VII Geodesics in Hyperbolic Surface

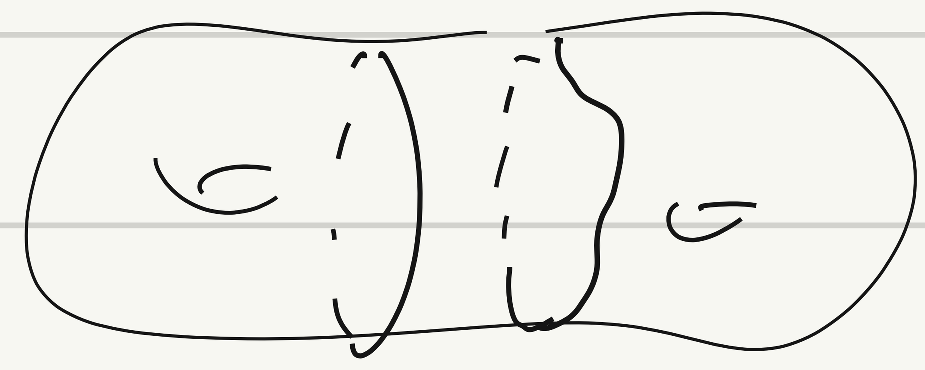
1. Homotopy class of loops (free homotopy). $S = \mathbb{H}/\Gamma$

γ, η two loops on S

$\gamma \sim \eta$ homotopic to each other if $\exists H : [0,1]\times[0,1] \to S$ s.t.

$\forall t \in [0,1]$, $H(t,0) = \gamma(t)$, $H(t,1) = \eta(t)$

~~$H(0,s) = H(1,s) = p$~~

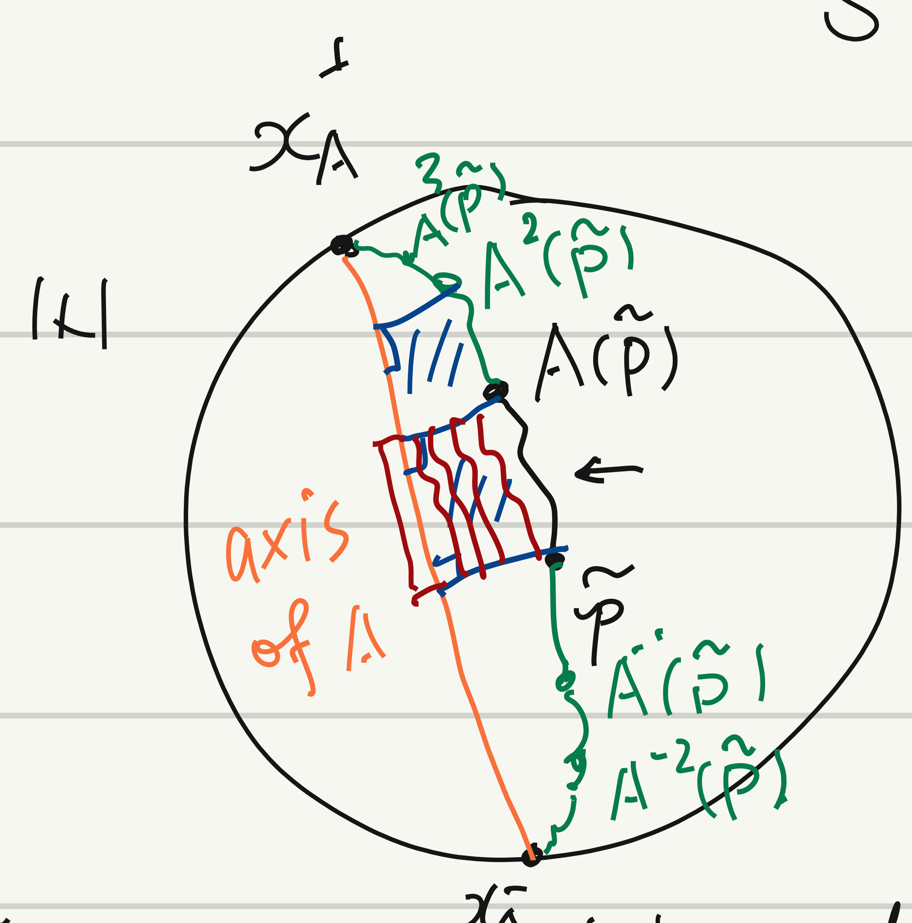


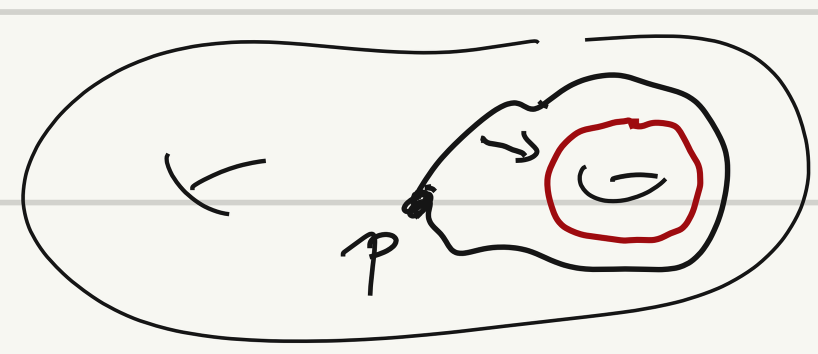
$[\gamma] := \{\gamma' \mid \gamma' \sim \gamma \;\; \gamma' \text{ loop on } S\}$

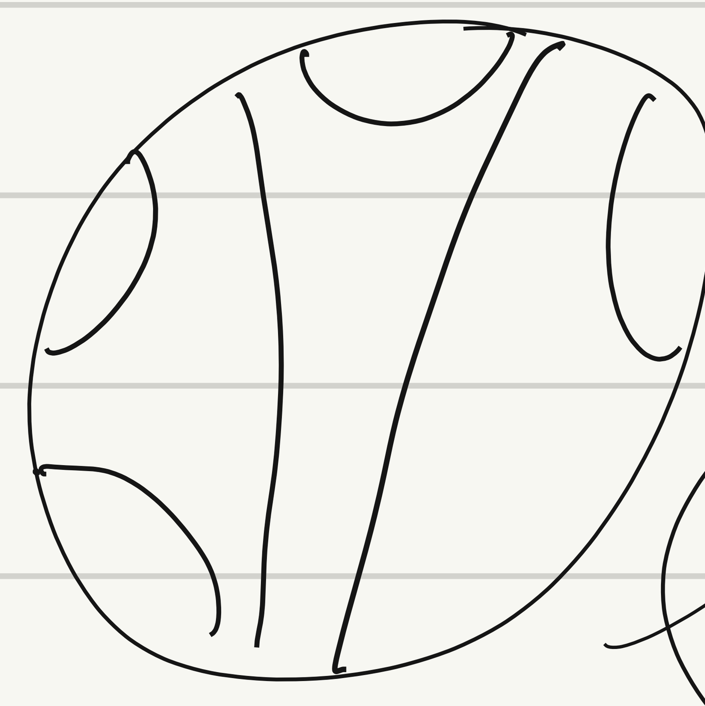
Prop: $\forall [\gamma] \neq [pt]$, $\exists !$ geodesic γ_0 in S, s.t. $\gamma_0 \in [\gamma]$.

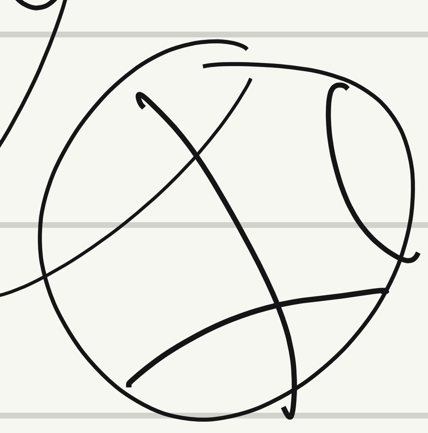
$$\begin{tikzcd} {[0,1]} \arrow[r, "\tilde{\gamma}", "\text{conti}"'] \arrow[dr, "\gamma"'] & \mathbb{H} \arrow[d] \\ & S = \mathbb{H}/\Gamma \end{tikzcd}$$

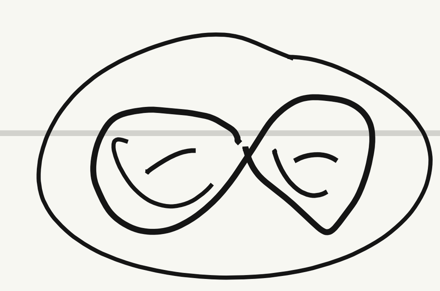
$\tilde{\gamma}$ is a lift of γ.











$A(\tilde p) \neq \tilde p$, A: hyperbolic elt.

2. Classification of geodesic.

γ geodesic on S $\qquad \gamma : \mathbb{R} \to S$

$\tilde{\gamma}$ lifts of γ in $\mathbb{H}$ $\leftarrow$

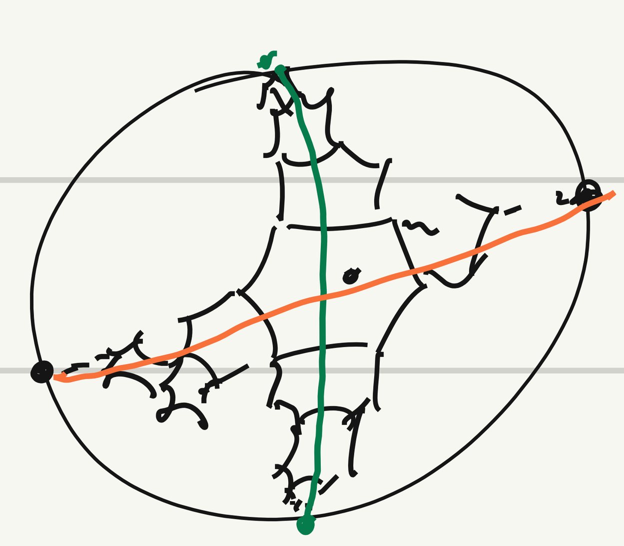
$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{\gamma}"] \arrow[dr, "\gamma"'] & \mathbb{H} \arrow[d] \\ & S \end{tikzcd}$$

① closed geod vs ∞ geodesic.

$\Gamma = \langle A_1 \dots A_n \mid R_1 \dots R_s \rangle$

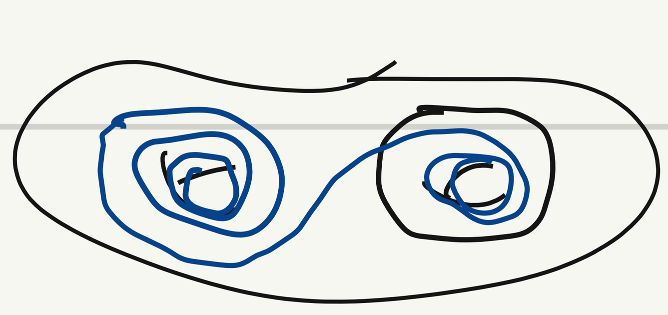
periodic: $\cdots W(A_1^{\pm 1} \dots A_n^{\pm 1}) W(A_1^{\pm 1} \dots A_n^{\pm 1}) \cdots$ $\quad \cdots AAA \cdots$

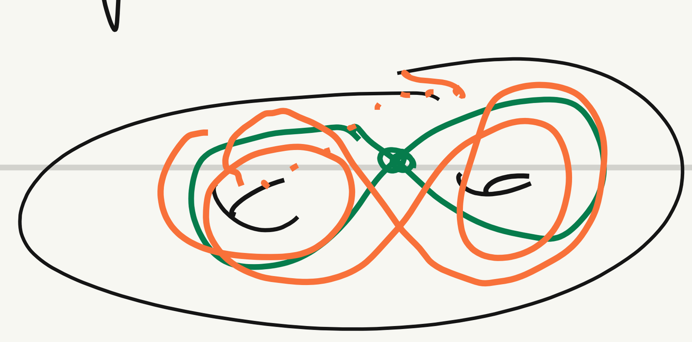
closed geod. $\neq R_s$

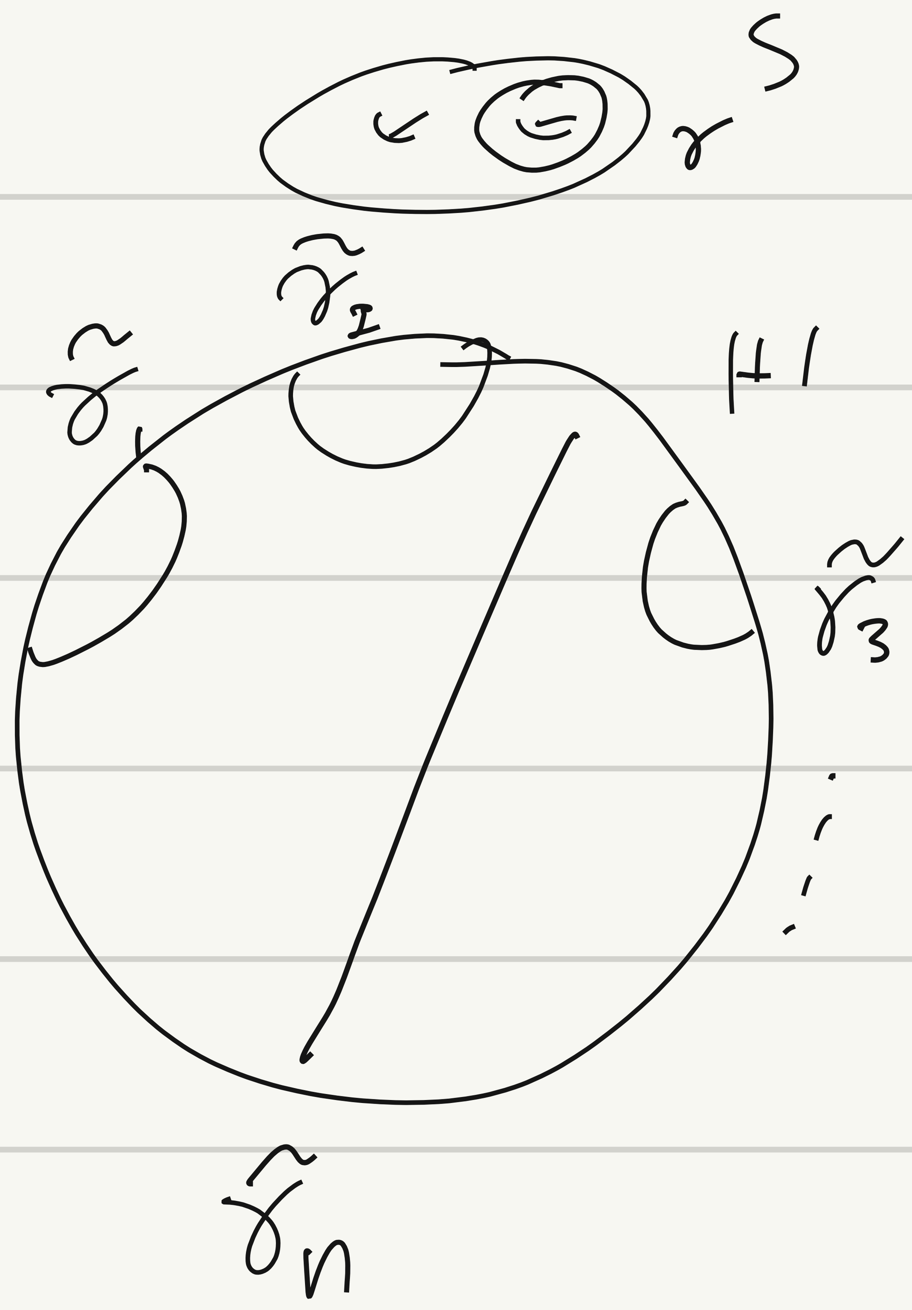


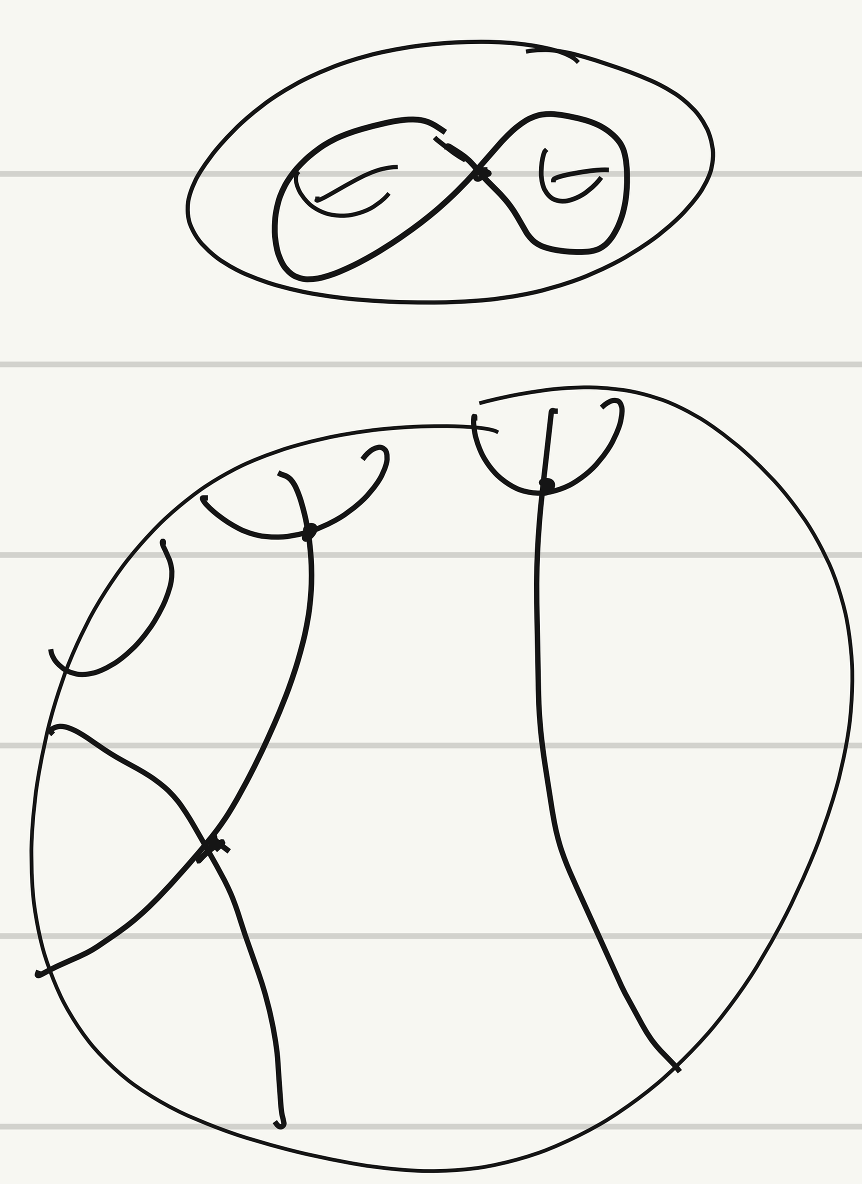
non periodic ∞ word of $A_1^{\pm 1} \dots A_n^{\pm 1}$

② simple geod vs geod with self-intersection.



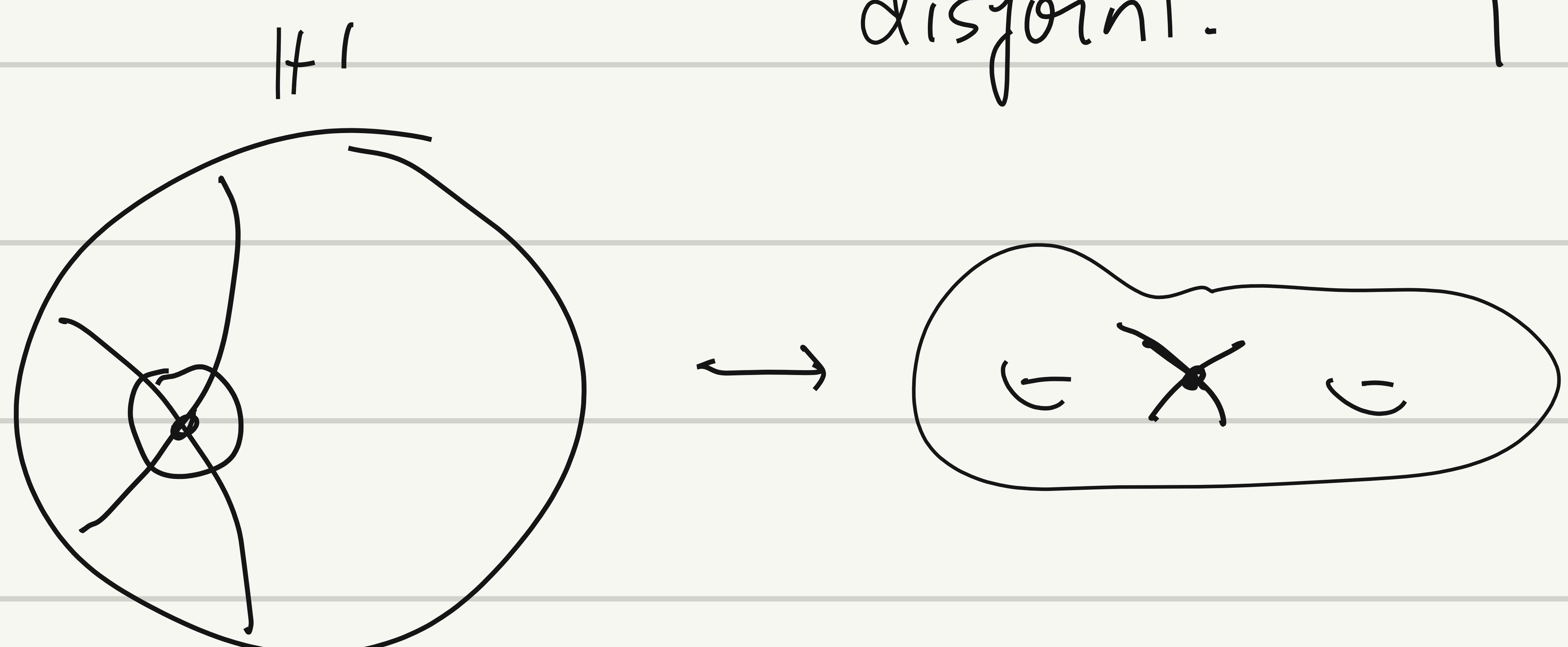






If γ simple, $\tilde{\gamma}$'s are pairwise disjoint.

If γ has self-intersection, $\exists\, \tilde{\gamma} \neq \tilde{\gamma}'$, $\tilde{\gamma} \cap \tilde{\gamma}' \neq \phi$.

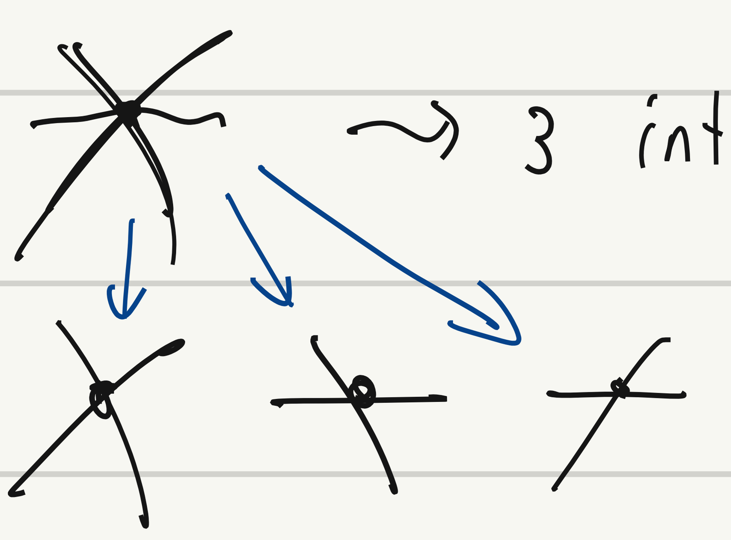


$\gamma \neq \eta$ geodesics in S $\quad \gamma : I_\gamma \to S \quad \eta : I_\eta \to S \quad I = \begin{cases} S^1 & \text{if closed} \\ \mathbb{R} & \text{if not} \end{cases}$

geometric intersection number. $i(\gamma, \eta) := \#\{(s,t) \in I_\gamma \times I_\eta \mid \gamma(s) = \eta(t)\}$

$\in \{0, 1, 2, \dots, \infty\}$

Rmk :

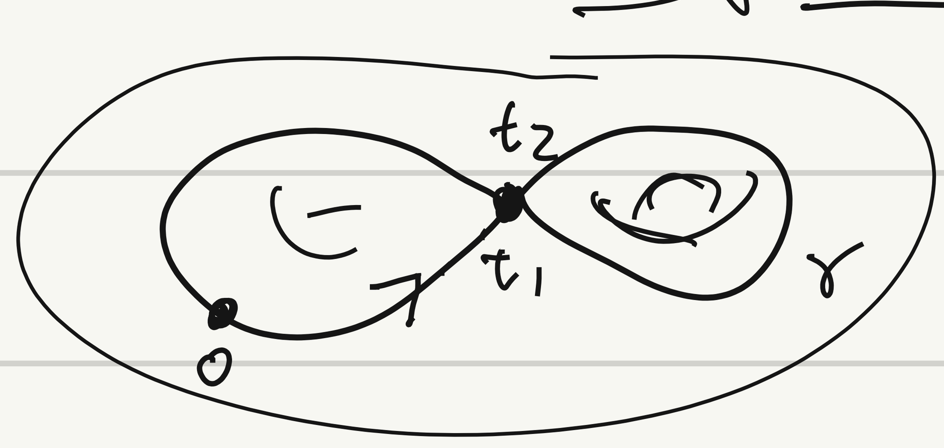


$\rightsquigarrow$ 3 intersection points

$$i(\gamma, \gamma) := \#\{(s,t) \in I_\gamma \times I_\gamma \mid \gamma(s) = \gamma(t),\ s \neq t\} = 2k.$$

γ is a <u>k-geodesic.</u> $\quad k = 0$, γ <u>simple geodesic.</u>

- Ex :



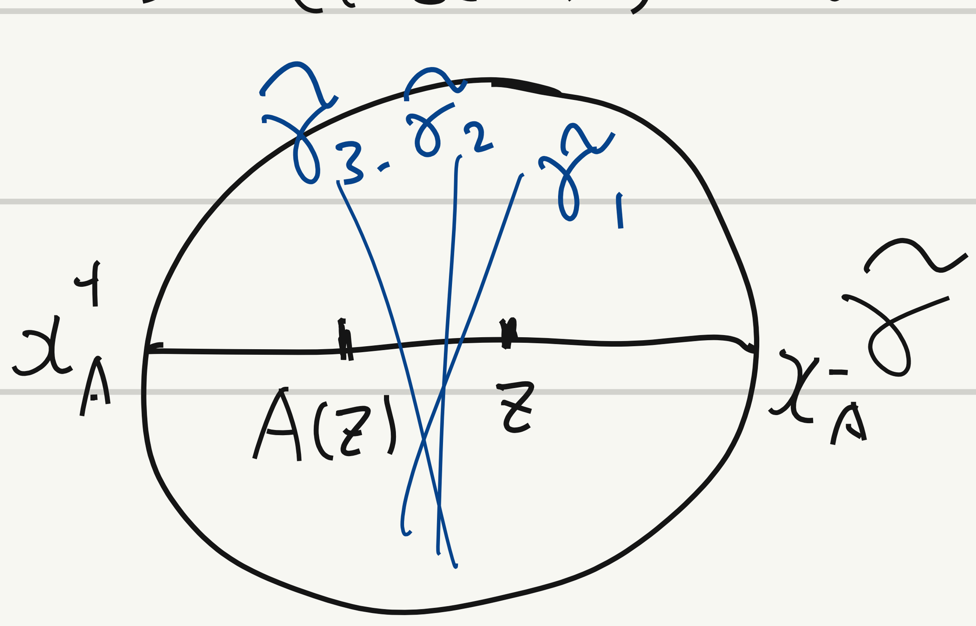
$I_\gamma = S^1$

$i(\gamma, \gamma) = 2.$

$(s, t) = (t_1, t_2)$

$(s, t) = (t_2, t_1).$

- If γ is closed, we can read k.



$\tilde{\gamma}$: lift of γ.

A.

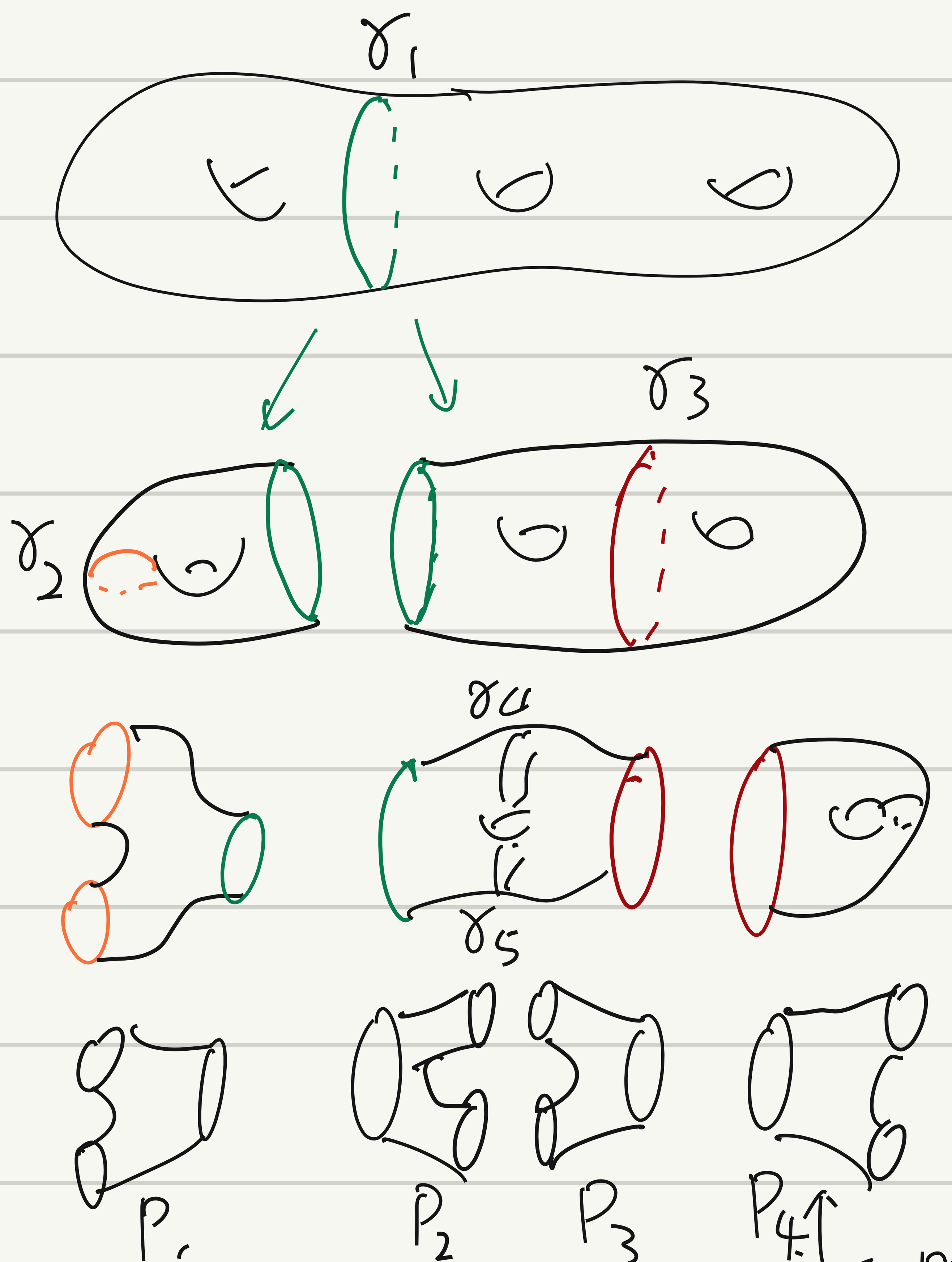
consider a period.

$[z, A(z)] \subset \tilde{\gamma}$.

$\#\{\tilde{\gamma}' \mid \tilde{\gamma}' \text{ lift of } \gamma,\ \tilde{\gamma}' \neq \tilde{\gamma},\ \tilde{\gamma}' \cap [z, A(z)] \neq \phi\}$

$= k.$

3. Simple closed geodesic.



S_g.
$\parallel$
$S = \mathbb{H}/\Gamma$.

pair of pants.

Def. A pants decomp of S is a maximal collection of pairwise disjoint simple closed geodesis on S

$\mathcal{P}(S) := \{\mathbb{P} \mid \mathbb{P}$ is a pants decomp of $S\}$

Prop. $\forall \mathbb{P} \in \mathcal{P}(S)$

$$|\mathbb{P}| = \frac{3}{2}|\chi(S)|$$

$\#$ of pants $= |\chi(S)|$

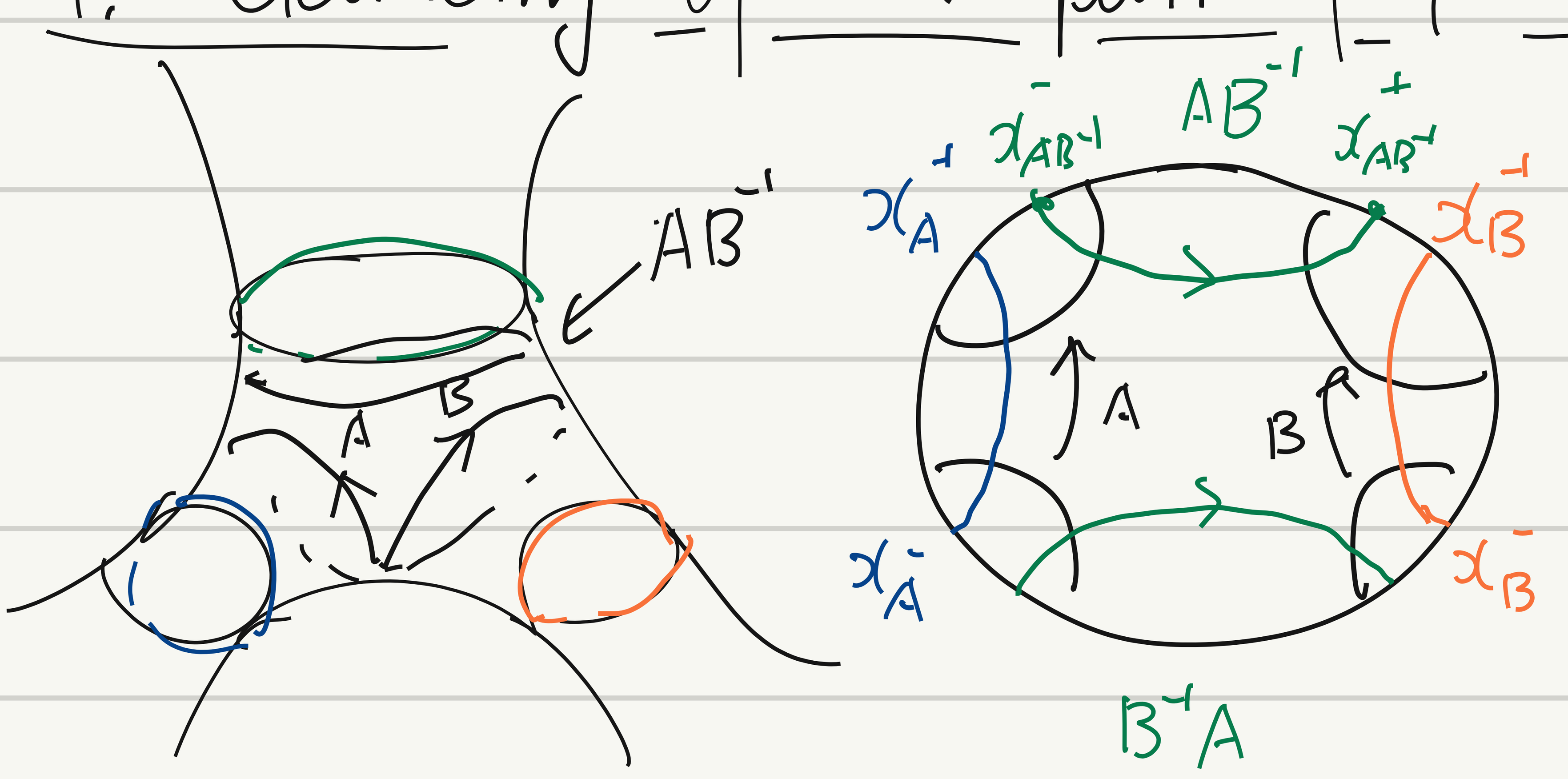
$\underbrace{\{\gamma_1 \dots \gamma_6\}}_{\{P_1, P_2, P_3, P_4\}}$ pants decomposition of S.

$\{P_1, P_2, P_3, P_4\}$ pants.

Ex. $g = 3$, $\chi(S_3) = 2 - 2\times 3 = -4$.

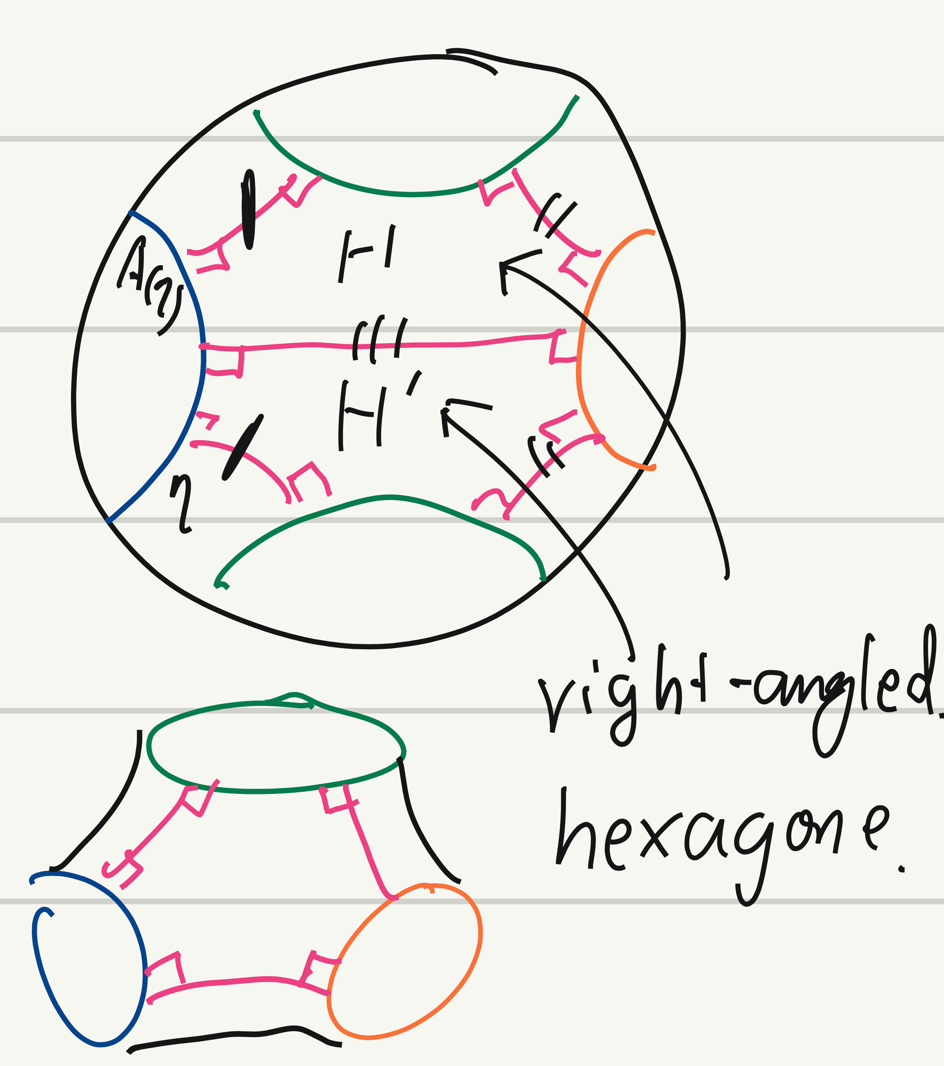
$|\mathbb{P}| = \frac{3}{2}\times 4 = 6$ $\qquad \#$ of pants $= 4$.

4. Geometry of a pair of pants

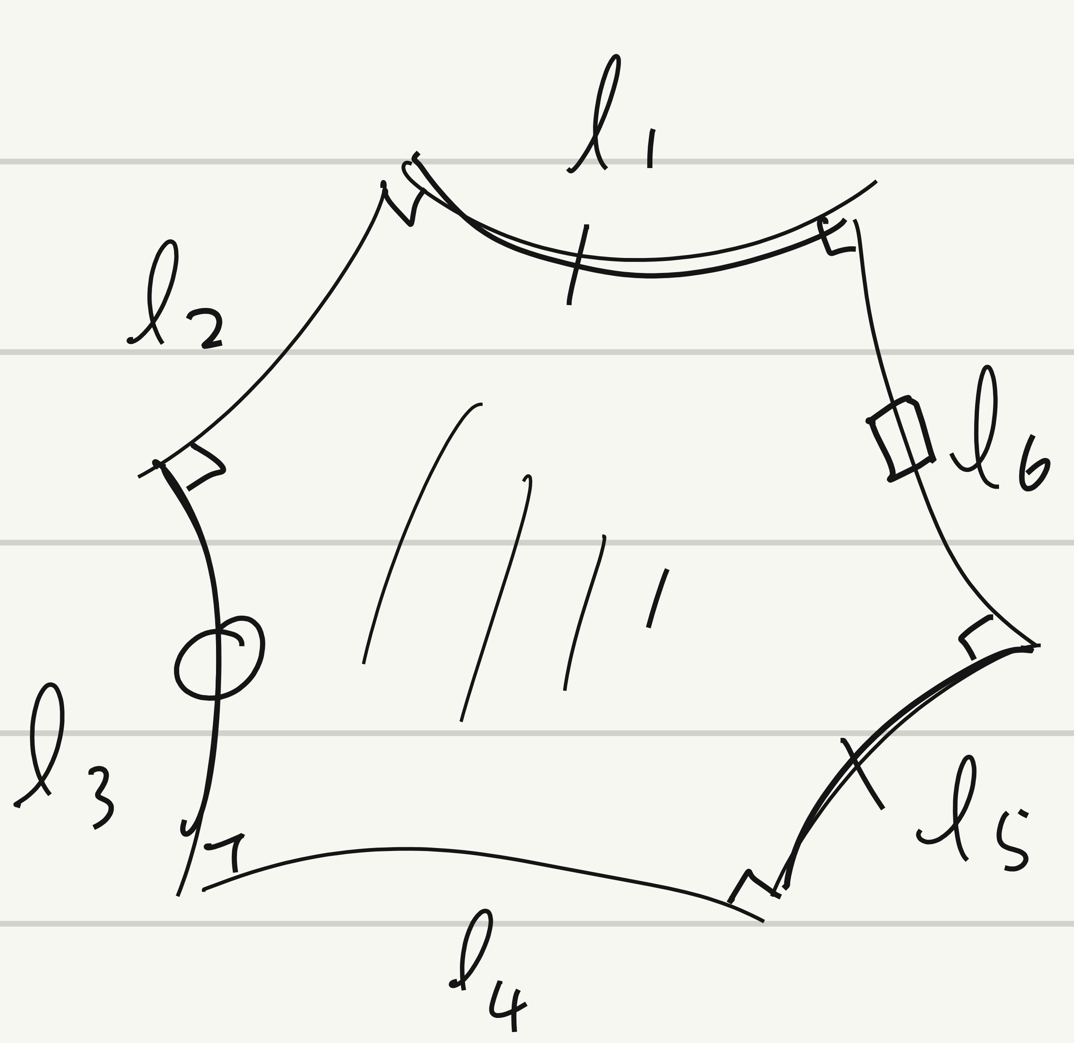


$\Gamma = \langle A, B\rangle$

$\mathbb{H}/\Gamma =$ 



right-angled hexagone.



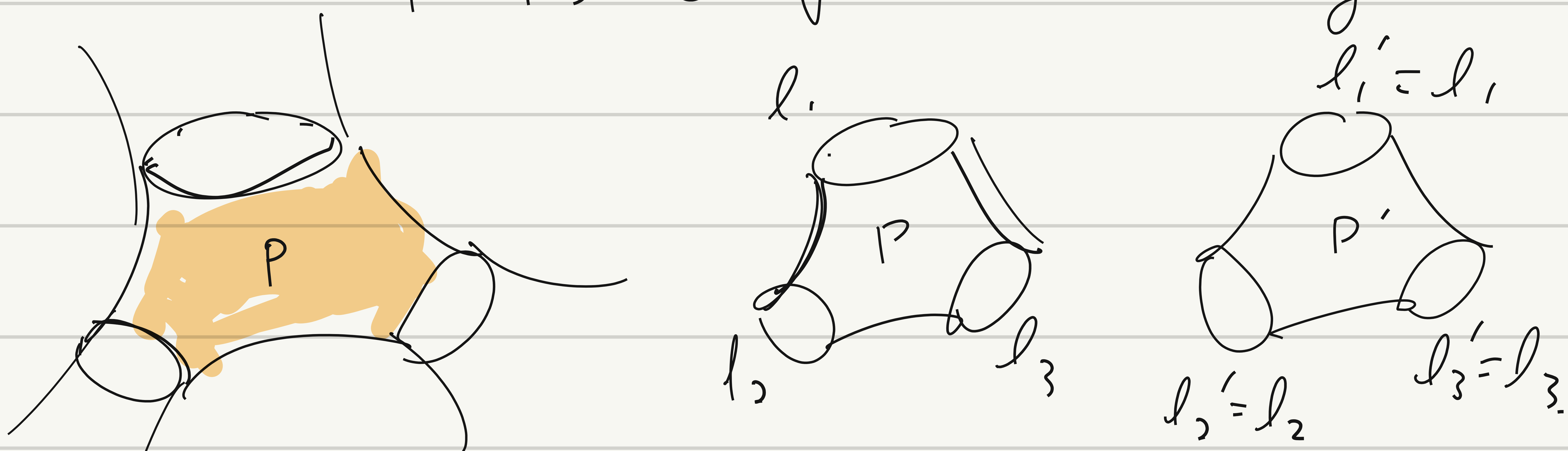
$$\operatorname{ch} l_3 = -\operatorname{ch} l_1 \operatorname{ch} l_5 + \operatorname{sh} l_1 \operatorname{sh} l_5 \operatorname{ch} l_6.$$

$$\operatorname{ch} l_6 = \frac{\operatorname{ch} l_3 + \operatorname{ch} l_1 \operatorname{ch} l_5}{\operatorname{sh} l_1 \operatorname{sh} l_5}$$

unique up to isometry

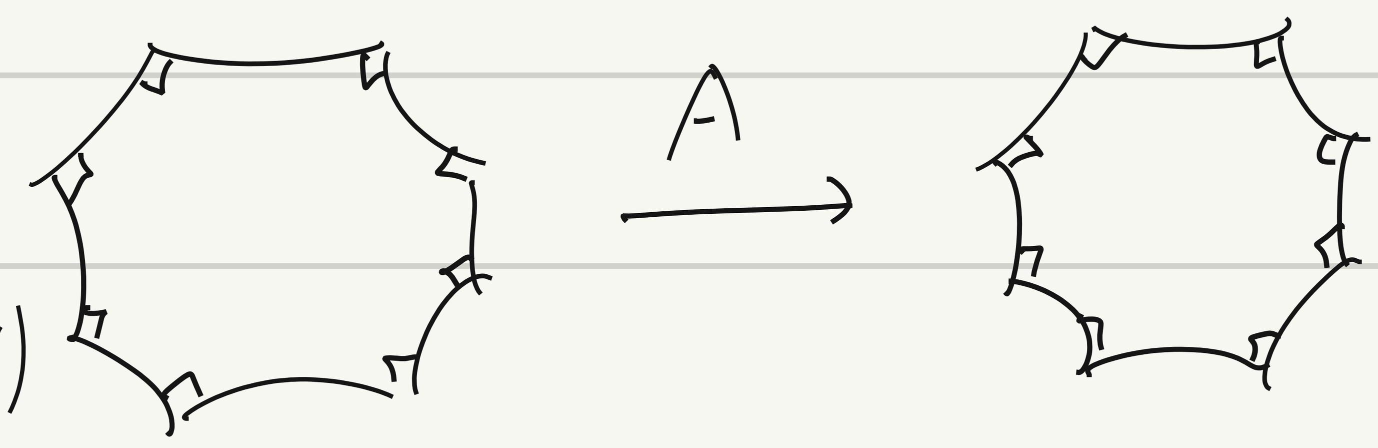
Prop. Geometry on $\mathbb{H}$ is determined by 3 of its side lengths.

Cor: Geometry on a P is determined by the lengths of its 3 geodesic boundary.

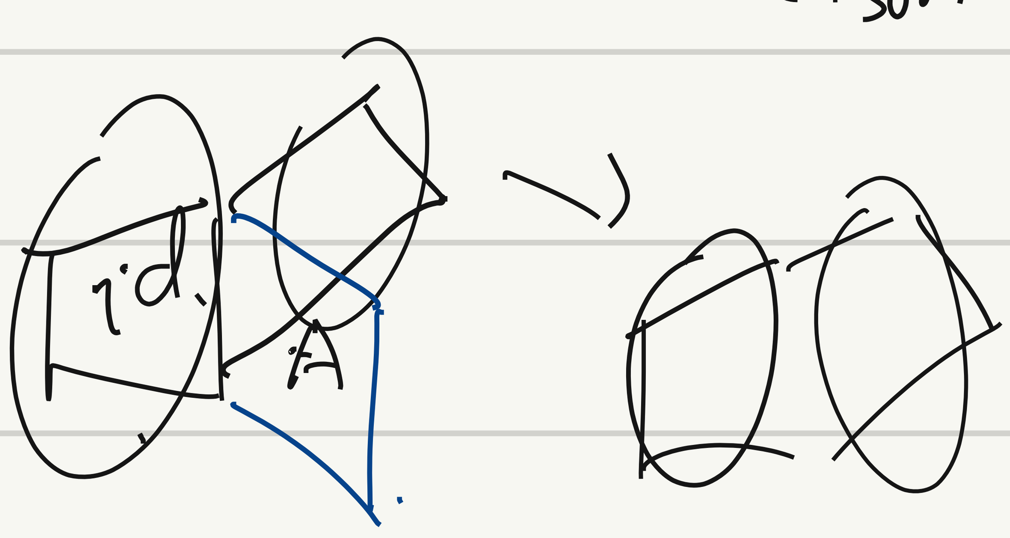


$\exists f: P \to P'$ isometry.

$\exists A \in \mathrm{Isom}(\mathbb{H})$

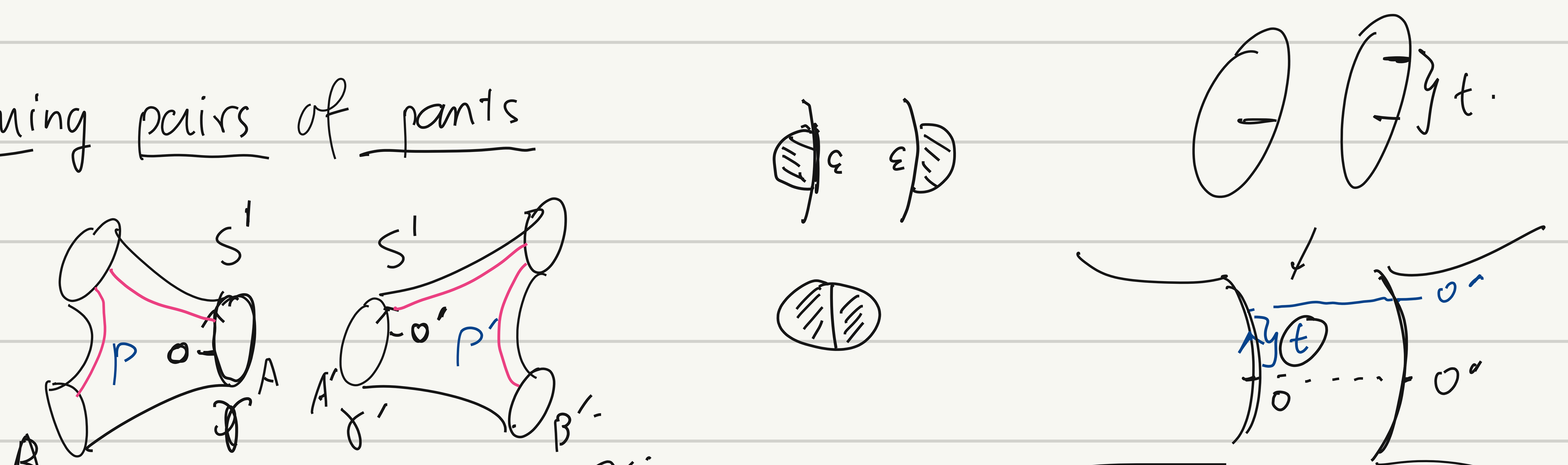


D(P) D(P')



$\underline{\Gamma}$ $\quad \underline{\Gamma'} = \underline{A}\,\underline{\Gamma}\,\underline{A^{-1}}$

4. Gluing pairs of pants

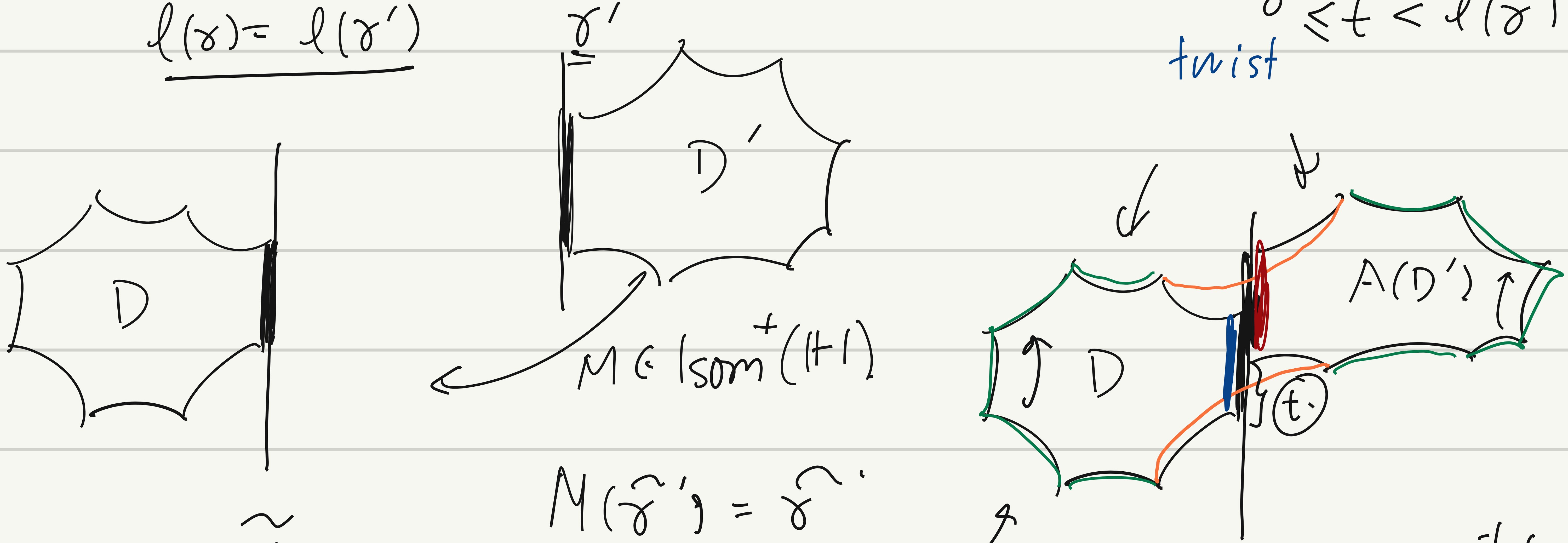


$\underline{l(\gamma) = l(\gamma')}$

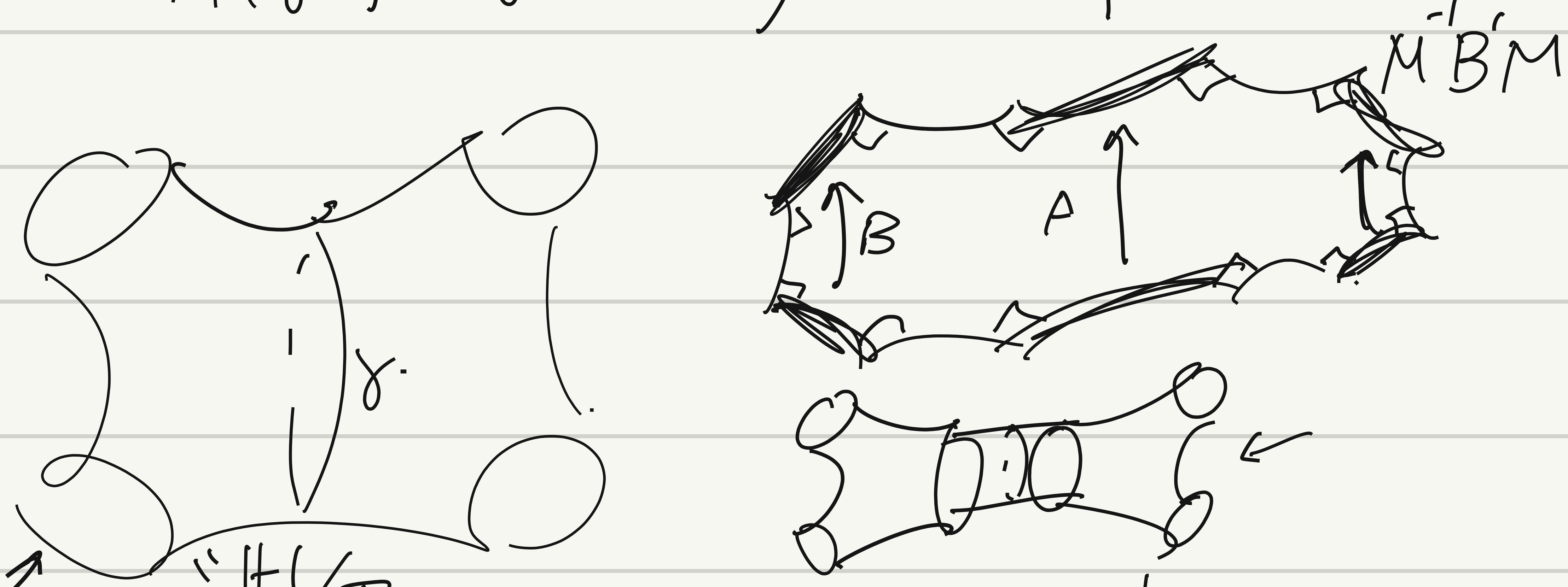
$0 \le t < l(\gamma)$

twist

$\mathbb{H}$.



$M(\tilde{\gamma}') = \tilde{\gamma}$



$\Gamma \qquad \Gamma' = \langle A', B' \rangle$

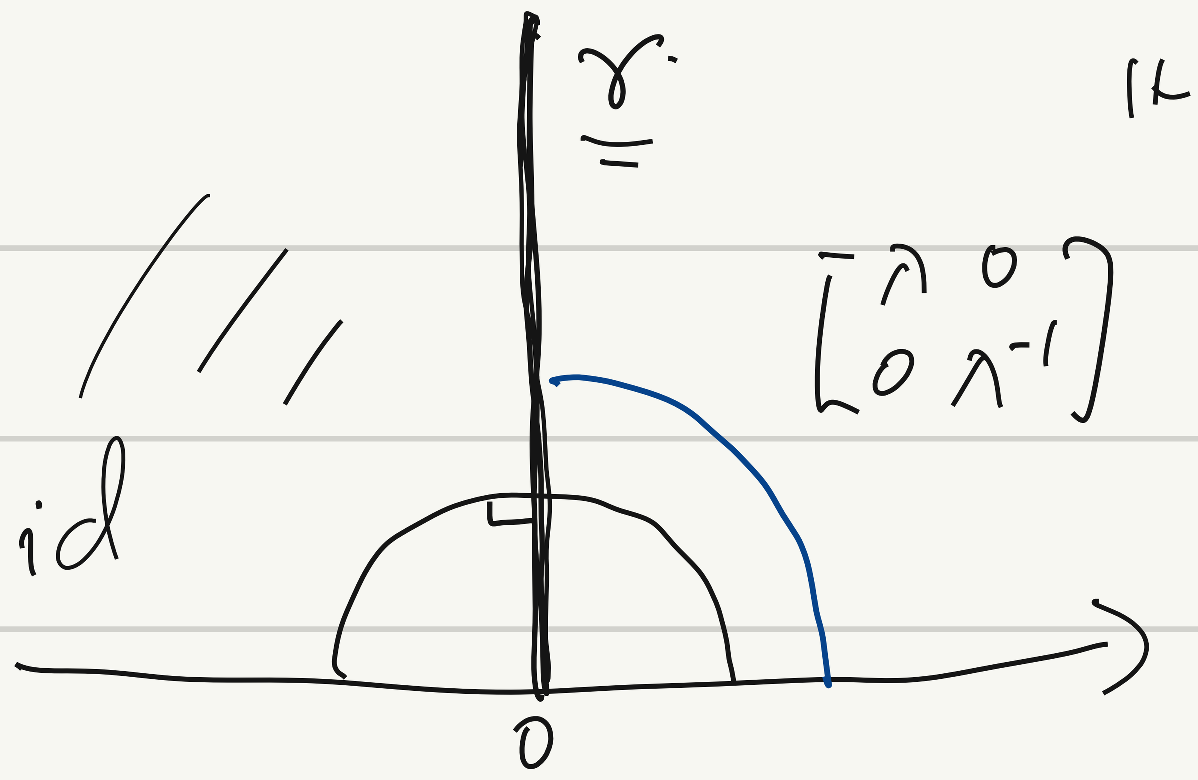
$\Gamma \qquad M^{-1}\Gamma' M$

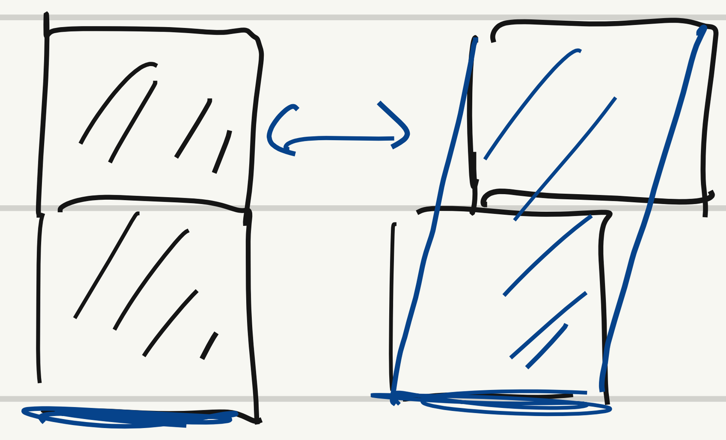
$= \langle \underline{A}, B \rangle \qquad \langle \underline{A}, M^{-1}B'M \rangle$

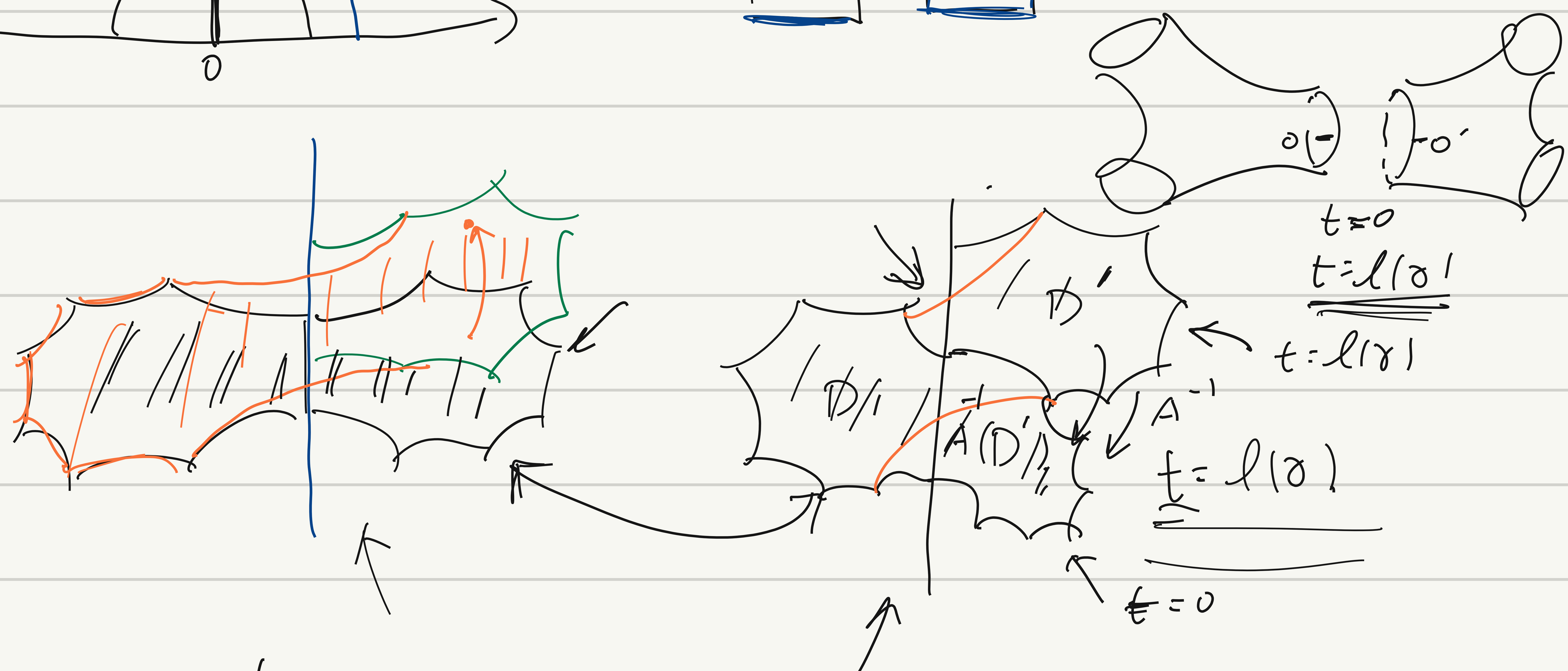
$\Gamma *_{\langle A \rangle} M^{-1}\Gamma' M = \Gamma_0$

$\leftarrow$ amalgamated free product.

Van-Kampen theorem.







$T \qquad T' = \langle A', B' \rangle$

$T \qquad M^{-1}T'M$

$= \langle A, B \rangle \qquad \langle A, M^{-1}B'M \rangle$

$T \ast_{\langle A \rangle} M^{-1}T'M = T_0$

$\parallel$

$\langle A, B, M^{-1}B'M \rangle$

$T_0 = \langle A, B, M^{-1}B'M \rangle$

$\langle A, B, A^{-1}M^{-1}B'MA \rangle$

$(T_0^t, \mathcal{A})$ marked discrete group

t generator of T_0

marking

Prop: Subgroups of $PSL(2,\mathbb{R})$: $0 \leq t < \ell(\gamma)$.

Marked subgroups of $PSL(2,\mathbb{R})$ $t \in \mathbb{R}$.

$(T_0^{t=0}, \mathcal{A}_0) \neq (T_0^{t=\ell(\gamma)}, \mathcal{A}_1)$

$T_0^{t=0} = T_0^{t=\ell(\gamma)}$

$\mathbb{H}/T = S.$

$A_i = \rho(a_i)$. as marking of $\rho(\pi_1(S))$

$\exists\, \rho : \pi_1(S) \to PSL(2,\mathbb{R})$

$\langle a_1 \dots a_n \mid R_1 \dots R_s \rangle$

marking of $\pi_1(S)$

$t = \ell(\gamma) + \varepsilon.$

$T^t = T^\varepsilon$

as subgroup of $PSL(2,\mathbb{R})$

$\mathbb{F}_2 = \langle a, b \rangle$

$\mathbb{F}_2 = \langle a, \underset{c}{ab} \rangle$

$w \in \mathbb{F}_2.$

$w(a,b)$

$w(a,c)$

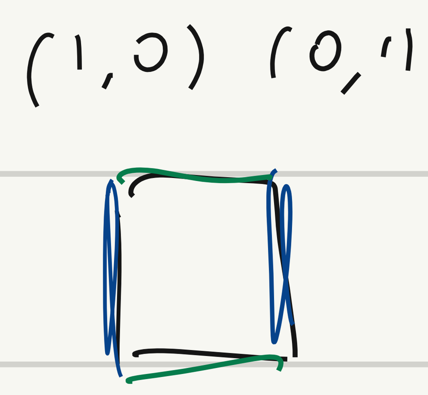
image

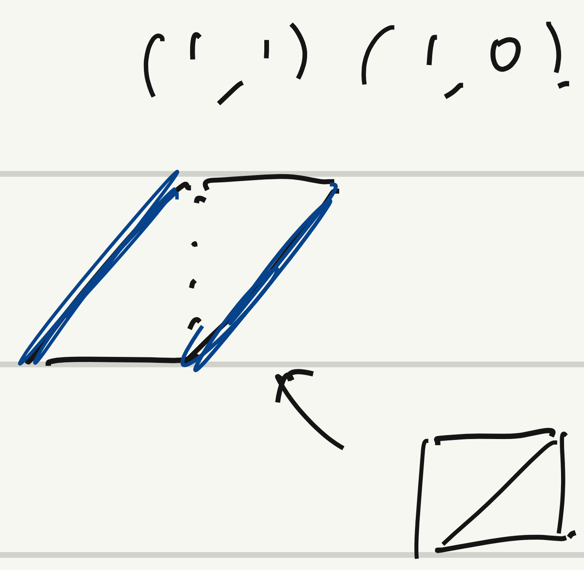
$\rho_1 : \pi_1(S) \to PSL(2,\mathbb{R})$

$\rho_2 : \pi_1(S) \to PSL(2,\mathbb{R})$

$T_1 = \rho_1(\pi_1(S)) \qquad A_1 = \rho_1(a_1) \dots A_n = \rho_1(a_n)$

$T_2 = \rho_2(\pi_1(S)) \qquad B_1 = \rho_2(a_1) \dots B_n = \rho_2(a_n)$





$\pi_1(T) = \mathbb{Z}^2$.

Prop: Subgroups of $PSL(2,\mathbb{R})$: $0 \le t < l(\gamma)$.

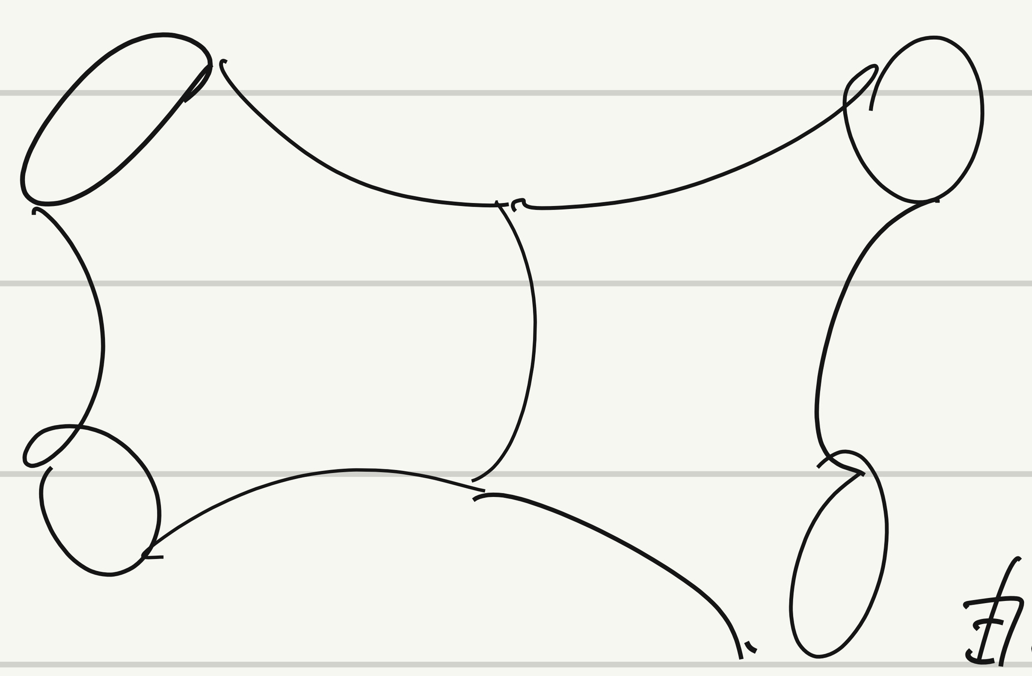
$\forall\ \underline{t \ne t'} \in [0, l(\gamma))$. $\Gamma^t \ne \Gamma^{t'}$ | $t = t' + l(\gamma)$. $\Gamma^t = \Gamma^{t'}$

Marked subgroups of $PSL(2,\mathbb{R})$ $t \in \mathbb{R}$.

$\forall t \ne t' \in \mathbb{R}$.

$\Gamma^t = \rho_t(\pi_1(S))$ $(\Gamma^t, \rho_t(a_1) \dots \rho_t(a_n))$

$\Gamma^{t'} = \rho_{t'}(\pi_1(S))$ $\ne$ $(\Gamma^{t'}, \rho_{t'}(a_1) \dots \rho_{t'}(a_n))$

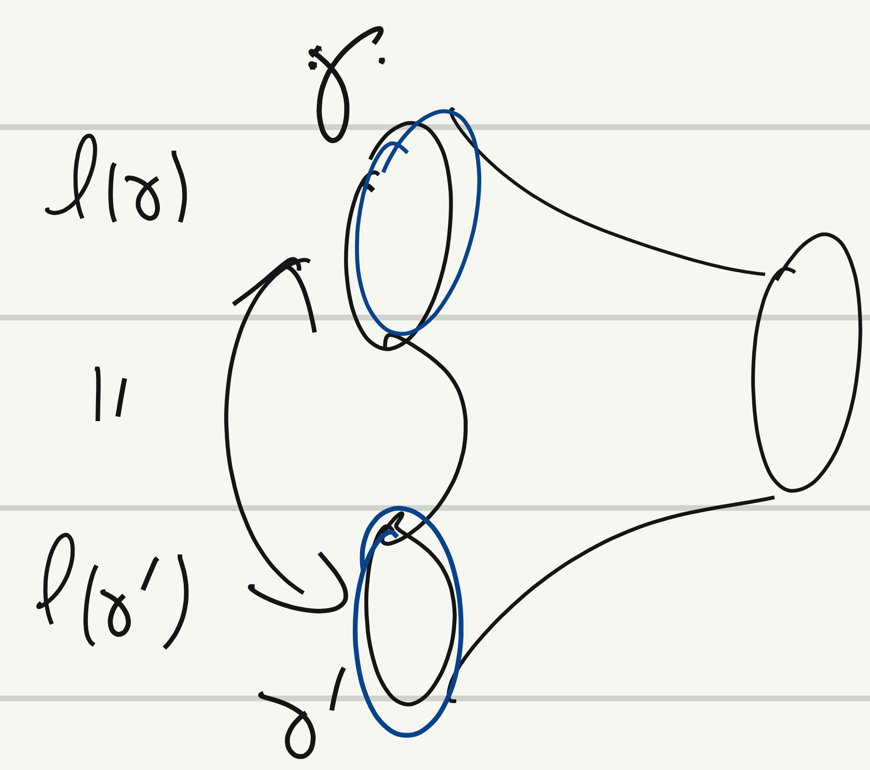


$t \ne t' \in [0, l(\gamma))$

$\nexists f: S^t \to S^{t'}$ isometry

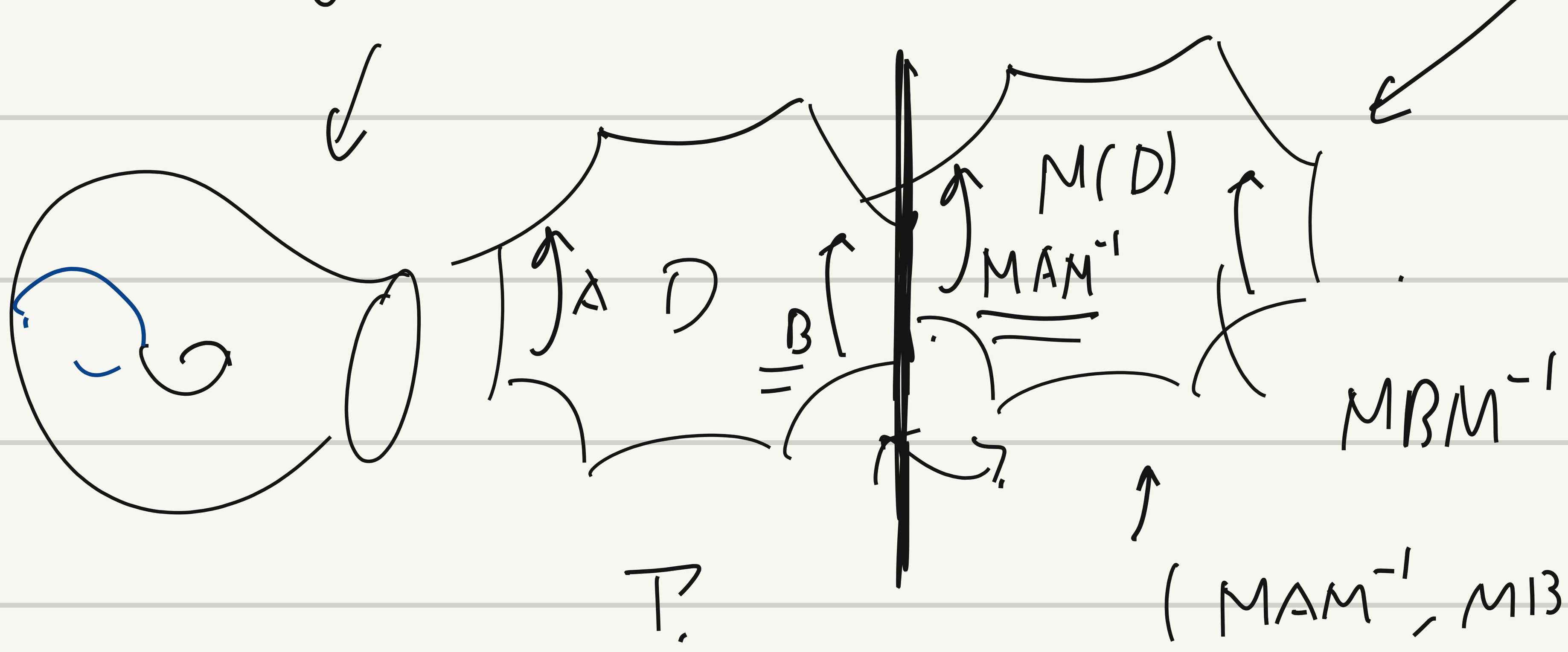
$t = t' + l(\gamma)$!

$\exists f: S^t \to S^{t'}$ isometry $\boxed{f \not\simeq id}$ marking.



HNN extension. $\Gamma = \langle A, B \rangle$

$\underset{\langle A \rangle}{*}\Gamma = \Gamma' = \langle A, B, M \mid \underline{\underline{B = MAM^{-1}}} \rangle$



$\boxed{MAM^{-1} = B}$

$\pi_1(S) \to \pi_1(P_1) \dots \pi_1(P_n)$

amalgamation

HNN-extension.

Prop. $\{P_1 \dots P_n\}$ parts in a pants decomp of $S = \mathbb{H}/\Gamma(S)$

$\underline{\Gamma_1} \dots \underline{\Gamma_n} < PSL(2,\mathbb{R})$

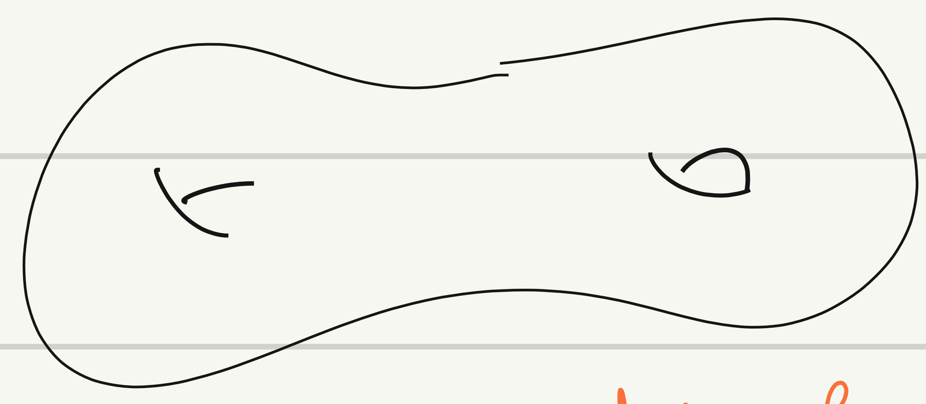
$\underline{\underline{\Gamma(S)}}$ can be obtained by taking amalgamation and HNN extension on $\Gamma_1 \dots \Gamma_n$ (up to conj)

Prop. Geometry on $\underline{S \text{ (marked)}}$ can be determined by
$\underbrace{(l_1, \ldots, l_{3g-3}, t_1, \ldots, t_{3g-3})}$ $\in \underset{\text{length para}}{\mathbb{R}_+^{3g-3}} \times \underset{\text{twist para.}}{\mathbb{R}^{3g-3}}$

6. Collar lemma :

γ simple closed geodesic in S.

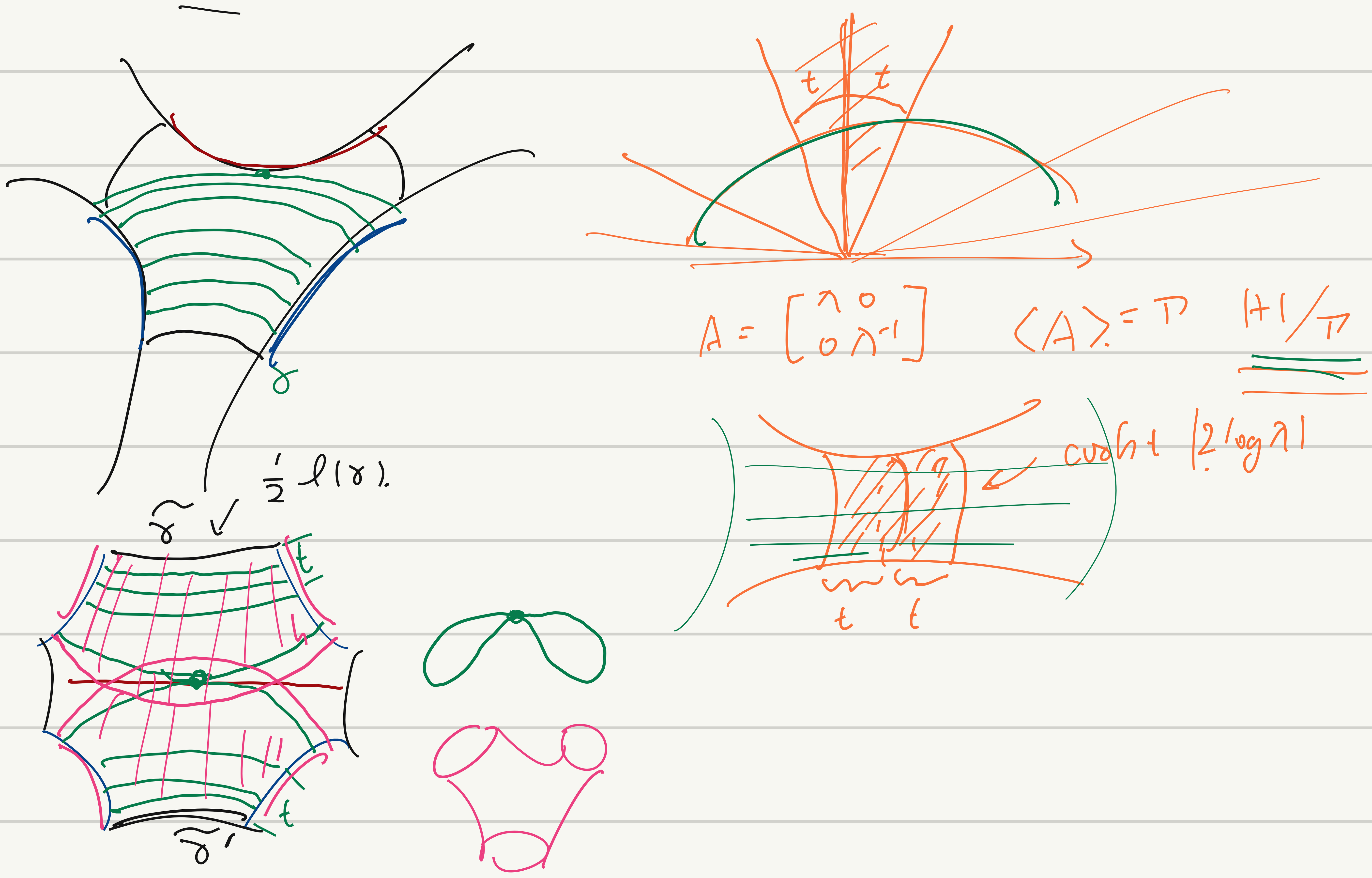
$\{p \in S \mid d_S(p, \gamma) < \underline{t}\} = \underline{C_\gamma(t)}$



Def : $C_\gamma(t)$ is a t-collar of $\underline{\gamma}$ if $\underline{C_\gamma(t)}$ is topologically a cylinder.

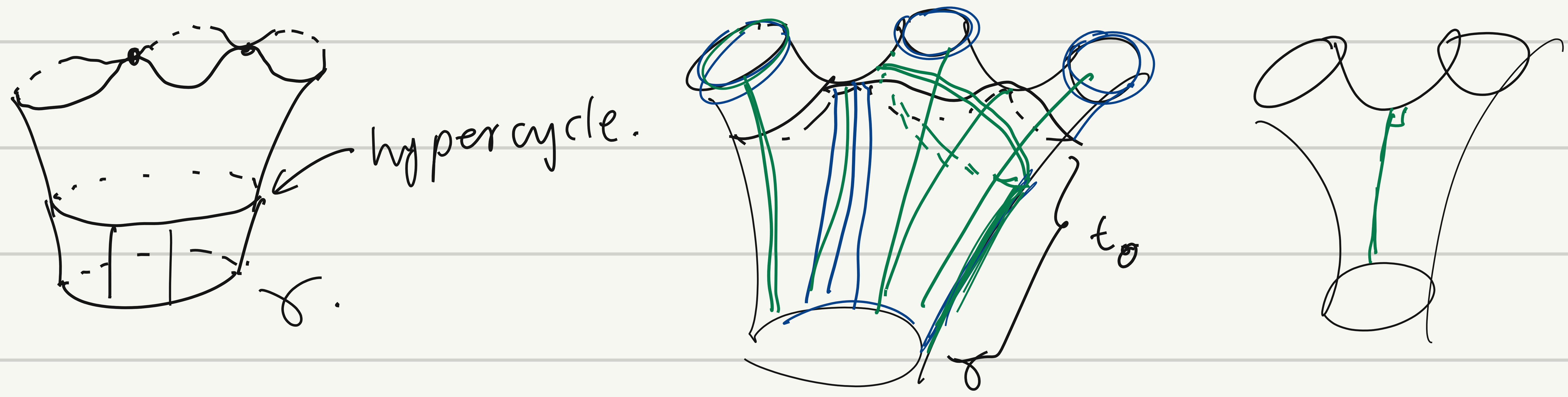
width of $C_\gamma(t)$

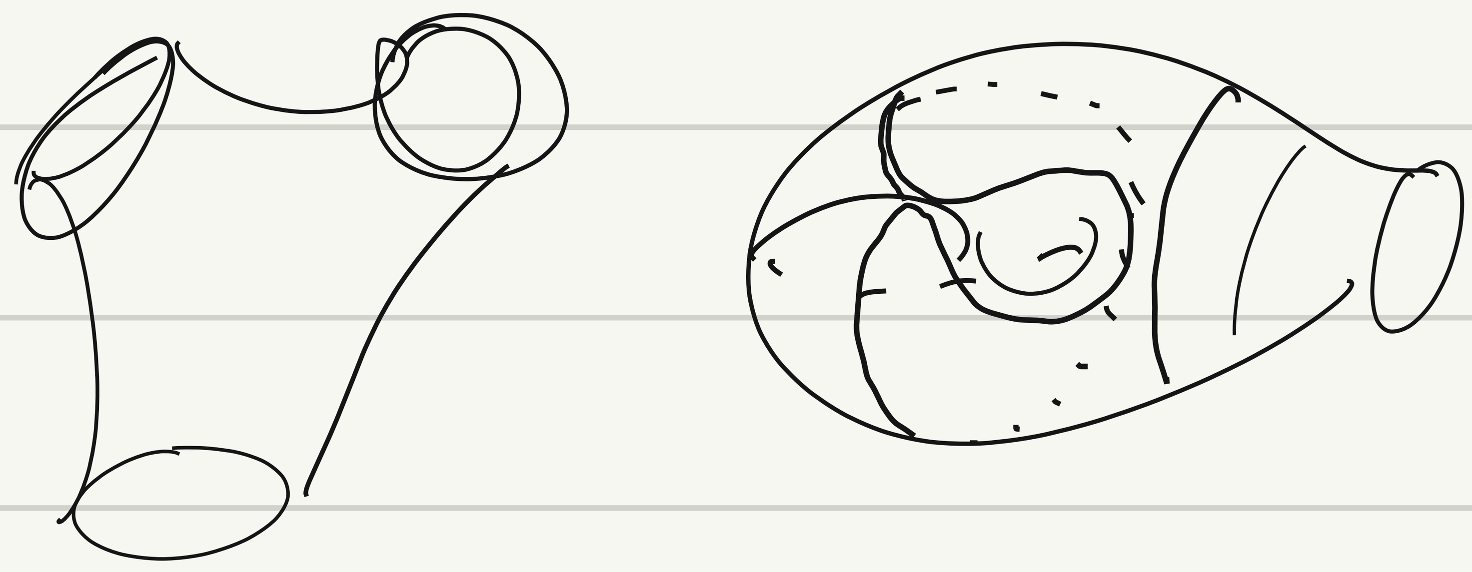
(loops of S in $C_\gamma(t) \to \pi_1(C_\gamma(t)) = \mathbb{Z}$).

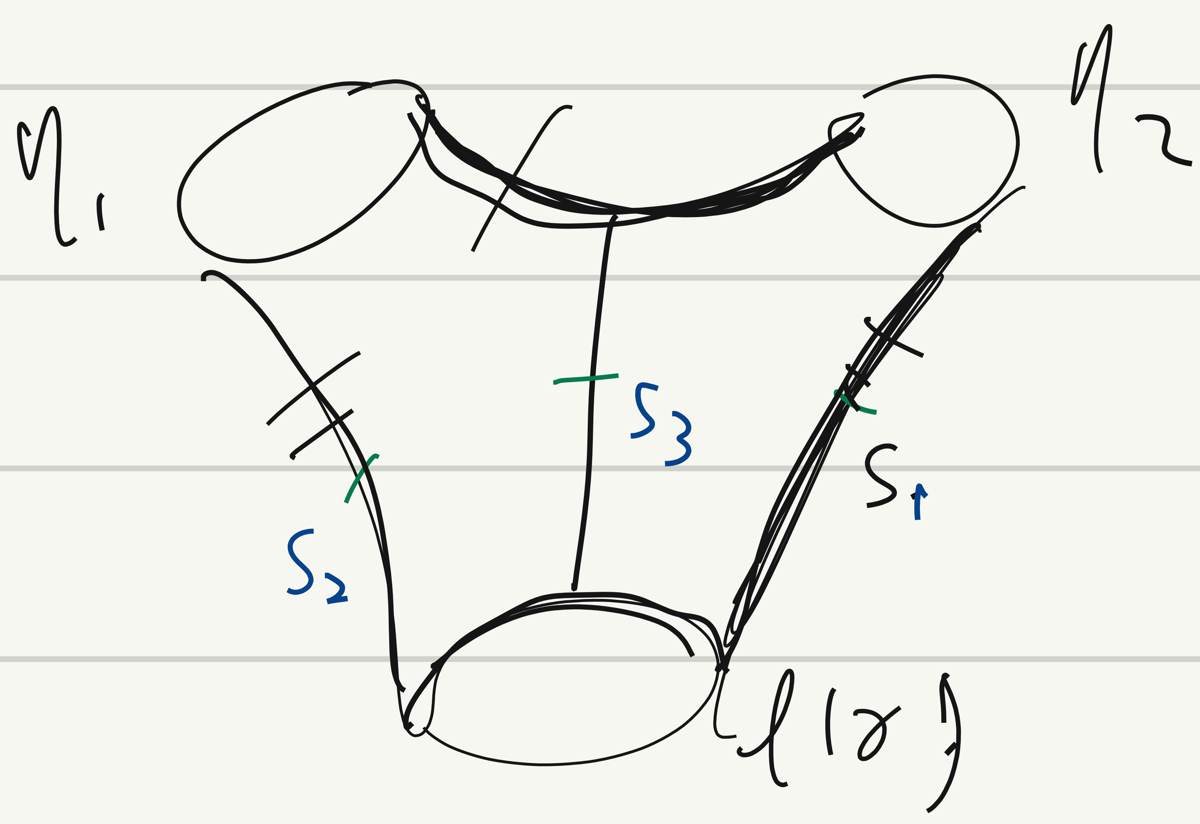


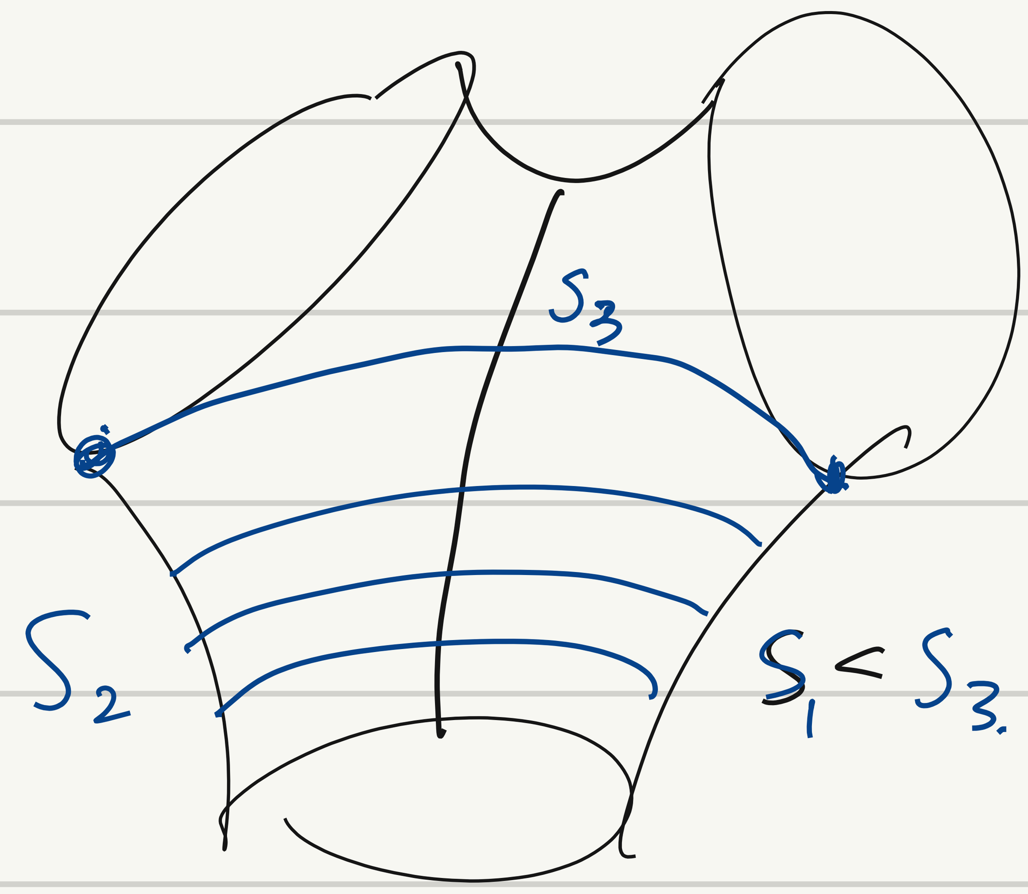
Collar lemma : If $l(\gamma)$ is the length of γ in $\underline{S}$, $\exists$ a collar of γ $\boxed{C_\gamma(t)}$ $\underline{t = \operatorname{arcsinh} \frac{1}{\sinh \frac{l(\gamma)}{2}}}$ $\boxed{\text{indep of } S}$

Proof :





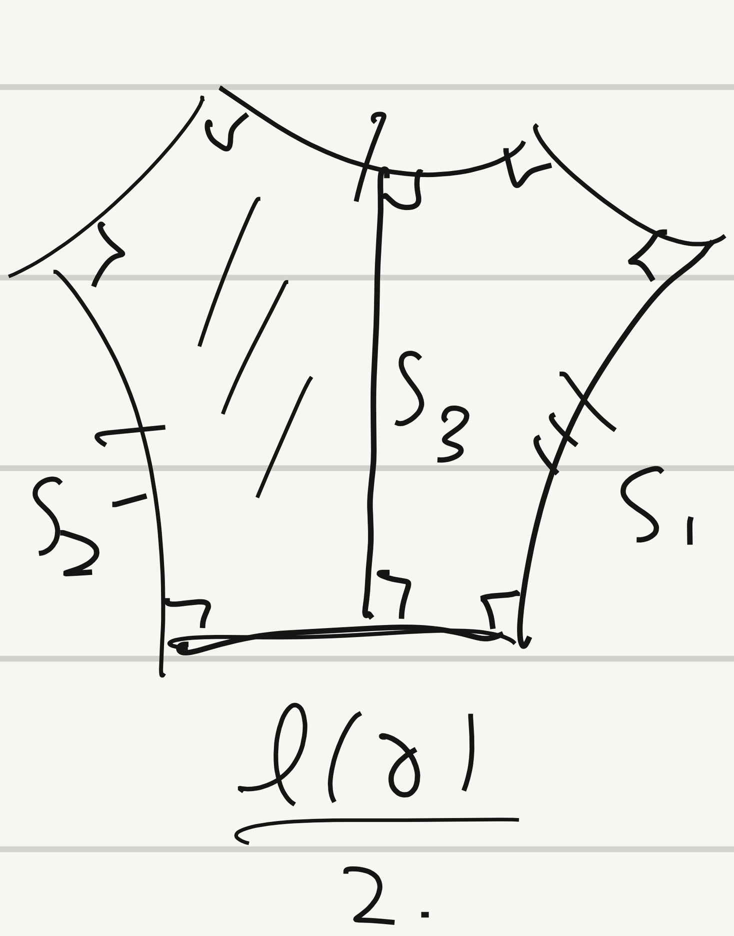


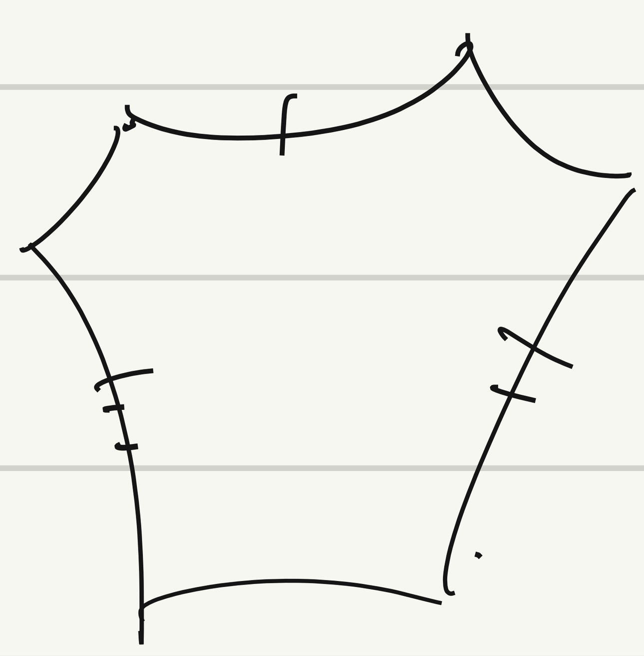


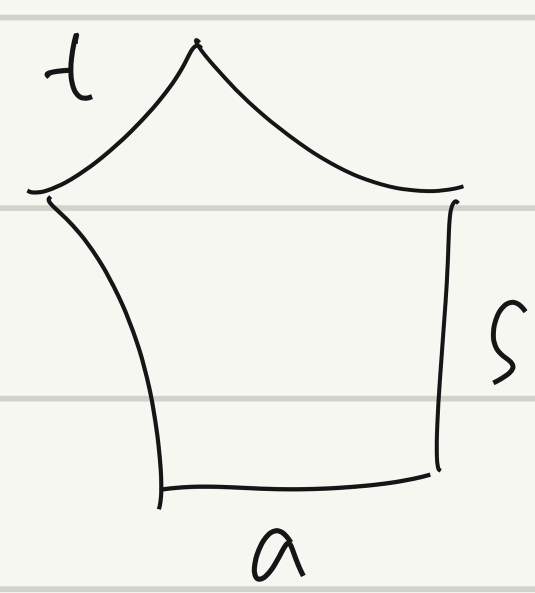
$t < \min\{s_1, s_2, s_3\}$

$$\inf_{p}\{\min\{s_1(p), s_2(p), s_3(p)\}\}$$

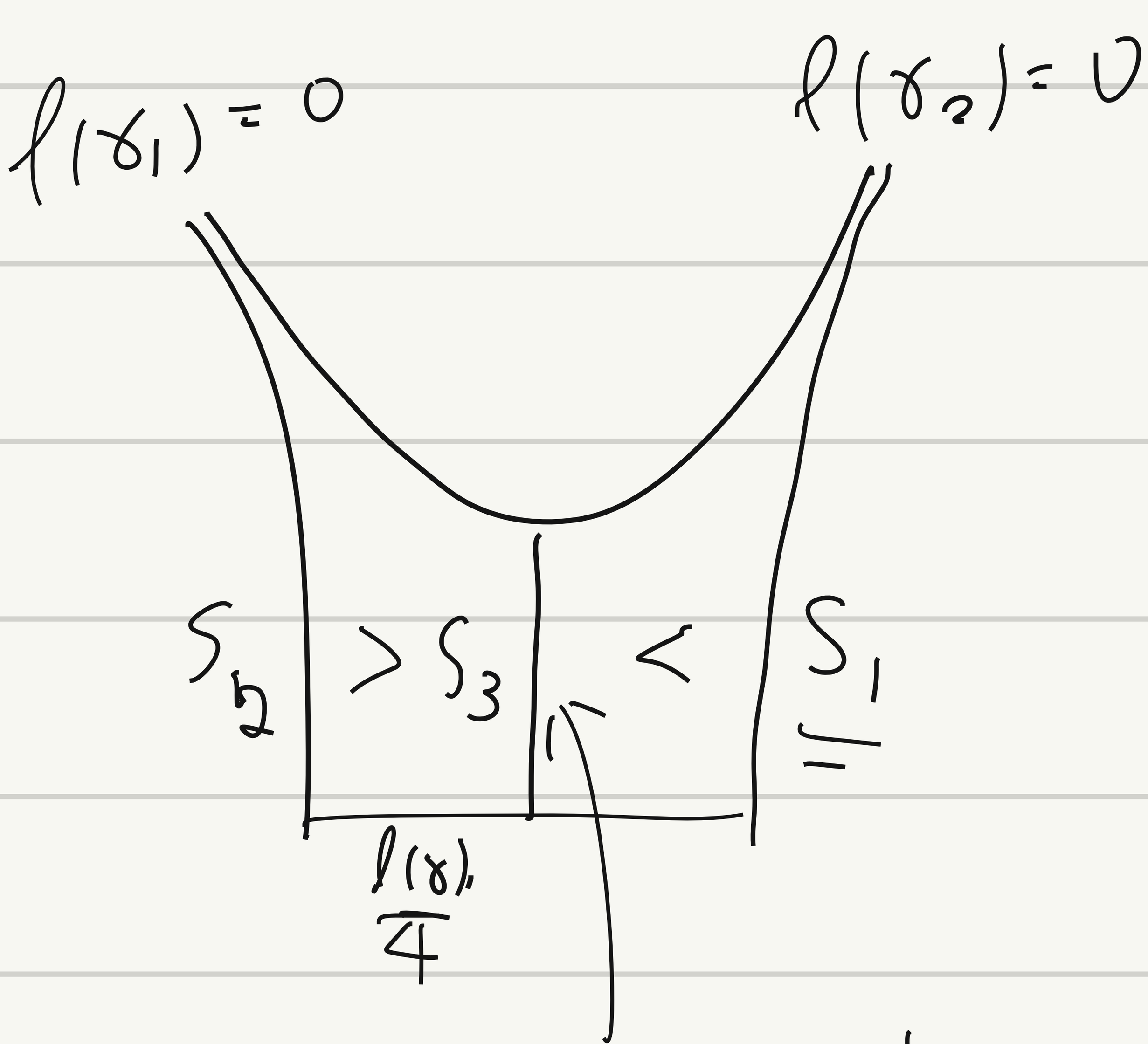
with one boundary length $= \ell(\gamma)$







$\sinh s \sinh a = \cosh t$

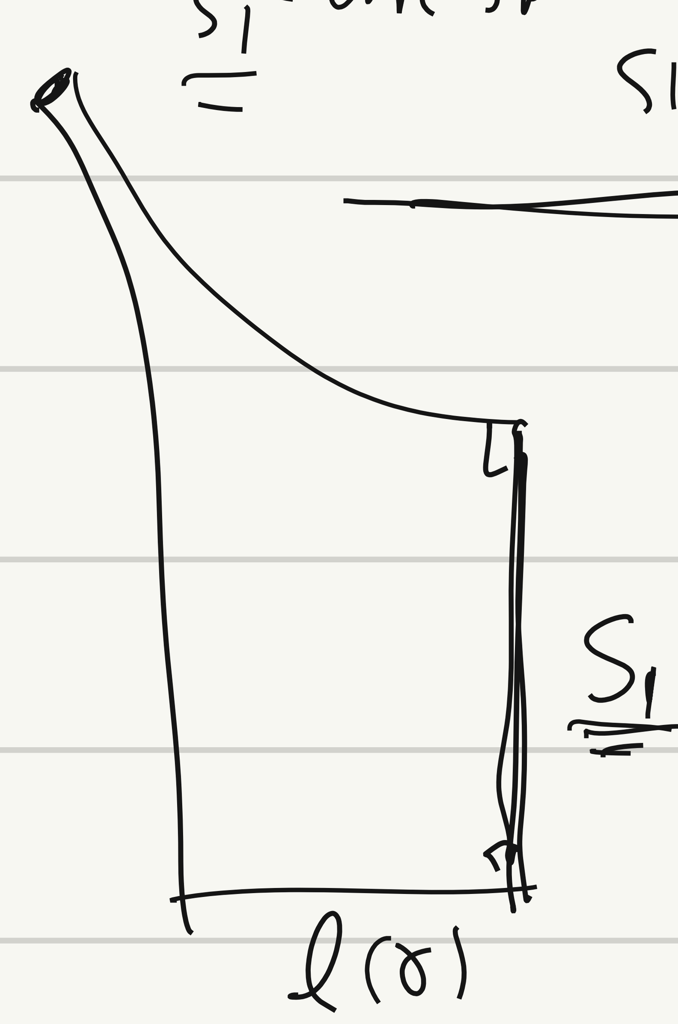


$s_3 = \operatorname{arcsinh} \frac{1}{\sinh \frac{\ell(\gamma)}{4}}$

$0 < \ell(\eta_1) \ll 1$

$\ell(\eta_2) \gg 0$

$s_1 = \operatorname{arcsinh} \frac{1}{\sinh \frac{\ell(\gamma)}{2}}$



$\inf \min = \left|\operatorname{arcsinh} \frac{1}{\sinh \frac{\ell(\gamma)}{2}}\right|$

6. "Collar lemma" for cusp.

Prop: H horocycle, $\ell(H) \leq 2$ $\Rightarrow$ H is embedded in S.