INTRODUCTION TO HYPERBOLIC SURFACES: SOLUTIONS

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This is an unofficial solution for the exercises of the short course, Introduction to hyperbolic surfaces, which is organized by Qiongling Li in the summer of 2021. More information can be found on `http://www.cim.nankai.edu.cn/2021/0611/c11453a372030/page.htm`.

1. Exercises I

We consider points and paths in the upper half plane $\mathbb{H}$. We use $l_{\mathbb{H}}$ and $l_{\mathbb{E}}$ as notations for the hyperbolic length and the Euclidean length respectively.

Exercise 1.1. (Easy) Let I denote the horizontal segment connecting i and $2+i$. Let $y>0$, and γ_y denote the path which is the union of the following three Euclidean segments:
- the vertical segment connecting i and iy,
- the horizontal segment connecting iy and $2+iy$,
- the vertical segment connecting $2+i$ and $2+iy$.

a) Find a parametrization of I and a parametrization of γ_y.
b) Compute $l_{\mathbb{H}}(I)$ and $l_{\mathbb{H}}(\gamma_y)$.
c) Find $y_0>0$, such that γ_{y_0} is the shortest among all γ_y's for $y>0$.

Solution.
a)

$$I(t)=i+2t \quad t\in[0,1],$$

$$\gamma_y(t)=\begin{cases} i[(y-1)t+1] & t\in[0,1]\\ 2(t-1)+iy & t\in(1,2]\\ 2+i[(1-y)t+3y-2] & t\in(2,3]\end{cases}$$

b)

$$l_{\mathbb{H}}(I)=\int_0^1\frac{|I'(t)|}{\mathrm{Im}I(t)}\mathrm{d}t=\int_0^1 2\mathrm{d}t=2,$$

$$\begin{aligned} l_{\mathbb{H}}(\gamma_y)&=\int_0^1\frac{|\gamma_y'(t)|}{\mathrm{Im}\gamma_y(t)}\mathrm{d}t+\int_1^2\frac{|\gamma_y'(t)|}{\mathrm{Im}\gamma_y(t)}\mathrm{d}t+\int_2^3\frac{|\gamma_y'(t)|}{\mathrm{Im}\gamma_y(t)}\mathrm{d}t\\ &=\int_0^1\frac{|y-1|}{(y-1)t+1}\mathrm{d}t+\int_1^2\frac{2}{y}\mathrm{d}t+\int_2^3\frac{|1-y|}{(1-y)t+3y-2}\mathrm{d}t\\ &=2|\log y|+\frac{2}{y}.\end{aligned}$$

c) When $y>1$,

$$\frac{\mathrm{d}}{\mathrm{d}y}l_{\mathbb{H}}(\gamma_y)=\frac{2(y-1)}{y^2}>0,$$

and when $0<y<1$,

$$\frac{\mathrm{d}}{\mathrm{d}y}l_{\mathbb{H}}(\gamma_y)=-\frac{2(y+1)}{y^2}<0,$$

so $y_0=1$ minimize $l_{\mathbb{H}}(\gamma_y)$. □

Exercise 1.2. (Normal) Let I denote the horizontal segement connecting i and $2+i$ as above. Let $y>0$, and η_y denote the path which is the union of the following two segments:
- the Euclidean segment connecting i and $1+iy$,
- the Euclidean segment connecting $1+iy$ and $2+i$.

a) Find a parametrization of η_y.
b) Compute $l_{\mathbb{H}}(\eta_y)$.
c) Compare $l_{\mathbb{H}}(\eta_y)$ for $y=2$ and $l_{\mathbb{H}}(I)$.

Solution.

a)

$$\eta_y(t) = \begin{cases} t + i[(y-1)t+1] & t \in [0,1] \\ t + i[(1-y)t + 2y - 1] & t \in (1,2] \end{cases}$$

b)

$$\begin{aligned} l_{\mathbb{H}}(\eta_y) &= \int_0^1 \frac{|\eta_y'(t)|}{\mathrm{Im}\eta_y(t)}\mathrm{d}t + \int_1^2 \frac{|\eta_y'(t)|}{\mathrm{Im}\eta_y(t)}\mathrm{d}t \\ &= \int_0^1 \frac{\sqrt{y^2-2y+2}}{(y-1)t+1}\mathrm{d}t + \int_1^2 \frac{\sqrt{y^2-2y+2}}{(1-y)t+2y-1}\mathrm{d}t \\ &= \frac{2\sqrt{y^2-2y+2}}{y-1}\log y. \end{aligned}$$

c) When $y = 2$, $l_{\mathbb{H}}(\eta_y) = 2\sqrt{2}\log 2 < 2 = l_{\mathbb{H}}(I)$. □

Exercise 1.3. (Hard) Let N be a positive integer. Let I_N denote the horizontal segment connecting $-N+i$ and $N+i$.

a) Compute $l_{\mathbb{H}}(I_N)$.

b) Describe the geodesic γ_N connecting $-N+i$ and $N+i$, and compute $l_{\mathbb{H}}(\gamma_N)$.

c) Find a function $f : \mathbb{N}^+ \to \mathbb{R}$, such that

$$\lim_{N\to+\infty} \frac{f(N)}{l_{\mathbb{H}}(\gamma_N)} = 1.$$

Solution.

a) Set $I_N(t) := Nt + i$, where $t \in [-1,1]$, then

$$l_{\mathbb{H}}(I_N) = \int_{-1}^1 \frac{|I_N'(t)|}{\mathrm{Im}I_N(t)}\mathrm{d}t = \int_{-1}^1 \frac{N}{1}\mathrm{d}t = 2N.$$

b) γ_N is the minor arc of the circle centered at the origin with radius $\sqrt{N^2+1}$ which connects $-N+i$ and $N+i$. Set $\theta_N := \arctan\dfrac{1}{N}$ and $\gamma_N(\theta) := \sqrt{N^2+1}e^{i\theta}$, where $\theta \in [\theta_N, \pi - \theta_N]$, then

$$l_{\mathbb{H}}(\gamma_N) = \int_{\theta_N}^{\pi-\theta_N} \frac{|\gamma_N'(\theta)|}{\mathrm{Im}\gamma_N(\theta)}\mathrm{d}\theta = \int_{\theta_N}^{\pi-\theta_N} \csc\theta\mathrm{d}\theta = 2\log(\sqrt{N^2+1}+N).$$

c) Let $f := l_{\mathbb{H}}(\gamma_N)$. □

Exercise 1.4. (Normal) Let w and z be two points in $\mathbb{H}$. Let $\gamma : [a,b] \to \mathbb{H}$ be a regular path connecting w and z.

a) Show that for any $y > 0$, if for all $t \in [a,b]$, we have $\mathrm{Im}\gamma(t) \leqslant y$(i.e. γ is entirely below the horizontal line H_y), then we have

$$l_{\mathbb{H}}(\gamma) \geqslant \frac{l_{\mathbb{E}}(\gamma)}{y}.$$

b) Let $v = \mathrm{Im}w$. Show that for any $y > v$, if there exists a $t \in [a,b]$, such that $\mathrm{Im}\gamma(t) > y$(i.e. γ crosses H_y), we have

$$l_{\mathbb{H}}(\gamma) \geqslant \left|\log\frac{y}{v}\right|.$$

c) Use a) and b) to show that $d_{\mathbb{H}}(w,z) = 0$ if and only if $w = z$.

Solution.

a) $l_{\mathbb{H}}(\gamma) = \displaystyle\int_a^b \frac{|\gamma'(t)|}{\mathrm{Im}\gamma(t)}\mathrm{d}t \geqslant \int_a^b \frac{|\gamma'(t)|}{y}\mathrm{d}t = \frac{1}{y}\int_a^b |\gamma'(t)|\mathrm{d}t = \frac{l_{\mathbb{E}}(\gamma)}{y}$.

b) $\mathrm{Im}\gamma : [a,b] \to \mathbb{R}$ is continuous so that there exists a $t_0 \in [a,b]$ such that $\mathrm{Im}\gamma(t_0) = y$. Thus $l_{\mathbb{H}}(\gamma) \geqslant l_{\mathbb{H}}(\gamma|_{[a,t_0]}) \geqslant d_{\mathbb{H}}(w, H_y) = \left|\log\dfrac{y}{v}\right|$.

c) It's obvious that $w = z$ implies $d_{\mathbb{H}}(w,z) = 0$, so it's suffices to prove the inverse direction. Suppose $w \neq z$, then $d_{\mathbb{E}}(w,z) \neq 0$. Let $R = \mathrm{Im}w + \mathrm{Im}z$. Thus for each γ connects w and z,

$$l_{\mathbb{H}}(\gamma) \geqslant \min\left\{\frac{l_{\mathbb{E}}(\gamma)}{R}, \log\frac{R}{\mathrm{Im}w}\right\},$$

so $d_{\mathbb{H}}(w,z) = \inf\limits_{\gamma} l_{\mathbb{H}}(\gamma) \geqslant \min\left\{\inf\limits_{\gamma}\dfrac{l_{\mathbb{E}}(\gamma)}{R}, \log\dfrac{R}{\mathrm{Im}w}\right\} = \min\left\{\dfrac{d_{\mathbb{E}}(\gamma)}{R}, \log\dfrac{R}{\mathrm{Im}w}\right\} > 0.$ □

2. EXERCISES II

Let $l_{\mathbb{E}}$, $l_{\mathbb{H}}$ and $A_{\mathbb{H}}$ be the notations for the Euclidean length, the hyperbolic length and the hyperbolic area respectively. Let H_y be the horizontal line passing iy and V_x be the vertical geodesic ending at x and ∞.

Exercise 2.1. (Easy) Let C denote a circle in $\mathbb{H}$ with Euclidean center $z_{\mathbb{E}} = x + iy_{\mathbb{E}} \in \mathbb{H}$, of Euclidean radius r.

a) Compute the hyperbolic radius R of C in term of x, $y_{\mathbb{E}}$ and r.

b) For each $y_{\mathbb{E}}$, find r such that $l_{\mathbb{H}}(C) = l_{\mathbb{E}}(C)$.

Solution.

a) Let the hyperbolic center of C be $y_{\mathbb{H}}$. Since $\log \dfrac{y_{\mathbb{H}}}{y_{\mathbb{E}} - r} = R = \log \dfrac{y_{\mathbb{E}} + r}{y_{\mathbb{H}}}$, we get $y_{\mathbb{H}} = \sqrt{(y_{\mathbb{E}} - r)(y_{\mathbb{E}} + r)}$, $R = \dfrac{1}{2} \log \dfrac{y_{\mathbb{E}} + r}{y_{\mathbb{E}} - r}$.

b) $l_{\mathbb{H}}(C) = 2\pi \sinh \left(\dfrac{1}{2} \log \dfrac{y_{\mathbb{E}} + r}{y_{\mathbb{E}} - r} \right)$ and $l_{\mathbb{E}}(C) = 2\pi r$, hence $l_{\mathbb{H}}(C) = l_{\mathbb{E}}(C)$ is equivalent to $r = 0$ or $r^2 = y_{\mathbb{E}}^2 - 1$. So such $r > 0$ exists only when $y_{\mathbb{E}} > 1$ and in this case $r = \sqrt{y_{\mathbb{E}}^2 - 1}$. □

Exercise 2.2. (Easy) Recall the definitions and some properties of the hyperbolic cosine function and the hyperbolic sine functions: for x and y in $\mathbb{R}$, we have

$$\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2}, \\ \sinh x &= \frac{e^x - e^{-x}}{2}, \\ 1 &= \cosh^2 x - \sinh^2 x, \\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y, \\ \sinh(x+y) &= \cosh x \sinh y + \sinh x \cosh y. \end{aligned}$$

These functions can be extended to $\mathbb{C}$. Using $e^{i\theta} = \cos\theta + i \sin\theta$ to verify the following equalities:

a) For any $\theta \in [0, 2\pi]$, we have

$$\begin{aligned} \sinh(i\theta) &= i \sin\theta, \\ \cosh(i\theta) &= \cos\theta. \end{aligned}$$

b) For any $x \in \mathbb{R}$, we have

$$\begin{aligned} \sin(ix) &= i \sinh x, \\ \cos(ix) &= \cosh x. \end{aligned}$$

Solution.

a)

$$\begin{aligned} \sinh(i\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{2} = i\sin\theta, \\ \cosh(i\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{(\cos\theta + i\sin\theta + (\cos\theta - i\sin\theta)}{2} = \cos\theta. \end{aligned}$$

b)

$$\begin{aligned} \sin(ix) &= \frac{(\cos(ix) + i\sin(ix)) - (\cos(ix) - i\sin(ix))}{2i} = \frac{e^{-x} - e^{x}}{2i} = i\sinh x, \\ \cos(ix) &= \frac{(\cos(ix) + i\sin(ix)) + (\cos(ix) - i\sin(ix))}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x. \end{aligned}$$

□

Exercise 2.3. (Easy) Let C_R denote a circle in $\mathbb{H}$ of hyperbolic radius R. Let D_R denote the closed disk bounded by C_R. Let $l(R) = l_{\mathbb{H}}(C_R)$ and $A(R) = A_{\mathbb{H}}(D_R)$.

a) Compute the following limits:

$$\lim_{R \to 0} l(R) - A(R).$$

$$\lim_{R \to +\infty} l(R) - A(R).$$

b) Verify the following equality.

$$(l(R))^2 = 4\pi A(R) + (A(R))^2$$

Solution. $l(R) = 2\pi \sinh R$, $A(R) = 2\pi(\cosh R - 1)$. Thus

a)
$$\lim_{R\to 0} l(R) - A(R) = 2\pi \lim_{R\to 0}(\sinh R - \cosh R + 1) = 0,$$
$$\lim_{R\to +\infty} l(R) - A(R) = 2\pi \lim_{R\to +\infty}(\sinh R - \cosh R + 1) = 2\pi \lim_{R\to +\infty}(-e^{-R} + 1) = 2\pi.$$

b)
$$\begin{aligned}4\pi A(R) + (A(R))^2 &= 8\pi^2(\cosh R - 1) + 4\pi^2(\cosh^2 R + 1 - 2\cosh R)\\ &= 4\pi^2 \sinh^2 R\\ &= (l(R))^2.\end{aligned}$$

□

Remark 2.1. We denote by l the hyperbolic length of a closed curve bounding a simply connected region in $\mathbb{H}$, and by A the area of this region. The isoperimetric inequality for hyperbolic plane is as follows:
$$l^2 \geqslant 4\pi A + A^2.$$
Moreover, the equality holds if and only if the region is a disk.

Exercise 2.4. We would like to get the formula for the length of an arc in a circle, a horocycle or a hypercycle, and compare them.

a) (Easy) Let C be a circle in $\mathbb{H}$ of hyperbolic radius R. Let c be an arc on C with central angle θ. Compute the length of c in term of R and θ.

b) (Normal) We consider horocycles H_y's with center ∞. Let c be an arc on H_1 between V_0 and V_x.

i. Compute the length of c in term of x.

ii. Compute the distance R between H_y and H_1 in term of y.

iii. Let c_y denote the horocycle arc on H_y between V_0 and V_x. Compute the length of c_y in terms of R and x.

c) (Hard) Consider hypercycles of center V_0. We denote by L_θ the hypercycle having angle θ with V_0. We consider the radius geodesics γ_r which is a geodesic with Euclidean center 0 and Euclidean radius r.

i. Compute the distance R between L_θ and V_0 in term of θ.

ii. Compute the distance d between $\gamma_1 (r = 1)$ and γ_r in term of r.

iii. Compute the length of the arc c in L_θ between radius γ_1 and γ_r, in terms of θ and r.

iv. Rewrite the length c in term of R and d.

Solution.

a) Without loss of generality, we could let the hyperbolic center of C be ai. Set
$$re^{i\theta} = \frac{x + yi + ai}{x + yi - ai}$$
and consider the coordinate transformation $(x, y) \mapsto (r, \theta)$, we can get
$$x = a\frac{2r\sin\theta}{r^2 - 2r\cos\theta + 1}, y = a\frac{r^2 - 1}{r^2 - 2r\cos\theta + 1}.$$
And from Euclidean geometry we know that C is the curve $r = \dfrac{e^R + 1}{e^R - 1}, \theta = \theta, \theta \in [0, 2\pi]$ and the central angle of (r, θ_1) and (r, θ_2) on the hyperbolic circle C is $|\theta_1 - \theta_2|$. Then
$$\mathrm{d}x = a\frac{2\sin\theta(1 - r^2)\mathrm{d}r + 2r[\cos\theta(r^2 + 1) - 2r]\mathrm{d}\theta}{(r^2 - 2r\cos\theta + 1)^2},$$
$$\mathrm{d}y = a\frac{-2\cos\theta(r^2 + 1)\mathrm{d}r - 2r(r^2 - 1)\sin\theta\mathrm{d}\theta}{(r^2 - 2r\cos\theta + 1)^2}.$$
Thus the hyperbolic length of the tangent vector of C is
$$\sqrt{\frac{4r^2a^2\dfrac{[\cos\theta(r^2 + 1) - 2r]^2 + (r^2 - 1)^2\sin^2\theta}{(r^2 - 2r\cos\theta + 1)^4}}{a^2\dfrac{(r^2 - 1)^2}{(r^2 - 2r\cos\theta + 1)^2}}} = \frac{2r}{r^2 - 1} = \sinh R.$$
Hence $l_{\mathbb{H}}(c) = \theta\sinh R$.

b) i. It's obvious that $l_{\mathbb{H}}(c) = |x|$.
ii. For any point $z \in H_1$, $d_{\mathbb{H}}(z, H_y) = |\log y|$, so $R = d_{\mathbb{H}}(H_y, H_1) = |\log y|$.
iii. $l_{\mathbb{H}}(c_y) = \dfrac{|x|}{y}$, so when $y > 1$, $l_{\mathbb{H}}(c_y) = e^{-R}x$ and when $y < 1$, $l_{\mathbb{H}}(c_y) = e^{R}x$.

c) i. $d_{\mathbb{H}}(V_0, re^{i(\frac{\pi}{2}-\theta)}) = l_{\mathbb{H}}(\gamma_r) = -\log\dfrac{\cos\theta}{\sin\theta + 1}$, hence $R = d_{\mathbb{H}}(V_0, L_\theta) = -\log\dfrac{\cos\theta}{\sin\theta+1}$.
ii. The only geodesic which is orthogonal to both γ_1 and γ_r is the segment connecting i and ri, thus $d = |\log r|$.
iii. Let $c(t) = (rt + 1 - t)(\sin\theta + i\cos\theta), t \in [0, 1]$. Then

$$l_{\mathbb{H}}(c) = \int_0^1 \frac{|r-1|}{(rt+1-t)\cos\theta}\mathrm{d}t = |\log r|\sec\theta.$$

iv. $l_{\mathbb{H}}(c) = d\cosh R$.

□

3. Exercises III

For $x \in \mathbb{R}$, $y > 0$ and $r > 0$, we use H_y for the horizontal line passing iy, V_x for the vertical geodesic with end point x and ∞, and $C(x, r)$ for the circular geodesic with Euclidean center x and Euclidean radius r.

Exercise 3.1. Let $x \in (0, 1)$. Let γ_x denote the circular geodesic with end points x and $1/x$.
a) (Easy) Compute the formula for the reflection ι_x of $\mathbb{H}$ along γ_x.
b) (Easy) Show that

$$\lim_{x\to 0^+} \iota_x = \iota_0,$$

where ι_0 is the reflection along V_0, i.e. for any $z \in \mathbb{H}$, we have

$$\lim_{x\to 0^+} \iota_x(z) = \iota_0(z).$$

c) (Normal) Compute the distance $d(x)$ between γ_x and V_0.
d) (Normal) Let $d_0 > 0$ be a constant. Find the hyperbolic isometry f such that
- the axis of f is $C(0, 1)$;
- the translation distance $l(f)$ of f is d_0;
- the translation direction is from -1 to 1.

Solution.
a) $\iota_x(z) = \dfrac{\dfrac{x+1/x}{2}\bar{z} - 1}{\bar{z} - \dfrac{x+1/x}{2}} = \dfrac{(x^2+1)\bar{z} - 2x}{2x\bar{z} - (x^2+1)}$.

b) For any $z \in \mathbb{H}$, $\displaystyle\lim_{x\to 0^+} \iota_x(z) = \lim_{x\to 0^+} \frac{(x^2+1)\bar{z} - 2x}{2x\bar{z} - (x^2+1)} = -\bar{z} = \iota_0(z)$.

c) The only geodesic which is orthogonal to both V_0 and γ_x is $C(0, 1)$ and we can get the Euclidean central angle of its arc between V_0 and γ_x is $\theta_x := \arctan\dfrac{2x}{1-x^2}$. Therefore,

$$d(x) = \log\frac{\sin\theta_x + 1}{\cos\theta_x} = \log\frac{1+x}{1-x}.$$

d) f is the composition of the reflection along V_0 and γ_x in order. And $d_0 = l(f) = 2d(x)$ tells us $x = \dfrac{e^{d_0/2}+1}{e^{d_0/2}-1}$. Thus

$$f = \frac{(x^2+1)z + 2x}{2xz + (x^2+1)},$$

where $x = \dfrac{e^{d_0/2}+1}{e^{d_0/2}-1}$. □

Exercise 3.2. Consider the parabolic isometry T_t.
a) (Easy) Find $x \in \mathbb{R}$ such that $V_x = T_t(V_0)$.
b) (Easy) Compute the length l_y of the segment in H_y between V_x and V_0.
c) (Easy) Show

$$\lim_{y\to+\infty} l_y = 0,$$

and use it to conclude that the translation distance $l(T_t)$ of T_t is 0.

d) (Easy) Show that $l(T_t)$ is not realizable, i.e. there is no $z \in \mathbb{H}$ such that $l(T_t) = d_{\mathbb{H}}(z, T_t(z))$.

Solution.

a) It's trivial that $x = 0 + t = t$.

b) $l_y = \dfrac{x}{y}$.

c)

$$\lim_{y\to+\infty} l_y = \lim_{y\to+\infty} \frac{x}{y} = 0,$$

and for any $z =\in \mathbb{H}$, $d_{\mathbb{H}}(z, T_t(z)) = \dfrac{t}{\text{Im}z}$, so

$$l(T_t) = \inf_{z\in\mathbb{H}} d_{\mathbb{H}}(z, T_t(z)) = 0.$$

d) For any $z \in \mathbb{H}$, $d_{\mathbb{H}}(z, T_t(z)) = \dfrac{t}{\text{Im}z} > 0 = l(T_t)$, so $l(T_t)$ is not realizable. □

Exercise 3.3. (Easy) Let $z = x + iy \in \mathbb{H}$. Find the elliptic isometry whose fixed point is z with rotation angle π.

Solution. We know that $C(x, y)$ and V_x are two geodesics intersecting at z with intersection angle $\dfrac{\pi}{2}$. So the composition of the two reflections along these two geodesics is what we need. For instance, these reflections are

$$w \mapsto \frac{x\bar{w} + y^2 - x^2}{\bar{w} - x}, w \mapsto -\bar{w} + 2x$$

respectively. Thus their composition

$$w \mapsto \frac{-xw + y^2 + x^2}{-w + x}$$

is the transformation required. □

4. Exercises IV

For $\theta \in [0, 2\pi)$, $\lambda > 0$ and $t \in \mathbb{R}$, we consider

$$K_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, A_\lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \text{and} N_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Their corresponding Möbius transformations are ρ_θ, ϕ_λ and T_t respectively. Recall that Möbius transformations on $\mathbb{H}$ are orientation preserving isometries of $\mathbb{H}$.

Let B and C be matrices in $\text{SL}(2, \mathbb{R})$. Recall that B and C are similar to each other in $\text{SL}(2, \mathbb{R})$ if there is a matrix $P \in \text{SL}(2, \mathbb{R})$, such that $B = PCP^{-1}$, (i.e. B can be obtained by taking the conjugation of C by P). In the following, by being similar, we always mean being similar in $\text{SL}(2, \mathbb{R})$.

Exercise 4.1. Show that for any matrices B and C in $\text{SL}(2, \mathbb{R})$, we have

$$\text{(Easy)} \quad \text{tr}B = \text{tr}B^{-1},$$

$$\text{(Normal)} \quad \text{tr}B\text{tr}C = \text{tr}BC + \text{tr}BC^{-1}.$$

Solution. Let $B = (b_{ij})$ and $C = (c_{ij})$, and then

$$B^{-1} = \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix}, C^{-1} = \begin{bmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{bmatrix}$$

since $\det B = \det C = 1$, thus $\text{tr}B = b_{11} + b_{22} = \text{tr}B^{-1}$ and $\text{tr}BC + \text{tr}BC^{-1} = \text{tr}B(C + C^{-1}) = \text{tr}B\text{tr}C$. □

Exercise 4.2. Let $M \in \text{SL}(2, \mathbb{R})$. We would like to check if M and M^{-1} are similar to each other in a geometric way.

a) (Easy) Show that for any geodesic, there is a Möbius transformation exchanging its two end points.

b) (Normal) Use a) to show that any matrix M associated to a hyperbolic Möbius transformation is similar to its inverse M^{-1}.

c) (Hard) The orientation on H induces an orientation on each cycle and each horocycle (described by giving a positive rotation direction). Moreover this orientation on a cycle or a horocycle is preserved by orientation preserving isometries. Use this fact to show that

i. If $M = N_t$, then M is similar to M^{-1}, if and only if $t = 0$.
ii. If $M = K_\theta$, then M is similar to M^{-1}, if and only if $\theta \in \{0, \pi\}$.

Solution.
a) If the end points are $x_1, x_2 \in \mathbb{R}$, then

$$z \mapsto \frac{\frac{x_1+x_2}{2} z - \frac{x_1^2+x_2^2}{2}}{z - \frac{x_1+x_2}{2}}$$

is a Möbius transformation required. If the end points are $x \in \mathbb{R}$ and ∞, then

$$z \mapsto -\frac{1}{z-x} + x$$

is a Möbius transformation required.

b) Let the axis of the hyperbolic Möbius transformation f be γ and g be the transformation we found in a). Then for any $z \in \gamma$, $gfg^{-1}(z) \in \gamma$, $d_{\mathbb{H}}(gfg^{-1}(z), z) = d_{\mathbb{H}}(f^{-1}(z), z)$ and gfg^{-1} has the same translation direction with f^{-1}, thus $gfg^{-1} = f^{-1}$. This implies that if M induces a hyperbolic Möbius transformation, M is similar to its inverse M^{-1} or $-M^{-1}$. However, $\mathrm{tr}M^{-1} = \mathrm{tr}M > 2$, which means M cannot be similar to $-M^{-1}$, thus M is similar to M^{-1}.

c) i. When $t = 0$, $N_t = N_{-t}$. If N_t is similar to its inverse N_{-t} when $t \neq 0$, there exists a Möbius transformation such that $PT_t = T_{-t}P$. Consider any horocycle H_y, $P(H_y)$ is also a horizontal line and for any $z \in H_y$, $P(z + t) = P(z) - t$, which means P reserves the orientation of H_y, contradiction.

ii. $K_0 = K_{-0}$, $K_\pi = K_{-\pi}$. When $\theta \notin \{0, \pi/2, \pi, 3\pi/2\}$, we can consider a cycle C with hyperbolic center i, then K_θ induces the rotation on C with angle 2θ. If K_θ is similar to its inverse $K_{-\theta}$, there exists a Möbius transformation such that $P\rho_\theta = \rho_{-\theta}P$ and $P(C)$ is also a cycle. However, for an arbitrary $z \in C$, $P(z)$, $P(\rho_\theta(z)) = \rho_{-\theta}(P(z))$, $P(\rho_\theta^2(z)) = \rho_{-\theta}^2(P(z))$, thus z, $\rho_\theta(z)$ and $\rho_\theta^2(z)$ will have different orientation with $P(z)$, $P(\rho_\theta(z))$ and $P(\rho_\theta^2(z))$ (one is clockwise and the other is counterclockwise) which means that P reserves the orientation of C, contradiction. When $\theta = \pi/2$ or $\theta = 3\pi/2$, that means there exists a matrix

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

such that

$$P \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

However, that implies $a^2 + b^2 = -1$, also a contradiction. □

Exercise 4.3. We consider the matrix

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let f be the Möbius transformation associated to M. We would like to use hyperbolic geometry to find $J(M)$ the Jordan normal form of M.

a) (Easy) Use the trace to show that M is hyperbolic.
b) (Normal) Compute its eigenvalues μ and μ^{-1} with $\mu > 1$.
c) (Normal) Find the fixed points x_1 and x_2 of f with $x_1 < x_2$.
d) (Easy) Find the parabolic isometry T_t, such that $T_t(x_1) = -T_t(x_2)$, and write down the matrix corresponding N_t.
e) (Hard) Find an elliptic isometry sending $T_t(x_1)$ to 0 and $T_t(x_2)$ to ∞, and write down its matrix B.
f) (Easy) Compute $P = BN_t$ and verify that P satisfies:

$$PMP^{-1} = J(M)$$

.

g) (Easy) Compare $J(M)$ with A_μ.
h) (Easy) Let $P_\lambda = A_\lambda P$. Show that for any $\lambda > 0$, we have

$$P_\lambda M P_\lambda^{-1} = J(M)$$

.

Solution.

a) tr$M = 2 + 1 = 3 > 2$, hence M is hyperbolic.

b) $\det(xI_2 - M) = x^2 - 3x + 1 \Longrightarrow \mu = \dfrac{3+\sqrt{5}}{2}, \mu^{-1} = \dfrac{3-\sqrt{5}}{2}$.

c) $z = f(z) = \dfrac{2z+1}{z+1} \Longrightarrow x_1 = \dfrac{1-\sqrt{5}}{2}, x_2 = \dfrac{1+\sqrt{5}}{2}$.

d) $T_t(x_1) = -T_t(x_2) \Longrightarrow x_1 + t = -(x_2 + t) \Longrightarrow t = -\dfrac{x_1+x_2}{2} = -\dfrac{1}{2}$.

$$N_t = \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix}.$$

e) It's obvious that the rotation around $\dfrac{\sqrt{5}}{2}i$ with angle $\dfrac{\pi}{2}$ is the isometry required. For instance, it is

$$z \mapsto \frac{z + \dfrac{\sqrt{5}}{2}}{-\dfrac{2}{\sqrt{5}}z + 1}$$

and its corresponding matrix is

$$B = \begin{bmatrix} \sqrt{2}/2 & \sqrt{10}/4 \\ -\sqrt{10}/5 & \sqrt{2}/2 \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

f)

$$P = BN_t = \begin{bmatrix} \sqrt{2}/2 & (\sqrt{10} - \sqrt{2})/4 \\ -\sqrt{10}/5 & (\sqrt{10} + 5\sqrt{2})/10 \end{bmatrix}$$

and

$$PMP^{-1} = \begin{bmatrix} (3+\sqrt{5})/2 & 0 \\ 0 & (3-\sqrt{5})/2 \end{bmatrix} = J(M)$$

g) $J(M) = A_\mu$.

h) $P_\lambda M P_\lambda^{-1} = A_\lambda J(M) A_\lambda^{-1} = A_\lambda A_\mu A_\lambda^{-1} = A_\mu = J(M)$. □

Exercise 4.4. We consider the following 3 subgroups of $\mathrm{SL}(2, \mathbb{R})$:

$$K = \{K_\theta | \theta \in [0, 2\pi)\},$$
$$A = \{A_\lambda | \lambda > 0\},$$
$$N = \{N_t | t \in \mathbb{R}\}.$$

The KAN decomposition (also called Iwasawa decomposition) of $\mathrm{SL}(2, \mathbb{R})$ states that: every $M \in \mathrm{SL}(2, \mathbb{R})$ can be written as a product $K_\theta A_\lambda N_t$ in a unique way (i.e. θ, λ and t are unique).

We would like to show this in a geometric way.

a) (Normal) By considering the algorithm that used for determining an isometry, show that for any matrix $M \in \mathrm{SL}(2, \mathbb{R})$ with tr$M \geqslant 0$, we can find matrices K_θ, A_λ and N_t, such that $M = K_\theta A_\lambda N_t$, for some $\theta \in [0, \pi)$, $\lambda > 0$ and $t \in \mathbb{R}$.

b) (Easy) Show that $K_\pi = -I_2$. (Hence the associated Möbius transformation is the identity map.)

c) (Normal) Show that for any $\theta \in [0, 2\pi)$ and $t \in \mathbb{R}$. If $K_\theta N_t$ preserves the vertical geodesic V_0, then we have $\theta = 0, \pi/2, \pi$ or $3\pi/2$ and $t = 0$.

d) (Normal) Use c) to conclude that if $K_\theta A_\lambda N_t = A_\mu$, where $\theta \in [0, 2\pi)$, $\lambda > 0$, $\mu > 0$ and $t \in \mathbb{R}$, then we have $\theta = 0$, $\lambda = \mu$ and $t = 0$.

e) (Easy) Conclude that the KAN decomposition for any $M \in \mathrm{SL}(2, \mathbb{R})$ is unique.

Solution.

a) Let f be the isometry associated by M and $f^{-1}(0) = x$, $f^{-1}(\infty) = x'$, $f^{-1}(i) = w = u + iv$. Then $f = \rho_{\theta_0/2}\phi_{v^{-1}}T_{-u}$, where $\theta_0 = \arctan \dfrac{v}{\dfrac{x+x'}{2} - u} < 2\pi$. So $M = K_{\theta_0/2}A_{v^{-1/2}}N_{-u}$ or $M = -K_{\theta_0/2}A_{v^{-1/2}}N_{-u}$, but tr$(K_{\theta_0/2}A_{v^{-1/2}}N_{-u}) = \cos\dfrac{\theta_0}{2}(v^{1/2} + v^{-1/2}) \geqslant 0$, hence $M = K_{\theta_0/2}A_{v^{-1/2}}N_{-u}$.

b) Trivial.

c) $K_\theta N_t$ sends ∞ to $-\cot\theta$ and 0 to $\dfrac{t\cos\theta+\sin\theta}{-t\sin\theta+\cos\theta}$. To keep the end points of V_0, (θ,t) must be $(0,0)$, $(\pi/2,0)$, $(\pi,0)$, or $(3\pi/2,0)$.

d) $A_\mu = K_\theta A_\lambda N_t = K_\theta N_{\lambda^2 t} A_\lambda$, so $A_{\mu\lambda^{-1}} = K_\theta N_{\lambda^2 t}$. Because $A_{\mu\lambda-1}$ keeps V_0 and sends 0 to 0, ∞ to ∞, we can get $(\theta,\lambda^2 t)=(0,0)$ or $(\pi,0)$ by c). Hence $t=0$ due to $\lambda\neq 0$. When $\theta=\pi$, $A_\mu=-A_\lambda$ and it is a contradiction. When $\theta=0$, $A_\mu=A_\lambda$, and it implies that $\lambda=\mu$.

e) If there exists (θ_1,λ_1,t_1), (θ_2,λ_2,t_2) such that $K_{\theta_1}A_{\lambda_1}N_{t_1}=K_{\theta_2}A_{\lambda_2}N_{t_2}$, then

$$K_{\theta_1-\theta_2}A_{\lambda_1}N_{t_1-t_2}=A_{\lambda_2}.$$

By d) we can get $(\theta_1,\lambda_1,t_1)=(\theta_2,\lambda_2,t_2)$. Hence the KAN decomposition is unique. □

5. Exercises V

Exercise 5.1. We consider the map $f_{\mathbb{D}}(z)=(z-i)/(z+i)$ from $\mathbb{H}$ to $\mathbb{D}$, and the matrix

$$A_{\mathbb{D}}=\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}$$

a) (Easy) Show that f can be extended to $\hat{\mathbb{R}}$, and sends $\hat{\mathbb{R}}$ to the unit circle.

b) (Easy) Show that $A_{\mathbb{D}}\mathrm{SL}(2,\mathbb{R})A_{\mathbb{D}}^{-1}$ i.e.

i. For any matrix $A\in\mathrm{SL}(2,\mathbb{R})$, we have $A_{\mathbb{D}}AA_{\mathbb{D}}^{-1}\in\mathrm{U}(1,1)$;

ii. For any matrix $B\in\mathrm{U}(1,1)$, there is a matrix $A\in\mathrm{SL}(2,\mathbb{R})$ such that $A_{\mathbb{D}}AA_{\mathbb{D}}^{-1}=B$.

Solution.

a) $f_{\mathbb{D}}(z)=\dfrac{1-i/z}{1+i/z}$, so for an arbitrary sequence $\{z_n\}\subset\mathbb{H}$ where $|z_n|\to+\infty$, $f_{\mathbb{D}}(z_n)\to 1$. Thus we can take $f(\infty)=1$. And for any $a\in\mathbb{R}$, $f_{\mathbb{D}}(a)=\dfrac{a^2-1+2ai}{a^2+1}$, so $f_{\mathbb{D}}$ sends $\hat{\mathbb{R}}$ onto the unit circle.

b) i. For any matrix

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in\mathrm{SL}(2,\mathbb{R}),$$

$$A_{\mathbb{D}}AA_{\mathbb{D}}^{-1}=\begin{bmatrix}\dfrac{(a+d)+(b-c)i}{2} & \dfrac{(a-d)-(b+c)i}{2}\\ \dfrac{(a-d)+(b+c)i}{2} & \dfrac{(a+d)-(b-c)i}{2}\end{bmatrix}\in\mathrm{U}(1,1).$$

ii. For any matrix

$$B=\begin{bmatrix}z & w\\ \bar{w} & \bar{z}\end{bmatrix}\in\mathrm{U}(1,1),$$

from i we can know the matrix

$$A=\begin{bmatrix}\mathrm{Re}(z+w) & \mathrm{Im}(z-w)\\ \mathrm{Im}(z+w) & \mathrm{Re}(z-w)\end{bmatrix}\in\mathrm{SL}(2,\mathbb{R})$$

satisfies $A_{\mathbb{D}}AA_{\mathbb{D}}^{-1}=B$. □

Exercise 5.2. Let γ and η be a pair of disjoint geodesics.

a) (Normal) Using the extreme value theorem and the convexity of the distance function, show that for any $z\in\gamma$ the distance $d_{\mathbb{H}}(z,\eta)$ can be realized by a unique point $w_z\in\eta$.

b) (Normal) Using the extreme value theorem and the convexity of the distance function, show that $\inf\{d_{\mathbb{H}}(z,\eta)|z\in\gamma\}$ can be realized by a unique point $z_0\in\gamma$.

c) (Easy) Conclude that the distance $d_{\mathbb{H}}(\gamma,\eta)$ are realized by a unique pair of points $(z_0,w_{z_0})\in\gamma\times\eta$.

Solution.

a) Let s be the arc length parameter of η and set $f(s)=d_{\mathbb{H}}(z,\eta(s))$. By the convexity of the distance function we know that f is a convex function. For any M large enough, we can get an interval $[a,b]\subset\mathbb{R}$ such that for any $x\in\mathbb{R}\setminus[a,b]$, $f(x)>M$. Then by the extreme value theorem, f must have a minimum on $[a,b]$, namely $f(s_0)$. By the convexity of f, $f(s_0)$ must be the global minimum and s_0 must be the unique point realize this minimum. Hence $\eta(s_0)$ is the required point.

b) Let t be the arc length parameter of γ and set $g(t)=d_{\mathbb{H}}(\gamma(t),\eta)$. Then by the extreme value theorem and the similar argument above, we can get there must be a t_0 realize the minimum of g. If there exist a $t_1\neq t_0$ also realize the minimum, then by the convexity of distance function, we have $g(\lambda t_0+(1-\lambda)t_1)<g(t_0)$ for any $\lambda\in(0,1)$, a contradiction.

c) For any $(z, w) \in \gamma \times \eta$, $d_{\mathbb{H}}(z, w) \geqslant d_{\mathbb{H}}(z, w_z) \geqslant d_{\mathbb{H}}(z_0, w_{z_0})$. And by the uniqueness claimed in a) and b), the '=' holds if and only if $(z, w) = (z_0, w_{z_0})$. □

Exercise 5.3. (Hard) Let $z_1, \cdots, z_n$ be n distinct points in $\mathbb{H}$ with $n > 2$. We define a function d on $\mathbb{H}$ as follows:

$$d(z) = \sum_{j=1}^{n} d_{\mathbb{H}}(z, z_j).$$

Using the extreme value theorem and the convexity of distance function, show that the infimum of $d(\mathbb{H})$ can be realized by a unique point in $\mathbb{H}$.

Solution. Let $B(z, r)$ denote the hyperbolic disc with center z and radius r. We can choose $R \in (d(z_1), +\infty)$ large enough such that

$$U := \bigcup_{j=1}^{n} B(z_j, R)$$

is connected and simply connected. Then for any point $z \in \mathbb{H} \setminus \bar{U}$, $d(z) > d_{\mathbb{H}}(z, z_1) > R > d(z_1)$, so if the minimum exists, it must be realized in $\bar{U}$. Use the extreme value theorem with d on $\bar{U}$, we can get the minimum can be indeed realized. If there exists two different points $z \neq w$ realize the minimum, then by the convexity of distance function, any point z' lying on the geodesic segment connecting z and w, there exists a $\lambda \in (0, 1)$ such that

$$d(z') = \sum_{j=1}^{n} d_{\mathbb{H}}(z', z_j) < \sum_{j=1}^{n} (\lambda d_{\mathbb{H}}(z, z_j) + (1-\lambda) d_{\mathbb{H}}(w, z_j)) = \lambda d(z) + (1-\lambda) d(w).$$

This makes a contradiction. □

Exercise 5.4. We would like to compute some trigonometry formulas:

a) (Easy) Let $\alpha \in (0, \pi)$. Consider the triangle with vertices $z_1 = \infty$, $z_2 = i$ and $z_3 = e^{i\alpha}$. Let l denote the length of the side I_1. Use the distance formula to show:

$$\cosh l \sin \alpha = 1.$$

b) (Normal) Let $\alpha, \beta \in (0, \pi)$. Consider the triangle with vertices $z_1 = \infty$, $z_2 = e^{i\alpha}$ and $z_3 = e^{i(\pi-\beta)}$ with $\alpha < \pi - \beta$. Let l denote the length of the side I_1. Use a) to show the following relations

$$\cosh l = \frac{1 + \cos\alpha \cos\beta}{\sin\alpha \sin\beta},$$

$$\sinh l = \frac{\cos\alpha + \cos\beta}{\sin\alpha \sin\beta},$$

Solution.

a) $l = d_{\mathbb{H}}(i, e^{i\alpha}) = \log \dfrac{\cos\alpha + 1}{\sin\alpha}$, so

$$\cosh l = \frac{\dfrac{\cos\alpha + 1}{\sin\alpha} + \dfrac{\sin\alpha}{\cos\alpha + 1}}{2} \frac{\dfrac{\cos\alpha + 1}{\sin\alpha} + \dfrac{1 - \cos\alpha}{\sin\alpha}}{2} = \frac{1}{\sin\alpha}.$$

b) Let $l_1 = d_{\mathbb{H}}(z_2, i)$ and $l_2 = d_{\mathbb{H}}(z_3, i)$. By a) we can get $\cosh l_1 = \csc\alpha$, so $\sinh l_1 = \cot\alpha$. And similarly, $\cosh l_2 = \csc\beta$ and $\sinh l_2 = \cot\beta$. Hence

$$\cosh l = \cosh(l_1 + l_2) = \frac{1 + \cos\alpha \cos\beta}{\sin\alpha \sin\beta}$$

and

$$\sinh l = \sinh(l_1 + l_2) = \frac{\cos\alpha + \cos\beta}{\sin\alpha \sin\beta}.$$

□

6. EXERCISES VI

Exercise 6.1. (Easy) Let $\theta = \alpha\pi$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Use definition to show that the group generated by

$$\rho_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

does not act properly discontinuously on $\mathbb{H}$.

Solution. By $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we get the group Γ generated by ρ_θ is an infinite group. Let K be an arbitrary hyperbolic circle with center i, then for any $g \in \Gamma$, $g(K) = K$, so

$$\#\{g \in \Gamma | g(K) \cap K \neq \emptyset\} = +\infty,$$

i.e. this action is not properly discontinuous. □

Exercise 6.2. Consider the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ given by

$$\begin{aligned}(m, n) : \mathbb{R}^2 &\to \mathbb{R}^2, \\ (x, y) &\mapsto (x + m, y + n).\end{aligned}$$

Let T be the flat torus $\mathbb{R}^2/\mathbb{Z}^2$. Let D be the unit square determined by $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. Let $l(x, y)$ be the Euclidean line in $\mathbb{R}^2$ passing $(0, 0)$ and (x, y).

a) (Normal) Using $\mathbb{Z}^2$ action to send all points in $l(1, 2)$ to D. Draw the image.
b) (Normal) Using $\mathbb{Z}^2$ action to send all points in $l(3, 2)$ to D. Draw the image.
c) (Hard) What is the algorithm to draw the image of $l(p, q)$ with $\gcd(p, q) = 1$ in D?
d) (Hard) What could we say about the image of the line $l(1, \sqrt{2})$.

Solution.

a) It's the union of the Euclidean segments connecting the following pairs of points.
 i. $(0, 0)$ and $(1/2, 1)$,
 ii. $(1/2, 0)$ and $(1, 1)$.
b) It's the union of the Euclidean segments connecting the following pairs of points.
 i. $(0, 0)$ and $(1, 2/3)$,
 ii. $(0, 2/3)$ and $(1/2, 1)$,
 iii. $(1/2, 0)$ and $(1, 1/3)$,
 iv. $(0, 1/3)$ and $(1, 1)$.
c) By the Bezout's Theorem in Elementary Number Theory, there exist $m, n \in \mathbb{Z}$ such that $mp + nq = 1$. Thus the image of $l(p, q)$ in D is the union of $p + q - 1$ Euclidean segments whose initial points are

$$(0, r/p), (s/q, 0),$$

where $r \in [0, p-1] \cap \mathbb{Z}$, $s \in [0, q-1] \cap \mathbb{Z}$, with slope q/p.

d) The image of $l(1, \sqrt{2})$ is dense in D because of a quick application of Dirichlet's Approximation Theorem, i.e. there exists integers m, n such that $|m\sqrt{2} - n| < \varepsilon$ for any $\varepsilon > 0$. □

Exercise 6.3. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Let $\Gamma = \langle A, B \rangle$.

a) (Normal) Show that Γ is a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, and conclude that it acts properly discontinuously on $\mathbb{H}$.
b) (Normal) Find a fundamental domain of for the Γ-action.
c) (Normal) Compute the area of the surface $S = \mathbb{H}/\Gamma$.

Solution.

a) Because A, B are both in $\mathrm{SL}(2, \mathbb{Z})$, Γ is a subgroup of $\mathrm{SL}(2, \mathbb{Z})$ and also discrete by the discreteness of $\mathbb{Z}$. As a corollary, it acts properly continuously on $\mathbb{H}$.

b) Consider the domain U which is bounded by four geodesics whose end points are the following four pairs.
 i. -1 and ∞,
 ii. ∞ and 1,
 iii. 1 and 0,
 iv. 0 and -1.

Or namely, $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Let U_1 be the domain which is bounded by γ_1 and γ_2, U_2 be the domain which is outside γ_3 and γ_4. Let f, g be the Möbius transformation induced by A, B respectively and $G_1 = \langle f \rangle$, $G_2 = \langle g \rangle$. For each orbit of Γ, choose z be the point which is closest to i. Then

$$d_{\mathbb{H}}(z,i) \leqslant d_{\mathbb{H}}(f^{-1}(z),i) = d_{\mathbb{H}}(z,i+2)$$

implies that $\mathrm{Re}z \leqslant 1$. We can deduce that $z \in \bar{U}$ by the similar argument with f, g, g^{-1}. On the other hand, consider any element $g_n \cdots g_1 \in \Gamma$ where $g_k \in G_{i_k}$, $g_k \neq \mathrm{id}_{\mathbb{H}}$ and $i_k \neq i_{k+1}$ for any k. So $g_1(U) \subset g_1(U_1) \subset \mathbb{H} \setminus U_{i_1}$, by induction and $U_1 \cup U_2 = \mathbb{H}$, we can get $g_n \cdots g_1(U) \subset \mathbb{H} \setminus U_{i_m}$, thus $g_n \cdots g_1(U) \cap U = \emptyset$. Therefore, U contains exact one elements in every orbit, i.e. it is a fundamental domain.

c) The area of S is equal to the area of the fundamental domain, which is equal to 2π. □

Exercise 6.4. Let $\gamma(x,x')$ be a complete geodesic in H with end point x and x' and oriented from x to x'. a) (Normal) Find a pair of matrices A and B in $\mathrm{SL}(2,\mathbb{R})$ such that A sends $\gamma(0,1)$ to $\gamma(\infty,2)$, and B sends $\gamma(0,\infty)$ to $\gamma(1,2)$.

b) (Normal) Describe all solutions of a) using parameter(s). How many parameters are needed?

c) (Easy) For any (A,B) a solution of a), let Γ be the subgroup of $\mathrm{SL}(2,\mathbb{R})$ generated by A and B. Let

$$S(A,B) = \mathbb{H}/\Gamma(A,B).$$

Compute the area of $S(A,B)$ for any solution (A,B) of a).

Solution.

a)

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

b) If A send 0 to ∞ and 1 to 2, then

$$A = \begin{bmatrix} \dfrac{s}{\sqrt{s-2}} & -\sqrt{s-2} \\ \dfrac{1}{\sqrt{s-2}} & 0 \end{bmatrix}.$$

If A send 0 to 2 and 1 to ∞, B send, then

$$A = \begin{bmatrix} \dfrac{s}{\sqrt{2-s}} & -\dfrac{2}{\sqrt{2-s}} \\ \dfrac{1}{\sqrt{2-s}} & -\dfrac{1}{\sqrt{2-s}} \end{bmatrix}.$$

If B send 0 to 1 and ∞ to 2, then

$$B = \begin{bmatrix} 2\sqrt{t} & \dfrac{1}{\sqrt{t}} \\ \sqrt{t} & \dfrac{1}{\sqrt{t}} \end{bmatrix}.$$

If B send 0 to 2 and ∞ to 1, then

$$B = \begin{bmatrix} -\sqrt{-t} & \dfrac{2}{\sqrt{-t}} \\ -\sqrt{-t} & \dfrac{1}{\sqrt{-t}} \end{bmatrix}.$$

Two parmeters are needed.

c) The area of $S(A,B)$ is the area of the domain bounded by $\gamma(0,1)$, $\gamma(\infty,2)$, $\gamma(0,\infty)$, $\gamma(1,2)$. It is equal to 2π. □

7. Exercises VII

For $n > 0$ integer, we call a polygon an n-gon, if it has n vertices. We denote it by P_n.

For any n and g non-negative integers, we denote by $S_{g,n}$ an oriented topological surface of genus g with n boundary components. If there is no boundary components, we will simply denote the surface by S_g.

We denote by Möb($\mathbb{H}$) the group of Möbius transformations on $\mathbb{H}$.

Exercise 7.1. For a surface S, we denote by $\chi(S)$ its Euler characteristic.

a) (Easy) Compute $\chi(P_n)$.

b) (Easy) Consider the construction of a flat torus S_1 by gluing opposite sides of P_4. Compute $\chi(S_1)$.

c) (Easy) Consider the construction of a genus g surface S_g using a $4g$-gon. Compute $\chi(S_g)$.

d) (Normal) Let P_{4g} be the polygon used to construct S_g. By cutting out a triangle in the interior, we create a surface which is topological $S_{0,2}$ a sphere with two holes. Using the same gluing pattern as in c), we get the surface S_g. Compute $\chi(S_{0,2})$ and $\chi(S_{g,1})$.

e) (Normal) By cutting out n disjoint triangles from P_{4g} and keep the same gluing pattern as in c), we construct surfaces $S_{0,n+1}$ and $S_{g,n}$ before and after the gluing. Compute $\chi(S_{0,n+1})$ and $\chi(S_{g,n})$.

f) (Hard) We consider gluing surfaces along boundary to get new surfaces.

i. Using e), for $n > 1$, show that $\chi(S_{g,n}) = \chi(S_{g+1,n-2})$.

ii. Using e), for $n_1 > 0$ and $n_2 > 0$, show that $\chi(S_{g_1,n_1}) + \chi(S_{g_2,n_2}) = \chi(S_{g_1+g_2,n_1+n_2-2})$.

iii. Check both equalities still hold when either g, g_1 or g_2 is 0.

g) (Easy) Compute the number of pair of pants in a pants decomposition of S_g.

h) (Easy) Compute the number of curves used in a pair of pants decomposition of S_g.

i) (Easy) Based on the answers of the question g) and h), guess the answers of the same questions for $S_{g,n}$ with $n > 0$. Check if it is correct.

Solution.

a) $\chi(P_n) = n - n + 1 = 1$.

b) $\chi(S_1) = 1 - 2 + 1 = 0$.

c) $\chi(S_g) = 1 - 2g + 1 = 2 - 2g$.

d) $\chi(S_{0,2}) = (4g + 3) - (4g + 3 + 1) + 1 = 0$. $\chi(S_{g,1}) = 4 - (2g + 4) + 1 = 1 - 2g$.

e) $\chi(S_{0,n+1}) = (4g+3n)-(4g+3n+n)+1 = 1-n$. $\chi(S_{g,n}) = (1+3n)-(2g+3n+n)+1 = 2 - 2g - n$.

f) i. $\chi(S_{g,n}) = 2 - 2g - n = 2 - 2(g + 1) - (n - 2) = \chi(S_{g+1,n-2})$.

ii. $\chi(S_{g_1,n_1}) + \chi(S_{g_2,n_2}) = 2 - 2(g_1 + g_2) - (n_1 + n_2 - 2) = \chi(S_{g_1+g_2,n_1+n_2-2})$.

iii. It's obvious that when $g = 0$, $n > 0$, $\chi(S_{g,n}) = 2 - 2g - n = 2 - n$ by b) and e).

g) The Euler characteristic of a pants $S_{0,3}$ is -1, and when we glue a pair of pants' boundaries, the number of vertices and edges will minus one together. So if the number of pants is m, then $m = -m\chi(S_{0,3}) = -\chi(S_g) = 2g - 2$.

h) Let the number of curves used in a pair of pants decomposition be k. Every curve is used two times and every pants use three curves, hence $2k = 3(2g - 2)$ and $k = 3g - 3$.

i) The argument in g) and h) can be used similarly in these questions. If the number of pants is m, then $m = -m\chi(S_{0,3}) = -\chi(S_{g,n}) = 2g - 2 + n$. Let the number of new curves used in a pair of pants decomposition be k. Then $2k + n = 3(2g - 2 + n)$ implies that $k = 3g - 3 + n$.

□

Exercise 7.2. a) (Easy) For any fours pairwise distinct points x_1, x_2, x_3 and x_4 in $\partial\mathbb{H}$, we define the cross ratio to be

$$\mathbb{B}(x_1, x_2; x_3, x_4) = \frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_3)(x_2 - x_4)}.$$

Show that $\mathbb{B}$ is invariant under Möbius transformations, i.e. for any $f \in$ Möb($\mathbb{H}$) we have:

$$\mathbb{B}(x_1, x_2; x_3, x_4) = \mathbb{B}(f(x_1), f(x_2); f(x_3), f(x_4)).$$

b) (Easy) Let η denote the geodesic ending at x_1 and x_2, and η' denote the geodesic ending at x_3 and x_4. Show that

i. η intersects η' if and only if $\mathbb{B}(x_1, x_2; x_3, x_4) < 0$;

ii. η and η' are disjoint if and only if $\mathbb{B}(x_1, x_2; x_3, x_4) > 0$.

c) (Hard) Let $f \in$ Möb($\mathbb{H}$) and γ be a geodesic ending at x and x'. Show that f is hyperbolic if $\mathbb{B}(x, f(x'); x', f(x)) < 0$.

Solution.

a) Let f be

$$z \mapsto \frac{az + b}{cz + d},$$

then

$$
\begin{aligned}
&\mathbb{B}(f(x_1), f(x_2); f(x_3), f(x_4)) \\
=&\frac{\left(\frac{ax_1+b}{cx_1+d}-\frac{ax_4+b}{cx_4+d}\right)\left(\frac{ax_2+b}{cx_2+d}-\frac{ax_3+b}{cx_3+d}\right)}{\left(\frac{ax_1+b}{cx_1+d}-\frac{ax_3+b}{cx_3+d}\right)\left(\frac{ax_2+b}{cx_2+d}-\frac{ax_4+b}{cx_4+d}\right)} \\
=&\frac{[(ax_1+b)(cx_4+d)-(ax_4+b)(cx_1+d)][(ax_2+b)(cx_3+d)-(ax_3+b)(cx_2+d)]}{[(ax_1+b)(cx_3+d)-(ax_3+b)(cx_1+d)][(ax_2+b)(cx_4+d)-(ax_4+b)(cx_2+d)]} \\
=&\frac{(ad-bc)^2(x_1-x_4)(x_2-x_3)}{(ad-bc)^2(x_1-x_3)(x_2-x_4)} \\
=&\mathbb{B}(x_1, x_2; x_3, x_4).
\end{aligned}
$$

b) By a), we know that cross ratio is invariant under Möbius transformation, so without loss of generality, we can suppose $x_3 = 0$, $x_4 = \infty$. Then $\mathbb{B}(x_1, x_2; x_3, x_4) = \dfrac{x_2}{x_1}$, hence

i. η intersects η' if and only if $x_1x_2 < 0$, which is equivalent to $\dfrac{x_2}{x_1} < 0$.

i. η and η' are disjoint if and only if $x_1x_2 > 0$, which is equivalent to $\dfrac{x_2}{x_1} > 0$.

c) Let f be

$$z \mapsto \frac{az+b}{cz+d},$$

then

$$
\begin{aligned}
0 <&\mathbb{B}(x, f(x'); x', f(x)) \\
=&\frac{(x-f(x))(f(x')-x')}{(x-x')(f(x')-f(x))} \\
=&\frac{[cx^2+(d-a)x-b][cx'^2+(d-a)x'-b]}{(ad-bc)(x-x')^2},
\end{aligned}
$$

which means $g(x) = cx^2 + (d-a)x - b$ has two real zeroes. Thus f has two real fixed points, i.e. f is hyperbolic. $\square$

Exercise 7.3. (Normal) Consider the flat torus $S_1 = \mathbb{R}^2/\mathbb{Z}^2$. Let $l(x, y)$ be the line passing $(0, 0)$ and (x, y). The projection of $l(x, y)$ to S_1 is a simple geodesic, denoted by $\gamma(x, y)$. Moreover, $\gamma(x, y)$ is closed if and only if $(x, y) \in \mathbb{Z}^2$. Let (p, q) and (r, s) be two distinct points in $\mathbb{Z}^2$. Compute the intersection number between $\gamma(p, q)$ and $\gamma(r, s)$.

Solution. Consider all the lines parallel to $l(p, q)$ and passes through a point in $\mathbb{Z}^2$, then every intersection of this pencil of lines with the segment connecting $(0, 0)$ and (r, s) associates one and only one intersection between $\gamma(p, q)$ and $\gamma(r, s)$. On the other hand, every intersection between $\gamma(p, q)$ and $\gamma(r, s)$ associate $\gcd(r, s)$ intersections between the pencil and the segment above. By Bezout's theorem in Elementary Number Theory, the above pencil of lines is evenly distributed at intervals of $\dfrac{\gcd(p, q)}{\sqrt{p^2+q^2}}$. The distance between (r, s) and $l(p, q)$ is equal to $\dfrac{|ps-qr|}{\sqrt{p^2+q^2}}$, hence the intersection number between $\gamma(p, q)$ and $\gamma(r, s)$ is equal to

$$\frac{\frac{|ps-qr|}{\sqrt{p^2+q^2}}}{\frac{\gcd(p, q)}{\sqrt{p^2+q^2}}\gcd(r, s)} = \frac{|ps-qr|}{\gcd(p, q)\gcd(r, s)}.$$

$\square$

8. Exercises VIII

Let $g \geqslant 0$ and $n \geqslant 0$ be integers such that $2 - 2g - n < 0$. We denote by S_g a closed hyperbolic surface of genus g, and by $S_{g,n}$ a hyperbolic surface of genus g with n cusps.

Exercise 8.1. We would like to study short geodesics on hyperbolic surfaces of genus g.

a) (Easy) Let $p \in S_g$. The injective radius R_p at p is the maximal positive real number such that the interior of a hyperbolic disk of radius R_p can be mapped isometrically to the R_p-neighborhood of p. Show that the exists a constant $c_1 > 0$, such that for any S_g,

$$\min\{R_p | p \in S_g\} < c_1.$$

b) (Easy) Use a) to show that there is a constant $c_2 > 0$, such that on any S_g, there is a simple closed geodesic shorter than c_2.

c) (Normal) Use Collar lemma and b), show that there is a constant $c_3 > 0$, such that on any S_g, there exists a simple closed geodesic γ which has a collar of area greater or equal to c_3.

d) (Normal) Use Collar lemma to show that on any S_g, any two distinct simple closed geodesics of length 1 must be disjoint.

Solution.

a) The hyperbolic area of S_g is equal to $(4g-4)\pi$. Thus the area of R_p- neighborhood of p, which is equal to $2\pi(\cosh R_p - 1)$, must smaller than it. Hence $\cosh R_p < 2g-1$ and there exists a constant $c_1 > 0$ such that for any S_g and p, $R_p < c_1$.

b) Let the length of the shortest simple closed geodesic γ on S_g be l, then $l \leqslant 2R_p$ for any $p \in \gamma$, since there is a simple closed geodesic whose length is $2R_p$ on S_g. So $l \leqslant 2R_p < 2c_1 =: c_2$.

c) Use the notation in Exercise 2.4.c). The collar bounded by γ_r, γ_R, V_0, and L_θ $(r < R)$ has width

$$w_\theta := \log\cot\left(\frac{\pi}{4} - \theta\right)$$

and area

$$A_\theta := \int_r^R \int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} \frac{\mathrm{d}r\mathrm{d}\theta}{r\sin^2\theta} = \log\frac{R}{r}\tan\theta.$$

Let the shortest simple closed geodesic be γ with length $l(\gamma)$. By Collar lemma and the discussion above, γ has a collar with area

$$2l(\gamma)\sinh w_\theta = \frac{2l(\gamma)}{\sinh\frac{l(\gamma)}{2}}.$$

Because of $\dfrac{4x}{\sinh x}$ is decreasing monotonically with x when $x > 0$, so the area of its collar is greater or equal to $c_3 := \dfrac{2c_2}{\sinh\frac{c_2}{2}}$.

d) Let one of the geodesic whose length is 1 be γ and its collar be U. The only simple closed geodesic in U is γ. If a geodesic enters and leaves U is the same boundary, then it does not intersect with γ. If a geodesic enters and leaves U in the different boundary of U, then its length must longer than the distance of two boundaries of U, which is equal to 2arcsinh $\left(\dfrac{1}{\sinh 0.5}\right)$, which contradicts to that its length is 1. If a geodesic only enters U but not leaves, then it must tend to γ and has an infinite length, also a contradiction.

$\square$

Exercise 8.2. (Hard) Let p be a cusp on $S_{g,n}$. If a horocycle H centered at p is embedded in $S_{g,n}$, we call the part between H and its center p the cusp region, and denote it by $D_p(r)$ where r is the length of H. Use Collar Lemma for cusps to show that any complete geodesic on $S_{g,n}$ intersecting $D_p(1)$ has self-intersections.

Solution. By Collar Lemma for cusps, the cusp region $D_p(2)$ is isometry to the quotient of the region in $\mathbb{H}$ with imaginary part larger than 1 under the action of $z \mapsto z+2$. In this viewpoint, the boudary of $D_p(1)$, a horocycle whose length is 1, is the projection of $H_{\sqrt{2}}$. Without loss of generality, we consider a complete geodesic γ passing through $\sqrt{2}i$, its Euclidean equation is γ : $x^2+y^2-2x_0x = 2$. Under the translation $z \mapsto z+2$, it turns into γ' : $(x-2)^2-2x_0(x-2)+y^2 = 2$. γ' and γ intersects at the point $(x_0+1, \sqrt{x_0^2+1})$. Because $\sqrt{x_0^2+1} \geqslant 1$, the intersection lies in $D_p(2)$. Hence γ has a self-intersection. $\square$

9. Exercises IX

Let D denote the open set in $\mathbb{H}$ bounded by three geodesics $V_{1/2}$, $V_{-1/2}$ and $C(0,1)$:

$$D = \{z \in \mathbb{H} | \mathrm{Re} z \in \left(-\frac{1}{2}, \frac{1}{2}\right), |z| > 1\}.$$

We would like to show that D is a fundamental domain for $\mathrm{PSL}(2,\mathbb{Z})$ action on $\mathbb{H}$. For our convenience, we use $\mathrm{SL}(2,\mathbb{Z})$ during the proof.

Exercise 9.1. (Normal) Cosinder a matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
in $\mathrm{SL}(2,\mathrm{Z})$. Compute the imaginary part of $A(z)$.

Solution. Let $z = u + iv$, then
$$A(z) = \frac{a(u+iv)+b}{c(u+iv)+d} = \frac{(au+b)(cu+d)+acv^2+iv}{(cu+d)^2+(cv)^2}.$$
Hence
$$\mathrm{Im}A(z) = \frac{\mathrm{Im}z}{(c|z|)^2 + 2cd\mathrm{Re}z + d^2}.$$
□

Exercise 9.2. (Normal) Show that for $z \in D$, we have $|cz+d| \geqslant 1$.

Solution. For any $z \in D$, we have
$$|cz+d|^2 = (c|z|)^2 + 2cd\mathrm{Re}z + d^2 \geqslant c^2 + d^2 + 2cd\mathrm{Re}z.$$
If $cd = 0$, then
$$|cz+d|^2 \geqslant c^2 + d^2 \geqslant 1.$$
If $cd > 0$,
$$|cz+d|^2 > c^2 + d^2 - cd = (c-d)^2 + cd \geqslant cd \geqslant 1.$$
If $cd < 0$, then
$$|cz+d|^2 > c^2 + d^2 + cd = (c+d)^2 - cd > -cd \geqslant 1.$$
Hence $|cz+d| \geqslant 1$ and the '=' holds if and only if $c = 0$ and $|d| = 1$. □

Exercise 9.3. (Normal) Show that for any $A \in \mathrm{SL}(2,\mathbb{Z})$ and any $z \in D$, if $A(z) \in D$, then $A = \pm I_2$.

Solution. By using Exercise 9.1 and Exercise 9.2, we can get
$$\mathrm{Im}A(z) = \frac{\mathrm{Im}z}{|cz+d|^2} \leqslant \mathrm{Im}z.$$
However, in the other hand, $\mathrm{Im}z = \mathrm{Im}A^{-1}(A(z)) \leqslant \mathrm{Im}A(z)$ by the same argument since $A^{-1} \in \mathrm{SL}(2,\mathbb{Z})$ as well. Thus $\mathrm{Im}A(z) = \mathrm{Im}z$ and $|cz+d| = 1$. That means $c = 0$ and $|d| = 1$, then A acts as a translation $z \mapsto z + b/d$. But $|b/d| \geqslant 1$ means A will send z outside D, hence $b = 0$. Finally, $A = \pm I_2$. □

Exercise 9.4. Show that any point $z \in \mathbb{H}$ can always be sent to the region between $V_{1/2}$ and $V_{-1/2}$, $\mathrm{Re}z \in [-1/2, 1/2]$.

Solution. Let n be the integer which is closest to $\mathrm{Re}z$, then the transformation induced by
$$\begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$
is what we need. □

Exercise 9.5. (Easy) Use Exercise 9.4 and the matrix
$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
to show that any point $z \in \mathbb{H}$ with $\mathrm{Im}z \geqslant \sqrt{3}/2$ can always be sent into $\bar{D}$ by an element in $\mathrm{SL}(2,\mathbb{Z})$.

Solution. By Exercise 9.4, we can suppose $\mathrm{Re}z \in [-1/2, 1/2]$. If $z \in \bar{D}$ now, we have completed. If $z \notin \bar{D}$, then $B(z) = -1/z \in \bar{D}$. □

Exercise 9.6. (Normal) Show that for any point $z \in \mathbb{H}$ with $\mathrm{Re}z \in [-1/2, 1/2]$ and $\mathrm{Im}z < \sqrt{3}/2$, we have $\mathrm{Im}B(z) > \mathrm{Im}z$.

Solution. By the condition we know $|z| < 1$, hence $\mathrm{Im}B(z) = \dfrac{\mathrm{Im}z}{|z|^2} < \mathrm{Im}z$. □

Exercise 9.7. (Normal) Show that there exists a constant $\varepsilon > 0$, such that for any point z with

$$\mathrm{Re}z \in [-1/2, 1/2] \text{ and } \frac{\sqrt{3}}{2} - \varepsilon < \mathrm{Im}z < \frac{\sqrt{3}}{2},$$

we have

$$\mathrm{Im}B(z) > \frac{\sqrt{3}}{2}$$

Solution. Set $\varepsilon = \dfrac{\sqrt{3}-1}{2}$, i.e. $\dfrac{1}{2} < \mathrm{Im}z < \dfrac{\sqrt{3}}{2}$. Then

$$\mathrm{Im}B(z) = \frac{\mathrm{Im}z}{|z|^2} \geqslant \frac{\mathrm{Im}z}{\frac{1}{4} + (\mathrm{Im}z)^2} > \frac{\frac{\sqrt{3}}{2}}{\frac{1}{4} + \frac{3}{4}} = \frac{\sqrt{3}}{2}$$

since $\dfrac{x}{\frac{1}{4} + x^2}$ is decreasing monotonically with x when $x > \dfrac{1}{2}$. □

Exercise 9.8. (Hard) Let z be any point in $\mathbb{H}$. We construct a sequence of points in $\mathbb{H}$ in the following way.

a) Check if $z \in \bar{D}$, if yes, stop; otherwise, apply

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix},$$

for some $n \in \mathbb{Z}$ on z, so that $\mathrm{Re}z \in [-1/2, 1/2]$. We denote the new point by z_1. If z_1 is in $\bar{D}$, stop; otherwise go to step b).

b) Apply B and we get $z_2 = B(z_1)$. Check if z_2 is in $\bar{D}$. If yes, stop; otherwise back to a).

Show that this process will stop in finite time and we will get a point in $\bar{D}$.

Solution. If this process will not stop in finite time, then we can get an infinite sequence $\{z_n\} \subset \mathbb{H}$ such that $\mathrm{Im}z_n < \mathrm{Im}z_{n+1} < \dfrac{1}{2}$ and $\mathrm{Re}z_n \in [-1/2, 1/2]$. Let the supremum of $\{\mathrm{Im}z_n\}$ be y. Then there exists a z_n such that $\mathrm{Im}z_n > \dfrac{y}{2}$. Then

$$\mathrm{Im}z_{n+1} = \mathrm{Im}B(z_n) = \frac{\mathrm{Im}z_n}{|z_n|^2} \geqslant \frac{\mathrm{Im}z_n}{\frac{1}{4} + (\mathrm{Im}z_n)^2} > \frac{2y}{y^2+1} \geqslant \frac{8}{5}y > y$$

since $\dfrac{x}{\frac{1}{4} + x^2}$ is increasing monotonically with x when $0 < x < \dfrac{1}{2}$, contradiction. □

Exercise 9.9. (Easy) Conclude that D is a fundamental domain for PSL$(2, \mathbb{Z})$-action on $\mathbb{H}$.

Solution. By Exercise 9.8, every orbit contains a point in $\bar{D}$. By Exercise 9.3, every orbit contains at most one point in D. Hence D is a fundamental domain. □

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