Introduction to hyperbolic surfaces

Exercises VIII

Let \( g \geq 0 \) and \( n \geq 0 \) be integers such that \( 2 - 2g - n < 0 \). We denote by \( S_g \) a closed hyperbolic surface of genus \( g \), and by \( S_{g,n} \) a hyperbolic surface of genus \( g \) with \( n \) cusps.

1. We would like to study short geodesics on hyperbolic surfaces of genus \( g \).

   a) (Easy) Let \( p \in S_g \). The injective radius \( R_p \) at \( p \) is the maximal positive real number such that the interior of a hyperbolic disk of radius \( R_p \) can be mapped isometrically to the \( R_p \)-neighborhood of \( p \). Show that there exists a constant \( c_1 > 0 \), such that for any \( S_g \),
   
   \[
   \min\{R_p \mid p \in S_g\} < c_1.
   \]
   
   (Hint: The area of a hyperbolic surface of genus \( g \) is constant.)

   b) (Easy) Use a) to show that there is a constant \( c_2 > 0 \), such that on any \( S_g \), there is a simple closed geodesic shorter than \( c_2 \).

   c) (Normal) Use Collar lemma and b), show that there is a constant \( c_3 > 0 \), such that on any \( S_g \), there exists a simple closed geodesic \( \gamma \) which has a collar of area greater or equal to \( c_3 \).

   (Hint: A collar can be foliated by hypercycles which can be used to compute the area of the collar. Consider the monotonicity of the function \( f(x) = x/\sinh x \).)

   d) (Normal) Use Collar lemma to show that on any \( S_g \), any two distinct simple geodesics of length 1 must be disjoint.

   (Hint: \( \sinh 0.5 > 0.52 \).)

2. (Hard) Let \( p \) be a cusp on \( S_{g,n} \). If a horocycle \( H \) centered at \( p \) is embedded in \( S_{g,n} \), we call the part between \( H \) and its center \( p \) the cusp region, and denote it by \( D_p(r) \) where \( r \) is the length of \( H \). Use Collar Lemma for cusps to show that any geodesic on \( S_{g,n} \) intersecting \( D_p(1) \) have self-intersections.

   (Hint: Lift \( S_{g,n} \) to \( \mathbb{H} \), and assume that \( \infty \) is a lift of one cusp. Up to a conjugacy by an isometry, the cyclic group in \( \Gamma_{g,n} \) fixing \( \infty \) can be generated by \( f(z) = z + 2 \). Find the horizontal line \( H \) projected to the horocycle of length 2 and the horizontal line \( H' \) projected to the horocycle of length 1 on \( S_{g,n} \). Consider a geodesic intersecting the horizontal line of \( H' \). Find its image under \( \Gamma_{g,n} \) which intersects itself.)