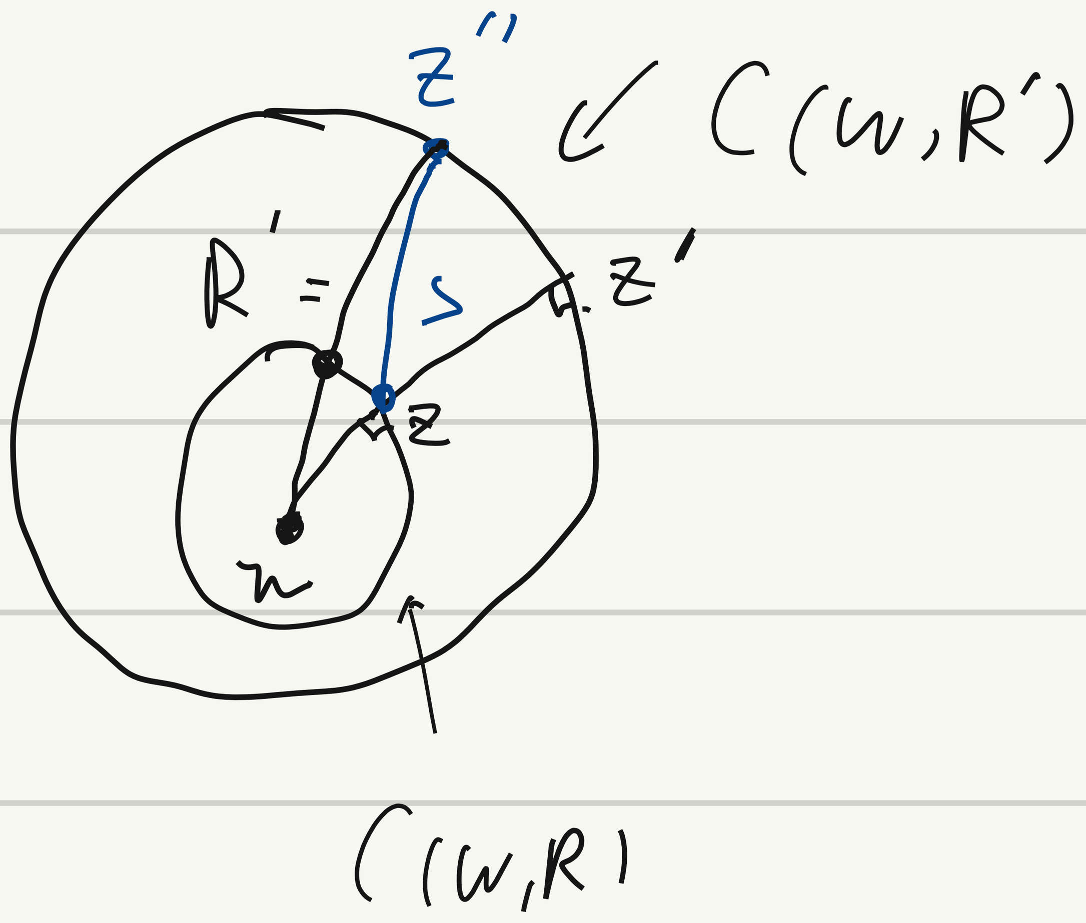


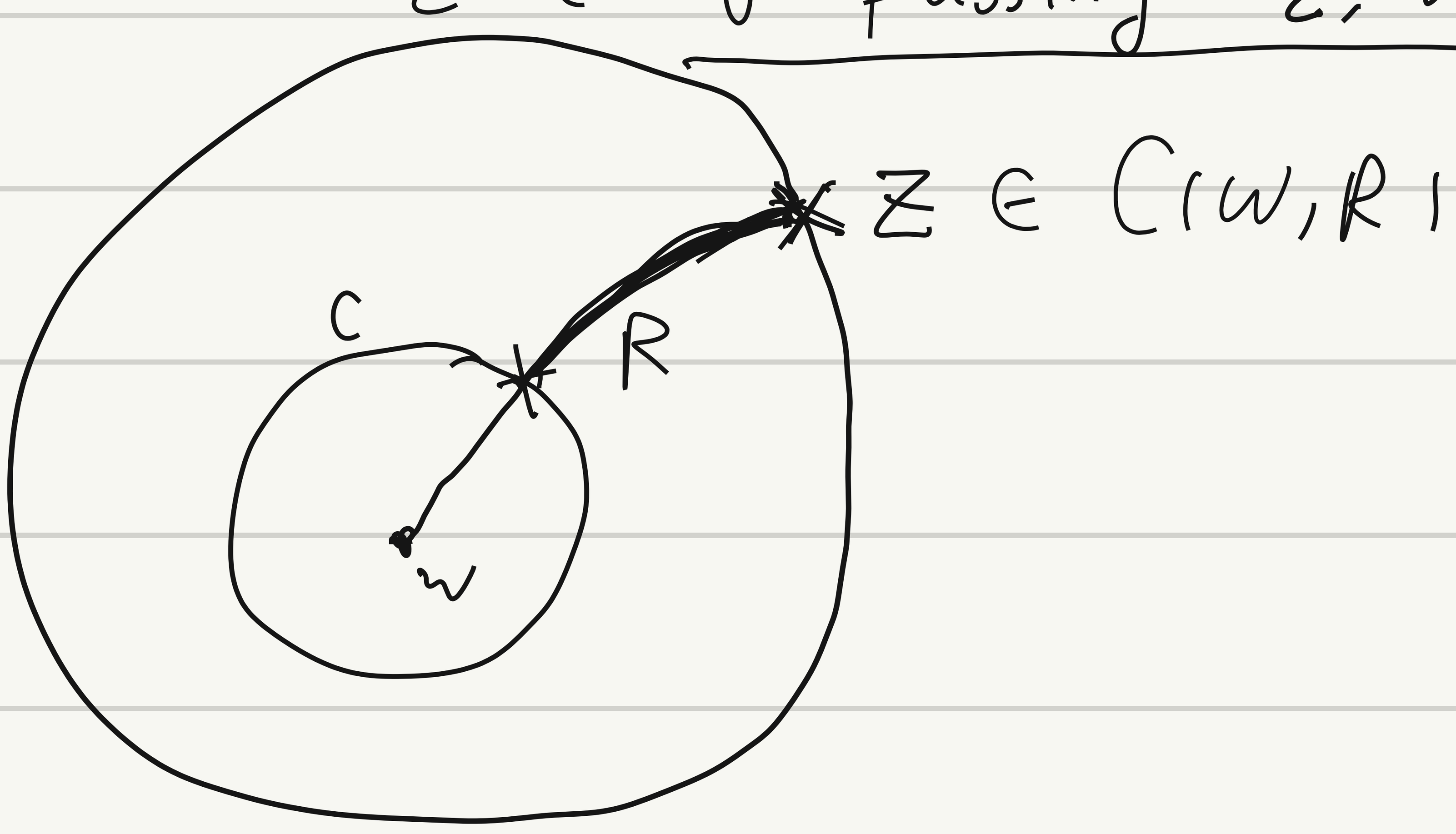
Prop: $d_{\mathbb{H}}(C(w,R), C(w,R')) = d_{\mathbb{H}}(z,z') \overset{*}{=} d_{\mathbb{H}}(z, C(w,R'))$
$= d_{\mathbb{H}}(z'', C(w,R))$

Proof: * Otherwise $\exists\, z'' \in C(w,R')$ s.t. $d_{\mathbb{H}}(z,z'') \le d_{\mathbb{H}}(z,z')$.

$$R' = d_{\mathbb{H}}(w,z') = d_{\mathbb{H}}(w,z) + d_{\mathbb{H}}(z,z')$$
$$\ge d_{\mathbb{H}}(w,z) + d_{\mathbb{H}}(z,z'')$$
$$> d_{\mathbb{H}}(w,z'') = R'$$

$R' > R'$ ↯. ∎

Cor: $\forall z \in \mathbb{H}$, $\forall C$ circle in $\mathbb{H}$ with center w
$d_{\mathbb{H}}(z,C)$ is realized by a unique $z' \in C$ s.t.
$z' \in \gamma$ passing z, w.



Consider $C(w,R)$ with $w = u+iv$.

$C(w,R) = \{ u + iv\,\mathrm{ch}R + (v\,\mathrm{sh}R)e^{i\theta} \mid \theta \in [0,2\pi] \}$

$$\cdot\ \ell_{\mathbb{H}}(C(w,R)) = \int_0^{2\pi} \frac{v\,\mathrm{sh}R}{v\,\mathrm{sh}R + (v\,\mathrm{sh}R)\sin\theta}\, d\theta \qquad \int \frac{a}{b + a\sin\theta}\, d\theta$$

$$= \int_0^{2\pi} \frac{\mathrm{sh}R}{\mathrm{ch}R + \mathrm{sh}R\sin\theta}\, d\theta$$

$$= \cdots = 2\pi\,\mathrm{sh}R > 2\pi R$$

$$A_{H}(\overline{D}(w,R)) = \int_0^R \ell_H(C(w,t))\,dt$$

$$= \int_0^R 2\pi \operatorname{sh} t\,dt = 2\pi(\operatorname{ch} R - 1) > \pi R^2$$