Introduction to hyperbolic surfaces

Exercises VIII

Let $g \geq 0$ and $n \geq 0$ be integers such that $2 - 2g - n < 0$. We denote by S_g a closed hyperbolic surface of genus g, and by $S_{g,n}$ a hyperbolic surface of genus g with n cusps.

1. We would like to study short geodesics on hyperbolic surfaces of genus g.

   a) (Easy) Let $p \in S_g$. The *injective radius* R_p at p is the maximal positive real number such that the interior of a hyperbolic disk of radius R_p can be mapped isometrically to the R_p-neighborhood of p. Show that the exists a constant $c_1 > 0$, such that for any S_g,

   $$\min\{R_p \mid p \in S_g\} < c_1.$$

   (Hint: The area of a hyperbolic surface of genus g is constant.)

   b) (Easy) Use a) to show that there is a constant $c_2 > 0$, such that on any S_g, there is a simple closed geodesic shorter than c_2.

   c) (Normal) Use Collar lemma and b), show that there is a constant $c_3 > 0$, such that on any S_g, there exists a simple closed geodesic γ which has a collar of area greater or equal to c_3.

   (Hint: A collar can be foliated by hypercycles which can be used to compute the area of the collar. Consider the monotonicity of the function $f(x) = x/\sinh x$.)

   d) (Normal) Use Collar lemma to show that on any S_g, any two distinct simple geodesics of length 1 must be disjoint.

   (Hint: $\sinh 0.5 > 0.52$.)

2. (Hard) Let p be a cusp on $S_{g,n}$. If a horocycle H centered at p is embedded in $S_{g,n}$, we call the part between H and its center p the cusp region, and denote it by $D_p(r)$ where r is the length of H. Use Collar Lemma for cusps to show that any geodesic on $S_{g,n}$ intersecting $D_p(1)$ have self-intersections.

   (Hint: Lift $S_{g,n}$ to $\mathbb{H}$, and assume that ∞ is a lift of one cusp. Up to a conjugacy by an isometry, the cyclic group in $\Gamma_{g,n}$ fixing ∞ can be generated by $f(z) = z + 2$. Find the horizontal line H projected to the horocycle of length 2 and the horizontal line H' projected to the horocycle of length 1 on $S_{g,n}$. Consider a geodesic intersecting the horizontal line of H'. Find its image under $\Gamma_{g,n}$ which intersects itself.)