Collar lemma: Let \( l(\gamma) \) be the length of a simple closed geodesic \( \gamma \) in \( S \). Then \( \exists \) a collar of \( \gamma \) with width \( \text{arcsinh} \left( \frac{1}{\sinh \frac{l(\gamma)}{2}} \right) \).

Proof:

Hence it is enough to consider only in a pair of pants.

Find a lower bound of \( \min(s, s', s'') \) for a fixed \( l(\gamma) \) and all possible \( a, b, c \in \mathbb{R}_+ \).

It is enough to study the right angled hexagon.

Lemma: The minimum of \( t \) is realized by \( P(0, 0, c) \).
\begin{proof}
\[
sinh \frac{a}{2} \sinh s = \cosh t
\]
(trigonometry formula for right angled pentagon)

\[t \text{ is strictly increasing as a function of } s.\]

\[
\sinh t_{\min} \sinh t_{\min} = 1
\]
(trigonometry formula for quadrilateral of angle \(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0\))

Moreover \(S = S' = \infty > t_{\min}\).
\end{proof}

Lemma the minimum of \(S\) and \(S'\) is realized by \(P(0, \infty, c)\)

\[
\min S' = \arcsinh \frac{1}{\sinh \frac{c}{2}}
\]

\begin{proof}
\[
\cosh S' = \frac{\cosh \left( \frac{a}{2} \cosh b + \cosh \frac{b}{2} \right)}{\sinh \frac{a}{2} \sinh b} \geq \frac{\cosh \frac{a}{2} \cosh \frac{b}{2}}{\sinh \frac{a}{2} \sinh \frac{b}{2}} \geq \frac{\cosh \frac{a}{2}}{\sinh \frac{a}{2}}
\]
(trigonometry formula for a right angled hexagon.
\end{proof}

\[
\text{Hence } \min \{S, S', t\} > \arcsinh \frac{1}{\sinh \frac{c}{2}} = \arcsinh \frac{1}{\sinh \frac{d}{2}}
\]

\[\Rightarrow \quad C_\sigma \left( \arcsinh \frac{1}{\sinh \frac{c}{2}} \right) \text{ is a collar of } \gamma.\]