Geodesic normal coordinates

Let $(\mathcal{U}, g)$ be a domain in $\mathbb{R}^n$ equipped with a metric g. Taking an orthormal basis $\{e_1, e_2, \ldots, e_n\} \subseteq T_p\mathcal{U}$ s.t. $\forall v \in T_p\mathcal{U}$,

$$v = \sum_{i=1}^{n} x_i e_i .$$

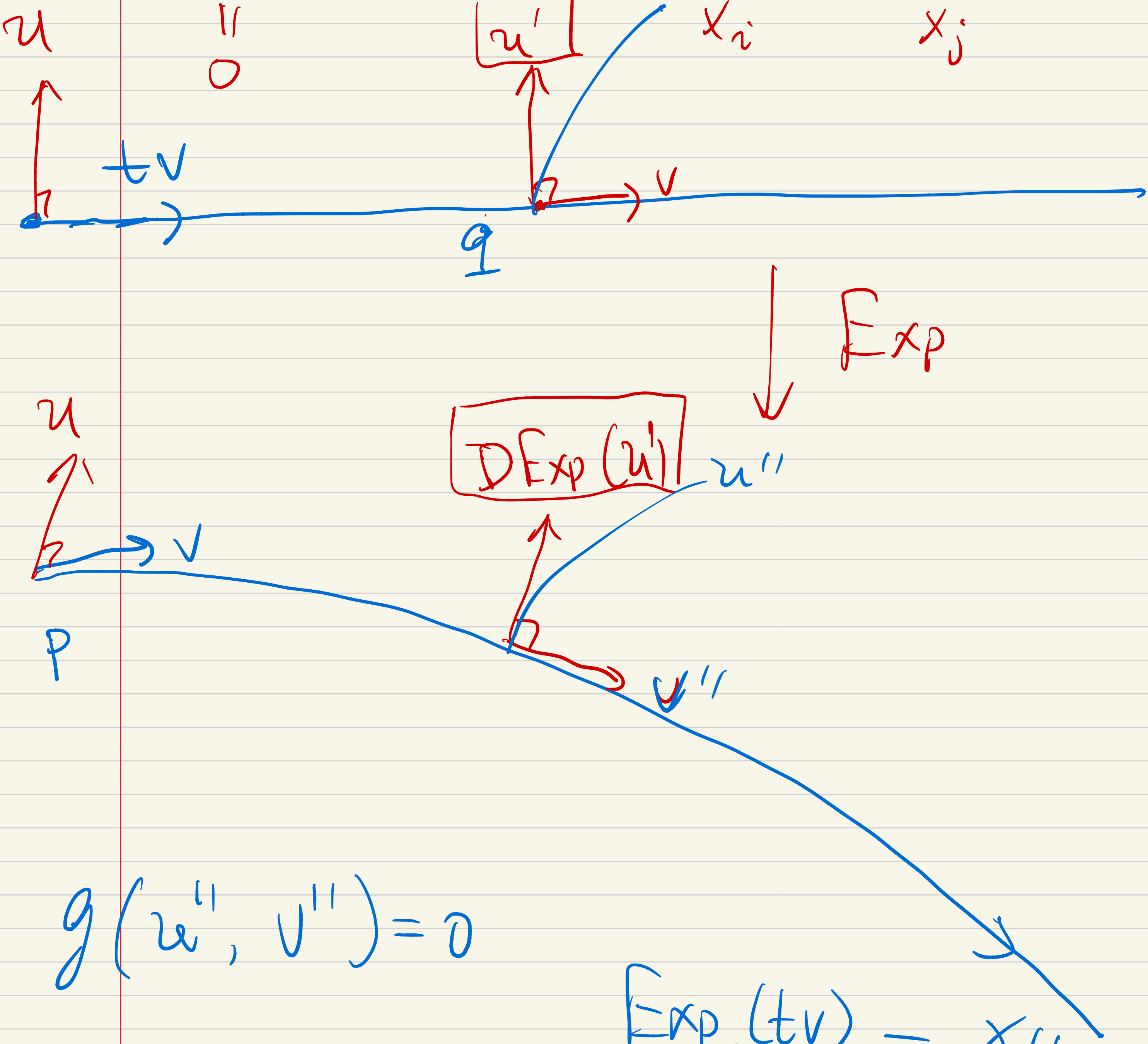
Since $\mathrm{Exp}_p : B_\delta(0^n) \to B_\delta(p)$ diffeomorphic when δ is sufficiently small, the aboves gives a system of coordinates, called geodesic normal coordinates.

We also denote: $\partial_i \equiv D\,\mathrm{Exp}_p(e_i)$

Check: In normal coordinates

- $\gamma(t) \equiv (t x_1, \ldots, t x_n)$ is a geodesic $\forall\ (x_1, \ldots, x_n) \in T_p\mathcal{U} \equiv \mathbb{R}^n$.
- $g_{ij}(p) = g_p(\partial_i, \partial_j) = \delta_{ij}$
- $\Gamma_{ij}^k(p) = 0$ ✓

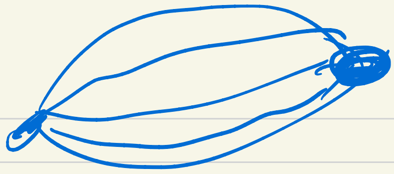
$$\underbrace{\cancel{\frac{d^2(tX_m)}{dt^2}}}_{=0} + \underbrace{\Gamma^m_{ij}}_{\boxed{u'}} \underbrace{\frac{d(tX_i)}{dt}}_{=X_i} \cdot \underbrace{\frac{d(tX_j)}{dt}}_{=X_j} = 0$$



$$g(u'', v'') = 0$$

$$\mathrm{Exp}_p(tv) = \gamma(t)$$

Jacobi field



In the above normal coordinates,

- $V_i(t,s) = (tx_1, \dots, t(x_i+s), \dots, tx_n)$
is a variation of $\gamma(t) = (tx_1, \dots, tx_n)$

Moreover, $\forall s$, $\gamma_s(t) = V(t,s)$ is a geodesic. So V is called a geodesic variation.

$J(t) \equiv \frac{\partial}{\partial s}\Big|_{s=0} V_i(t,s)$ called a Jacobi field. $\boxed{(0, \dots t, \dots 0)}$

In the above normal coordinates

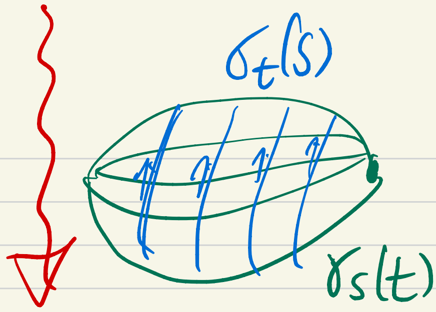
$$\boxed{J(t) = \frac{\partial}{\partial s}\Big|_{s=0} V_i(t,s) = t\,\partial_i}$$

One more example:

$V(t,s) = \mathrm{Exp}_p(t(u+sv))$,

$J(t) = \frac{\partial}{\partial s}\Big|_{s=0} V(t,s) = t\, D\mathrm{Exp}_{tu}(v)$.

Jacobi equation



Denote $\gamma_s(\cdot) \equiv V(\cdot, s)$, $\quad T_s \equiv \gamma_s'(\cdot)$

$\sigma_t(\cdot) \equiv V(t, \cdot)$, $\quad X_t \equiv \sigma_t'(\cdot)$

Notice that for any $s \in (-\epsilon, \epsilon)$,

$\underline{\nabla_{T_s} T_s \equiv 0}$. Then it follows that

$$0 \equiv \nabla_X \nabla_T T$$

$$= \nabla_X \nabla_T T - \nabla_T \nabla_X T + \nabla_T (\nabla_X T)$$

By the torsion-freeness of ∇,

$$\nabla_X T - \nabla_T X = [X, T]$$

$$= [DV(\partial_s), DV(\partial_t)]$$

$$\frac{\partial V}{\partial s} = DV\left(\frac{\partial}{\partial s}\right)$$

$$\frac{\partial V}{\partial t} = DV\left(\frac{\partial}{\partial t}\right)$$

$$= DV[\partial_s, \partial_t]$$

$$= 0.$$

Therefore,

$$\nabla_X \nabla_T T - \nabla_T \nabla_X T + \nabla_T \nabla_T X = 0.$$

By direct (but lengthy) computations, $\forall Z \in \mathfrak{X}(U)$,

$$\nabla_X \nabla_T Z - \nabla_T \nabla_X Z$$
$$= D\Gamma(X)(T, Z) - D\Gamma(T)(X, Z) + \Gamma(X, \Gamma(T, Z)) - \Gamma(T, \Gamma(X, Z)).$$

Evaluating the above equations at $S=0$,

$$\boxed{\nabla_{\gamma'} \nabla_{\gamma'} J + R(J, \gamma')\gamma' \equiv 0},$$

where

$$R_x(u, v)w \equiv D\Gamma_x(u)(v, w) - D\Gamma_x(v)(u, w)$$
$$+ \Gamma_x(u, \Gamma_x(v, w)) - \Gamma_x(v, \Gamma_x(u, w))$$

is called Riemann curvature tensor

Check: $\forall X, Y, Z \in \mathfrak{X}(U)$,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$= (\nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z) + (\nabla_Y \nabla_X Z - \nabla_{\nabla_Y X} Z)$$
$$\equiv \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z.$$

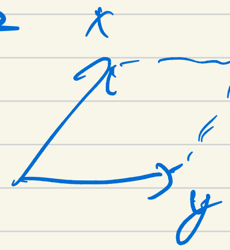
Definition of curvatures

- ## Riemann curvature tensor

$$R(X,Y,Z,W) = g(R(X,Y)Z, W)$$

- ## Sectional curvature

$$\sec(X,Y) = \frac{R(X,Y,Y,X)}{\|X\|^2\|Y\|^2 - g(X,Y)^2}$$



- ## Ricci tensor

$$\mathrm{Ric}(X,Y) = \sum_{j=1}^{n} R(E_j, X, Y, E_j)$$

$\{E_j\}_{j=1}^{n}$ Orthonormal Basis

- ## Scalar curvature

$$Sc = tr_g(\mathrm{Ric}) = \sum_{j=1}^{n} \mathrm{Ric}(E_j, E_j)$$