Morse Lemma

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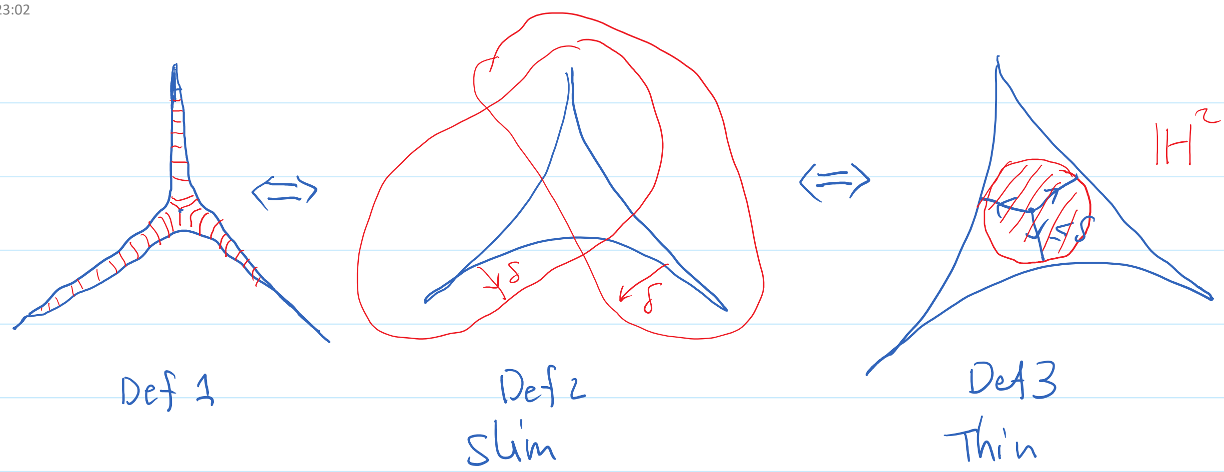


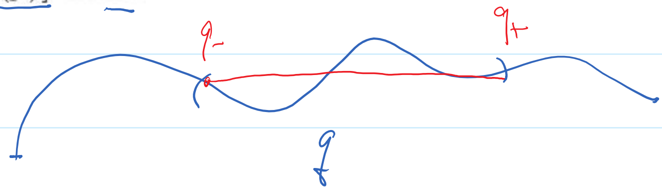
Definition 7.12. A path p in (X, d) is called a (λ, c)-*quasi-geodesic* for $\lambda \geq 1, c \geq 0$ if the following holds

$$d(q_-, q_+) \leq \mathrm{Len}(q) \leq \lambda d(q_-, q_+) + c$$

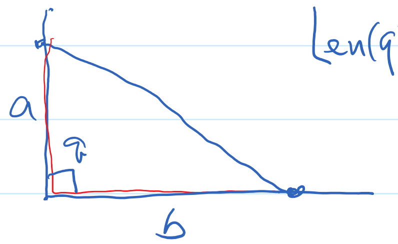
for any (connected) subpath q of p.

rectifiable

Remark. The length parametrization of p gives a (λ, c)-quasi-isometric embedding of the interval $[0, \mathrm{Len}(p)]$ in X.



Ex: 1) $\mathbb{E}^2$

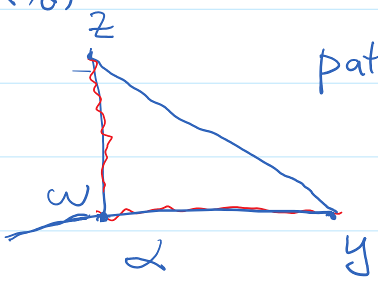


$\mathrm{Len}(q) = a + b \leq \sqrt{2}\sqrt{a^2+b^2}$

S $(\sqrt{2}, 0)$-quasi-geodesic

2) (X, d) $\quad d(z,w) \triangleq \inf\{d(z,w') : w' \in \alpha\}$

path $p \triangleq [z\ w][w\ y]$ is $(3,0)$-quasi-geodesic



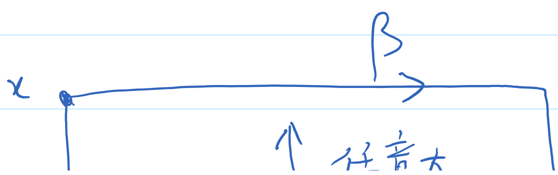
$\mathrm{Len}(p) \leq d(z,w) + d(w,y) \leq 3\, d(z,y) + 0$

$\leq d(z,y) + d(w,y)$

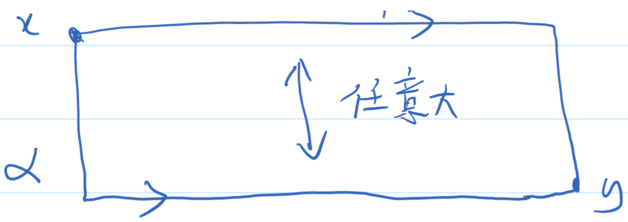
$\leq d(z,y) + d(z,w) + d(z,y)$

$\leq 2\, d(z,y) + d(z,y)$

3) $\mathbb{E}^2$



α, β $(\sqrt{2}, 0)$-q.geod



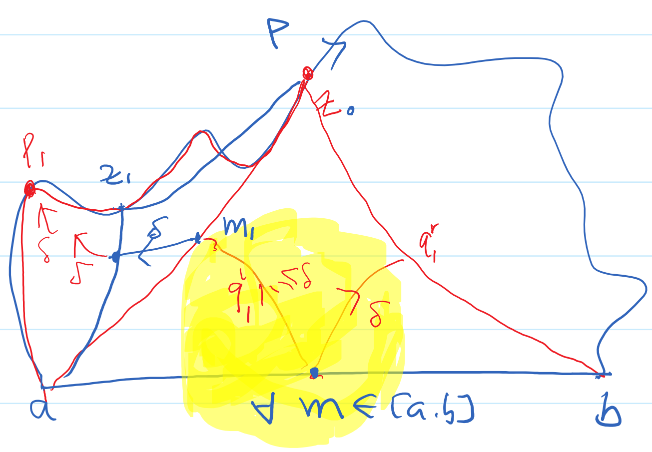
α, β $(\sqrt{2}, 0)$-q.geod

Lemma 7.10 (Exponential divergence). *Let p be a rectifiable path between a, b, and $[a, b]$ denote some geodesic between a, b. Then we have*

$$[a, b] \subset N_D(p)$$

where $D = 6\delta(\log_2 \boxed{Len(p)} + 1) + 2$.

Proof:



Let z_i be the middle pt of P_i

$\exists\, q_i^l$ s.t. $d(m, q_i^l) \le \delta$

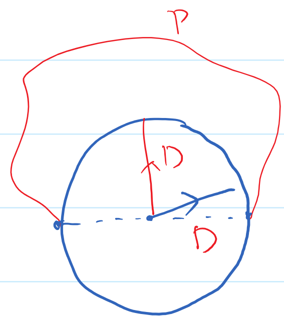
repeats $\log_2(len(p))$ times

Corollary: if $d(m, p) \le R$ for the middle pt $m \in [a, b]$

then $Len(p) \ge C_1 e^{C_2 R}$

for some universal constants. $C_1, C_2 > 0$.

Ex: $\mathbb{H}^2$



$len(p) \ge e^D$

Morse Lemma

Lemma 7.13 (Stability of quasi-geodesics). *For any $\lambda \ge 1, c \ge 0$, there exists $D = D(\delta, \lambda, c) > 0$ with the following property. Let p, q be two (λ, c)-quasi-geodesics in a δ-hyperbolic space (X, d). Then $p \subset N_D(q)$.*

Proof: Assume q to be geodesic.

Proof: Assume q to be geodesic.

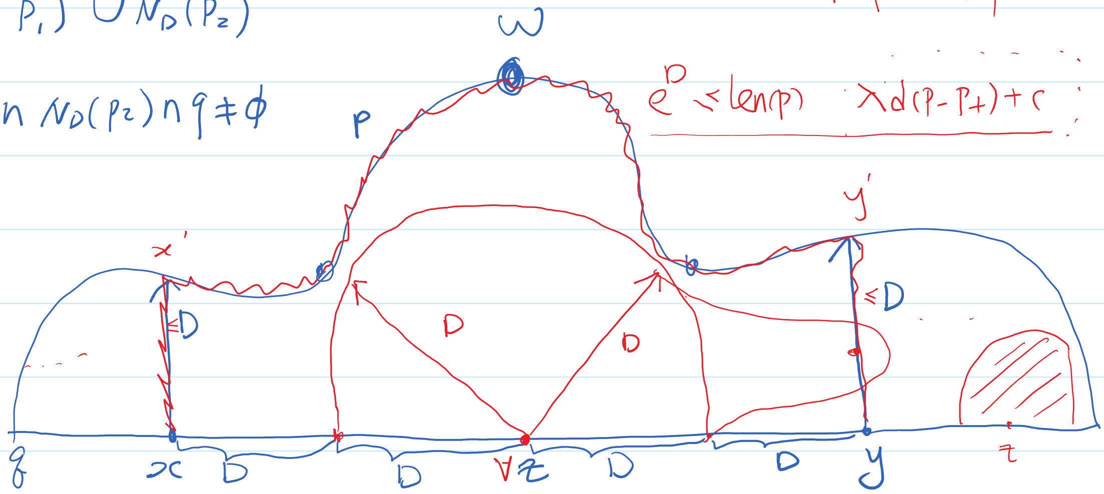
The goal is to bound:

$D \triangleq \max\{d(z,p) : z \in q\}$ $\Longrightarrow q \subseteq N_D(p)$

hope it depends on D

$q \subseteq N_D(p_1) \cup N_D(p_2)$

$\Rightarrow N_D(p_1) \cap N_D(p_2) \cap q \neq \phi$



$[xx'] \cap B(z,D) = \phi = [yy'] \cap B(z,D) \Rightarrow \alpha \cap B(z,D) = \phi$

Cut out the head and tail:

- $Len(\alpha) \leq 2D + [\lambda \cdot 6D + c]$

$$\leq 6\lambda D + c + 2D$$

- $Len(\alpha) \geq c_1 e^{c_2 D}$ [Exp. Divergence]

$\Rightarrow$ D is uniformly bounded by a constant depending only λ, c, δ.

双曲性的拟等距不变性

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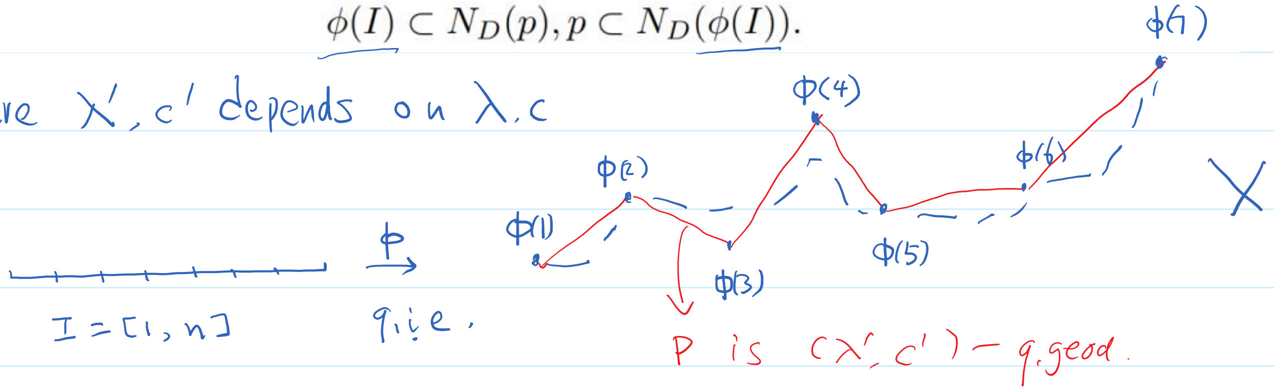
7.5. **Hyperbolicity is a quasi-isometric invariant.** We give another definition of a quasi-geodesic. A *parameterized quasi-geodesic* is a quasi-isometric embedding map of a finite or infinite interval of $\mathbb{R}$ in (X, d).

The following lemma implies that a parameterized quasi-geodesic can be converted to a continuous quasi-geodeisc without essensal lose.

Lemma 7.18. *Let $\phi : I \to X$ be a (λ, c)-quasi-isometric embedding map, where I is a finite or infinite interval in $\mathbb{R}$. Then there exist a (λ', c')-quasi-geodesic p and a constant $D > 0$ such that the following holds*

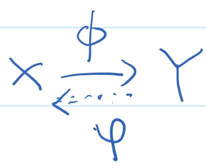
$$\phi(I) \subset N_D(p), p \subset N_D(\phi(I)).$$

where λ', c' depends on λ, c



Theorem 7.19. *Let X, Y be two proper length spaces. Assume that $\phi : X \to Y$ is a quasi-isometry. If X is hyperbolic, then Y is also hyperbolic.*

Proof: Any quasi-isometry has a quasi-inverse which is a Q.I. embedding:



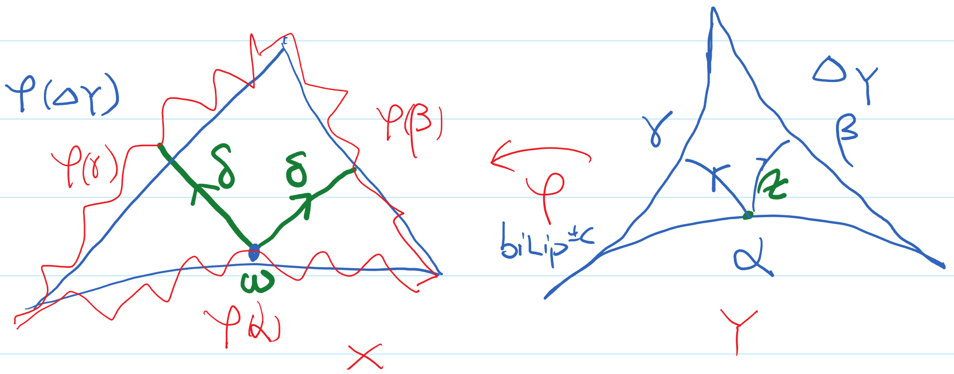
Let $\varphi : Y \to X$ be Q.I.E s.t.

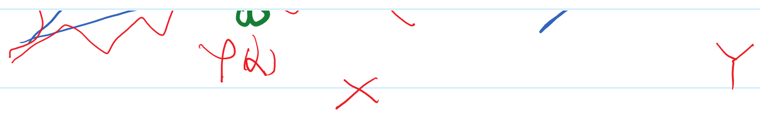
$d_Y(\phi \cdot \varphi(y), y) \leq R, \quad \forall y \in Y$

$d_X(\varphi \cdot \phi(x), x) \leq R, \quad \forall x \in X$

Let Δ_Y be geodesic triangle in Y

Then $\varphi(\Delta_Y)$ is quasi-geodesic triangle in X





By hyperbolicity of X and Morse Lemma,

$\varphi(\Delta_Y)$ has δ-center $w \in \varphi(\alpha)$

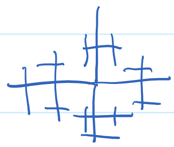
So Δ_Y has $(\lambda\delta+c)$-center $z \in \alpha$ $\lrcorner$

Definition 9.1. A finitely generated group G is called *hyperbolic* if there exists a finite generating set S such that the Cayley graph $\mathscr{G}(G,S)$ is δ-hyperbolic for some $\delta \geq 0$.

Remark. Since hyperbolicity is a quasi-isometric invariant (see Theorem 7.19), we see that the definition of a hyperbolic group does not depend on the choice of a generating set.

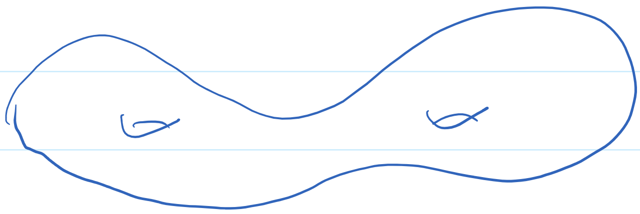
Example:

1) $\mathbb{F}_2$



2) π_1 (Compact Riem. mfld w/ neg. curvature).

π_1



is hyp. group

$\pi_1\Sigma \curvearrowright \mathbb{H}^2$ cocompact & proper

$\mathrm{Cay}(\pi_1\Sigma) \overset{Q.I.}{\sim} \mathbb{H}^2$ (Gromov-hyp)

$\Rightarrow$ $\mathrm{Cay}(\pi_1\Sigma)$ is δ-hyp.

Cor: If a group acts properly & cocompactly on proper hyperbolic length space then it is hyperbolic.

从局部到整体原则

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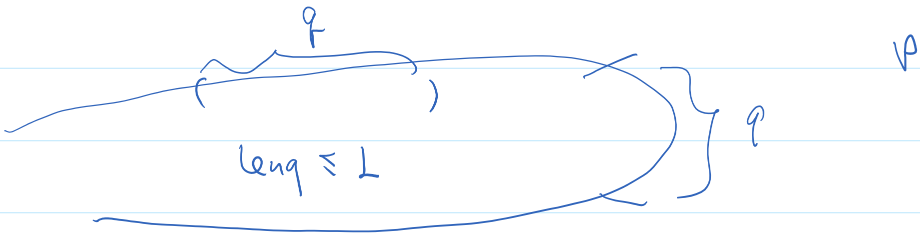
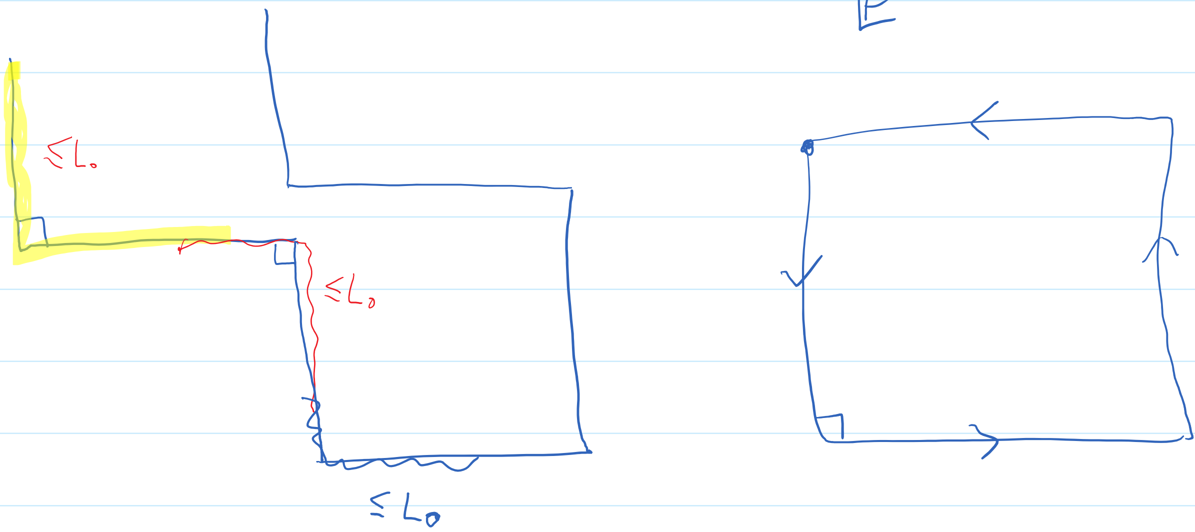
Definition 9.2. Let $L, \lambda, c > 0$. A path p in X is called a *L-local (λ, c)-quasi-geodesic* if every subpath of p with length L is a (λ, c)-quasi-geodesic.



Lemma 9.3 (Local=>Global)**.** *For any $\lambda, c > 0$, there exist $L_0 > 0$ and $\lambda', c' > 0$ with the following property.*

Fix any $L >$ $\boxed{L_0.}$ Let γ be a L-local (λ, c)-quasi-geodesic. Then γ is a (λ', c')-quasi-geodesic.

E^2



Rmk: This result is NOT true in E^2 !

Definition 3.1. Let G be a group, and $X \subset G$ be a subset. The *normal closure* of X, denoted by $\langle\langle X\rangle\rangle$, is the minimal normal subgroup containing X. Equivalently,

$$\begin{aligned}\langle\langle X\rangle\rangle &= \langle\{gxg^{-1} : x \in X, g \in G\}\rangle \\ &= \{(g_1 x_1^{\epsilon_1} g_1^{-1})\dots(g_n x_n^{\epsilon_n} g_n^{-1}) : n \in \mathbb{N}, x_i \in X, g_i \in G, \epsilon_i \in \{1,-1\}\}.\end{aligned}$$

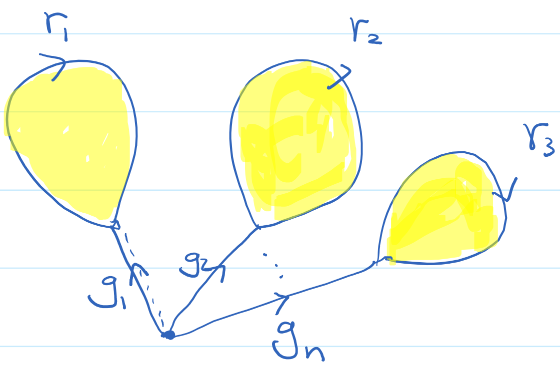
Group Presentation:

$$\langle X \mid R\rangle \qquad R \subseteq W(X)$$

This represents group $G \cong F(X)/\langle\langle R\rangle\rangle$.

Ex: $\langle a,b \mid ab=ba\rangle \cong \mathbb{Z}^2 = F_2/\langle\langle aba^{-1}b^{-1}\rangle\rangle$

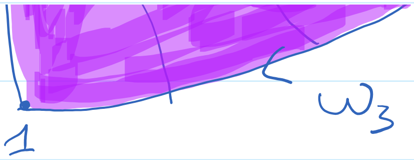
Rmk: Every word in $\langle\langle R\rangle\rangle$ traces a closed path in $Cay(G,X)$ whose label is the word.



10.1. **Hyperbolic groups are finitely presentable.** $N \to F(S) \twoheadrightarrow G$

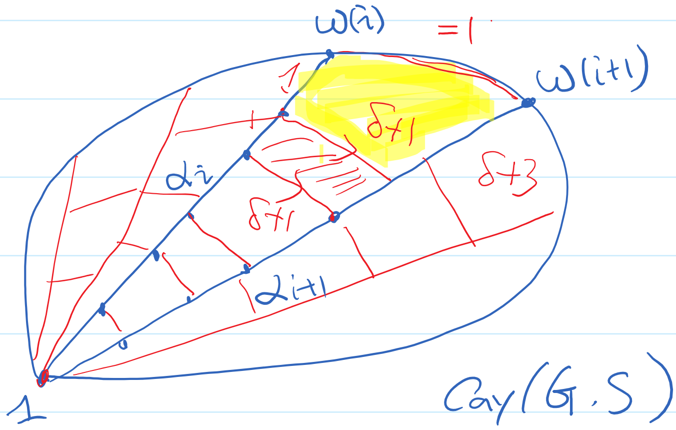
Theorem 10.1. *Let G be a hyperbolic group with a finite generating set S. Then there exists a finite set R of words in $\mathcal{W}(\tilde{S})$ such that $G \cong \langle S|R\rangle$.* $= F(S)/\langle\langle R\rangle\rangle$

$$\begin{aligned} w &= w_1 w_2 w_3 \\ &= w_1 (w_2 u) u^{-1} w_3 \\ &= \underbrace{w_1 (w_2 u) w_1^{-1}}_{\text{conjugacy}} \cdot w_1 u^{-1} w_3 \end{aligned}$$



conjugacy of $w_2 u$

δ-hyp



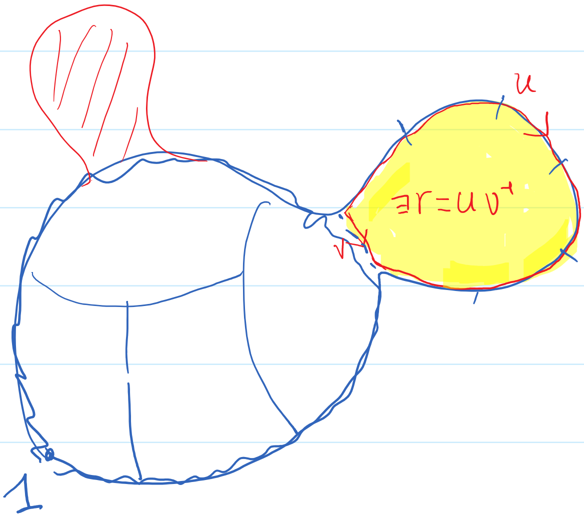
$\mathbb{R} = \{$ words over S of length $D\}$

$\| \boxed{2\delta+6}$

$\# \mathbb{R} < \infty$

Definition 10.2 (Dehn presentation). Let $G \cong \langle S|R\rangle$ be a group presentation. Assume that R contains all cyclic permutations of every $r \in R$.

The $\langle S|R\rangle$ is called a *Dehn presentation* if any $w \in \langle\langle R\rangle\rangle$, there exists a subword u of w and a relator $r \in R$ such that $r = uv^{-1}$ and $\mathrm{Len}(u) > \mathrm{Len}(v)$.



$\Rightarrow$ Area$(\omega) \leq |\omega|$

Dehn function

Thm: Linear isoperimetric inequality $\Rightarrow$ hyperbolicity.

Theorem 10.3 (Dehn presentation for hyperbolic groups). *A hyperbolic group has a Dehn presentation. Thus, word problem is solvable in hyperbolic groups.*

Proof: Let $\omega \equiv_G 1$ ($\in \langle\langle R\rangle\rangle$) be a word representing identity

Corollary 9.4. *There exists $L_0, \lambda, c > 0$ such that for any $L > L_0$, any L-local geodesic path is a (λ, c)-quasi-geodesic.*

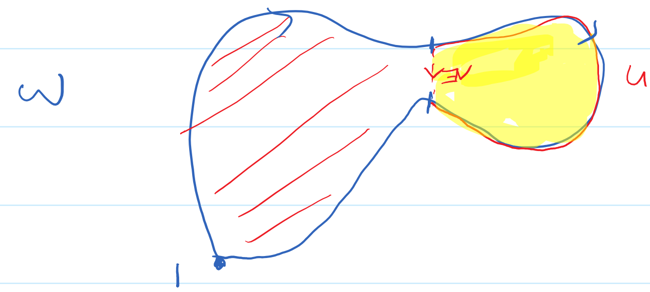
Set $\mathbb{R} \triangleq \{$ words over S of length $2L_0 + c\}$

Then $\langle S \mid R \rangle$ is Dehn presentation:

If w contains subword u of length L_0

s.t.

u is $\boxed{\text{NOT}}$ GEODESIC Word



$\text{Len } u = L_0$

$\Downarrow$

$\text{Len } v < L_0$

then we can shorten w by replacing u by a geodesic word r.

Otherwise:

$\boxed{\text{Any}}$ subword of W with L_0 is Geodesic

$\Downarrow$

$\Rightarrow$ w is L_0-local geodesic

By Cor. 9.4 we have

$$|w| \leq \lambda \cdot \boxed{0} + C \leq C.$$

$\Rightarrow$ $w \in R$.

共轭问题可解性

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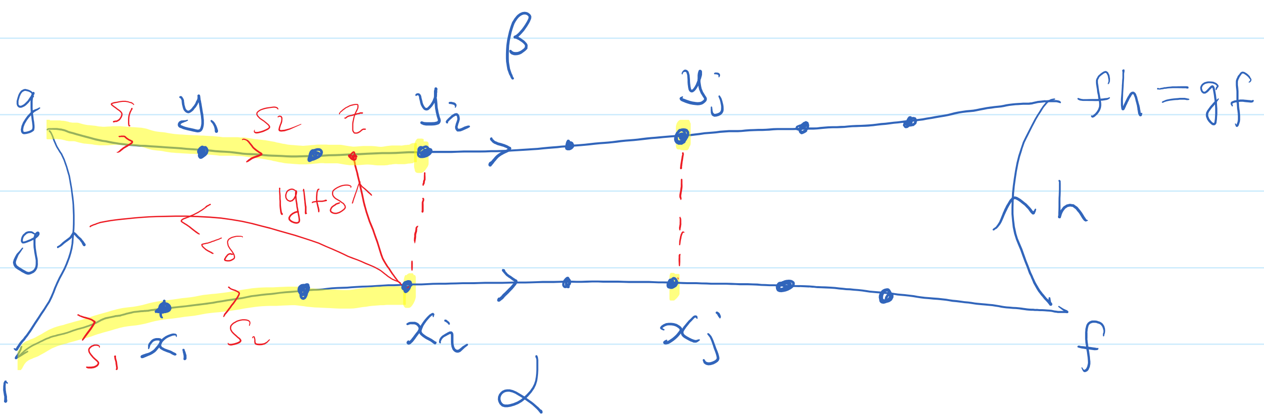
$W_1 \overset{conj}{=} W_2 \iff \exists\, u \in W(S)$, $u w_1 u^{-1} = w_2$

10.3. **Solving conjugacy problem.** Solving conjugacy problem depends on the following result.

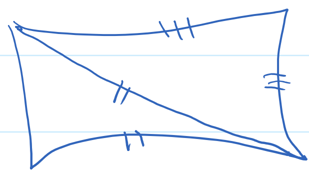
Lemma 10.4 (Bounding conjugators). *Assume that $g = fhf^{-1}$ for some $g, h, f \in G$. Then there exist a constant $D = D(|g|, |h|)$ and $f' \in G$ such that $d(1, f') \leq D$ and $g = f'hf'^{-1}$.*

Proof $\quad Lab(\alpha) = Lab(\beta) = f.$



Obs: $\exists D = D(|f|; |g|, \delta)$ s.t.

$$d(x_i, y_i) \leq D, \quad \forall i$$



$d(x_i, z \in \beta) \leq |g| + \delta$

$d(1, x_i) = d(g, y_i)$