群在树上的作用

2021年7月28日 9:14

Groups acting Simplicial Trees without inversions	$\longleftrightarrow$	Groups obtained by taking free product almagamations, HNN's

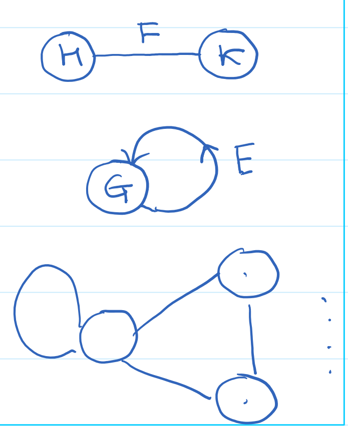
Bass-Serre

Bass-Serre tree $\longleftarrow$ **Graph of groups**

T ↻

$\Gamma = \underline{\pi_1}(\text{Graph of gps})$

$\xrightarrow{T/\Gamma}$



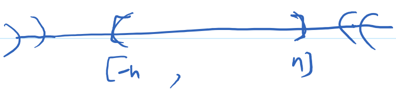
Groups acting on Trees with finite edge stabilizer	$\longleftrightarrow$	Groups splitting over finite groups.

Stalling's Group Ends Thm

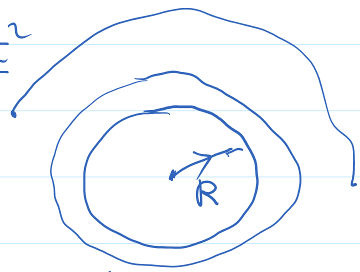
Assume Γ is NOT vir. $\mathbb{Z}$

Γ has ∞-many ends (of Cayley graph)

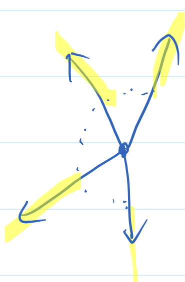
$\mathbb{R}^1$ # Ends = 2

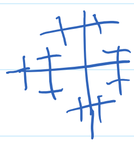


$\mathbb{E}^2$



#Ends = 1





#Ends = ∞

Ends compactification $\partial_E \Gamma$

$\updownarrow$

$\{$ one-ended infinite paths $\}/_{p \sim q}$

$p \sim q$ eventually belong to the same connected comp.

Rmk: • # Ends of group $\Gamma = \{0, 1, 2, +\infty\}$

• # Ends = 2 $\iff$ $\Gamma \cong$ vir. $\mathbb{Z}$.

- $\#\text{Ends} = \infty \iff \Gamma = H *_F K$ $\quad \#F < \infty$

  or $\Gamma = G *_{H \cong K}$ $\quad H \cong K$ is finite

- $\#\text{Ends}$ is Quasi-isometric invariant.

  So groups splitting over finite groups is Q.I. invariant

Thm: (Stallings) $\Uparrow$ graph of groups with finite vertex/edge subgroups

- (1) Group acting properly on a tree is virtually free.

  (virtually free group acts properly on a tree)

- (2) Group acting freely on a tree is a free group. (Easy)

Cor: $\#H < \infty$ $\#K < \infty$ $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ \implies H * K$ is virtually free

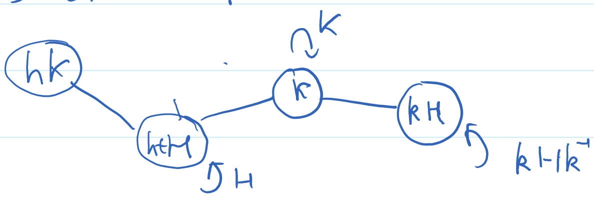
Le The kernel

$$N \to H * K \to H \times K$$

is a free group

proof: By Thm(2) it suffices to prove that $N \overset{\text{freely}}{\curvearrowright} T$

where T is B.S.-tree for $H * K$.



Suppose NOT: $\exists\, g \in N \setminus 1$ fixes a pt $x \in T$.

Then g is conjugated into H or K:

$$g \in cHc^{-1} \quad \text{or} \quad g \in cKc^{-1}, \quad c \in H * K$$

$c = h_1 k_1 h_2 k_2 \cdots$

However, the quotient map

$\pi|_H$ is injective. $\quad \pi : H * K \longrightarrow H \times K$

$\pi(g) = \pi(c) \cdot \pi(h) \cdot \pi(c^{-1}) \in H \times K$
$= \pi(h) = h$

sends every nontrivial element $\boxed{g = chc^{-1} \in cHc^{-1}}$

to non-trivial element $h \in H \setminus 1$!

This is the contradiction (Since $g \in N = \ker(\pi)$)

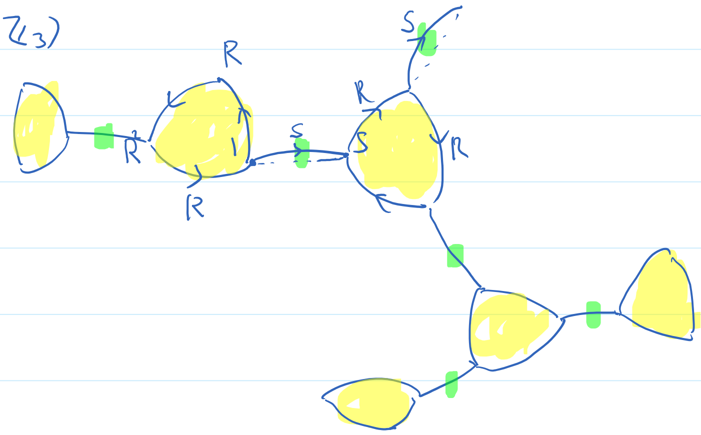
Ex: $\mathbb{Z}_2 * \mathbb{Z}_3$ is virtually free: $\exists\, F_2 < \mathbb{Z}_2 * \mathbb{Z}_3$ of finite index 6 |

$\langle S \rangle \quad \langle R \rangle$

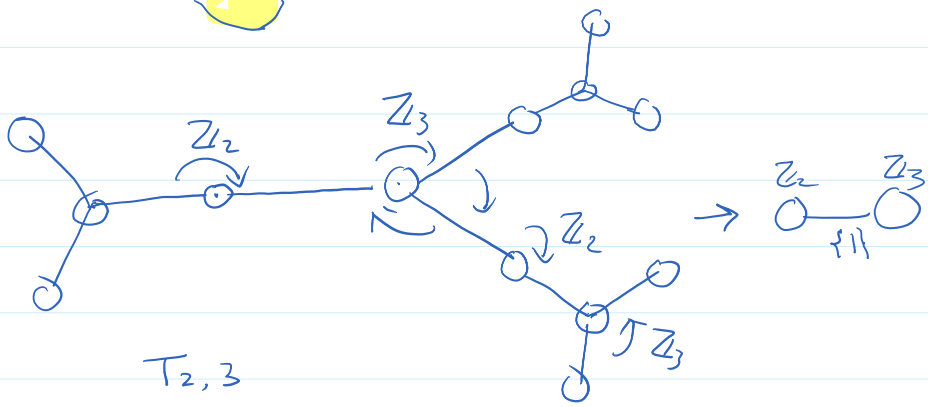
Ex: $\mathbb{Z}_2 * \mathbb{Z}_3$ is virtually tree: $\exists\, H_2 < \mathbb{Z}_2 * \mathbb{Z}_3$ of finite x 6 !

$\langle S \rangle$ $\langle R \rangle$

$\mathrm{Cay}(\mathbb{Z}_2 * \mathbb{Z}_3)$



BS Tree



$T_{2,3}$

模群

2021年7月12日 9:51

$PSL(2,\mathbb{Z}) \overset{\text{proper}}{\curvearrowright} \mathbb{H}^2$:

$= \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}$

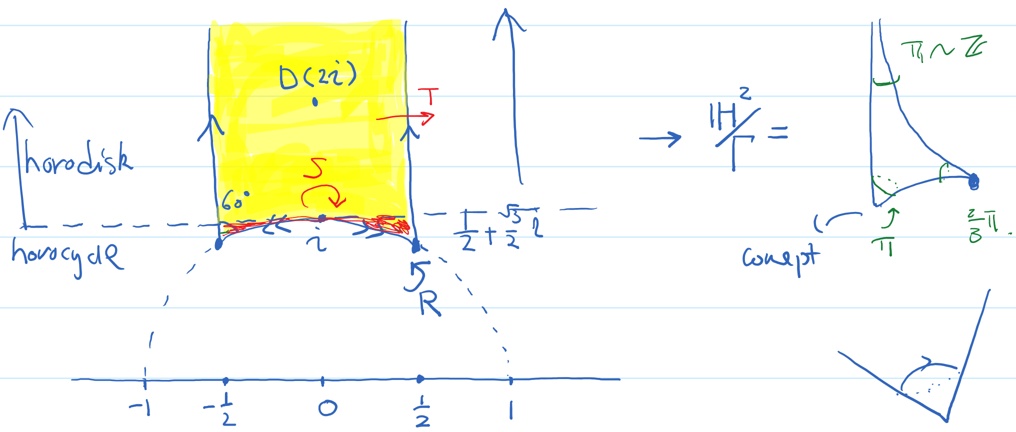
$= \langle T, S \mid S^2 = 1 \rangle$

$T = (z \mapsto z+1)$

$S = (z \mapsto \frac{-1}{z})$

· It is NOT cocompact (finite volume)

· Cusp



$\mathbb{F}_2 \overset{f.i.}{>} \mathbb{Z}_2 * \mathbb{Z}_3$

Question: Is $PSL(2,\mathbb{Z})$ hyperbolic group ?

Answer: Yes ! Indeed it is virtually $\mathbb{F}_2$!

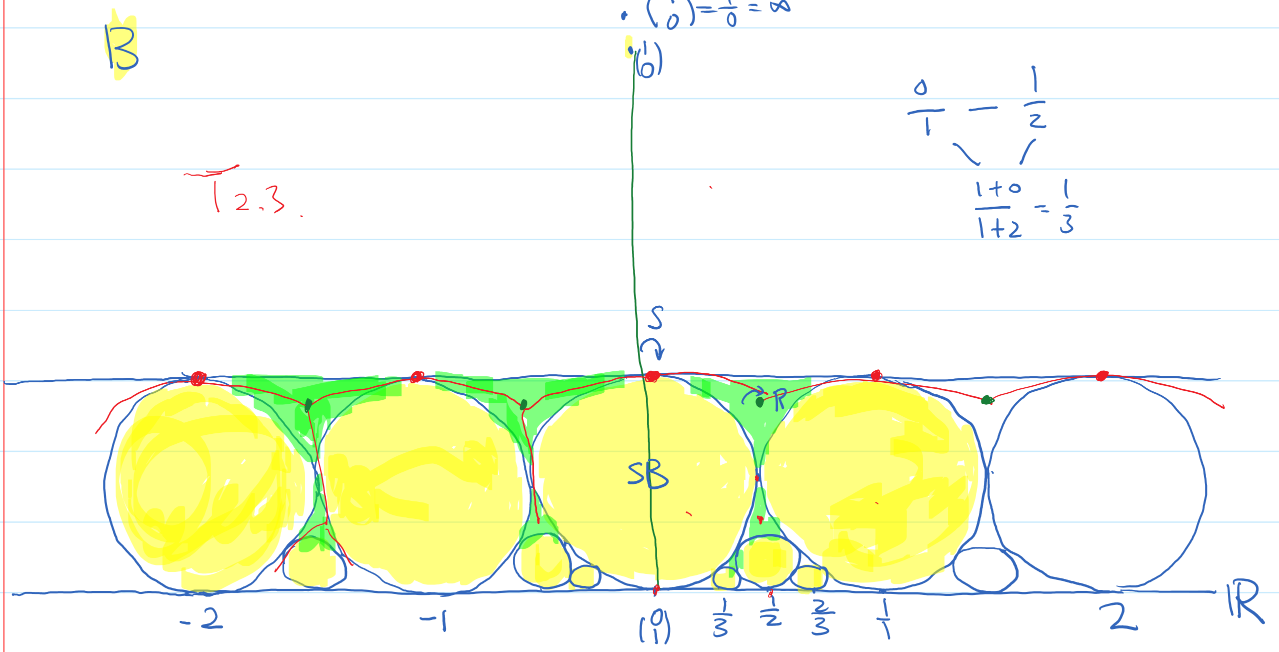
$z \mapsto (-\frac{1}{z}+1)$

Fact: $PSL(2,\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3 = \langle S : S^2=1 \rangle * \langle R = TS : R^3 = 1 \rangle$

- consider all $PSL(2,\mathbb{Z})$-translates of horodisk $B \triangleq \{(x,y) : y \geq 1\}$

This gives the following circle packing of $\mathbb{H}^2$ where circles are based at $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{Q}$ $\quad \frac{az+b}{cz+d}(\infty) = \begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{Q}$

· $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{0} = \infty$



- Two circles are tangent iff their bases $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}$ Satisfy

$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 1 \iff \begin{vmatrix} a & a+c \\ b & b+d \end{vmatrix} = 1$

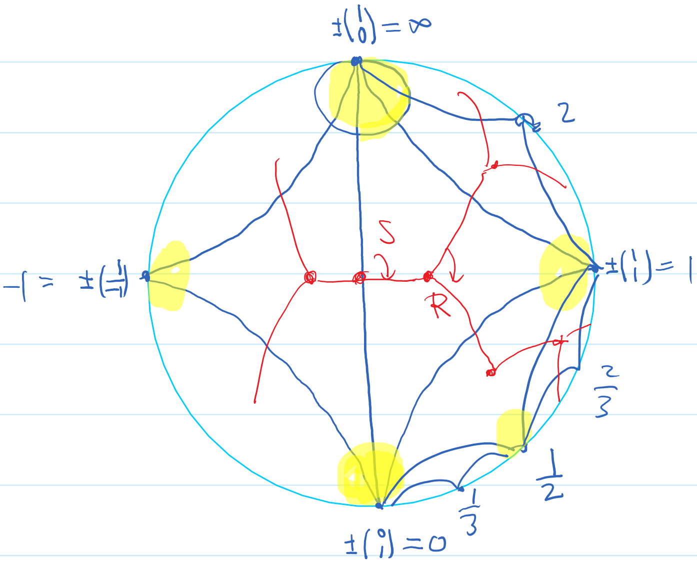
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$\pm\frac{a}{b} \in \mathbb{Q}$

- Three circles form a triangle iff

$$\begin{array}{ccccc} & & \pm\begin{pmatrix} a \\ b \end{pmatrix} & & \\ \begin{pmatrix} a-c \\ b-d \end{pmatrix} & & | & & \pm\begin{pmatrix} a+c \\ b+d \end{pmatrix} \\ & & \begin{pmatrix} c \\ d \end{pmatrix} & & \end{array}$$

- Collapsing each horodisk to its base gives the following Farey tessallation. (It is nerve of the above circle packing)



- $PSL(2,\mathbb{Z}) \circlearrowright$ the dual graph of the Farey tessallation $= T_{2,3}$

$\rightsquigarrow \quad PSL(2,\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3.$

组合极限球

2021年7月29日 16:08

Let Γ be a graph $\rightsquigarrow$ Hyperbolic space $\mathcal{H}(\Gamma)$

s.t. • Gromov bdry is just one pt

• if Γ is a line

then $\mathcal{H}(\Gamma) \overset{Q.I.}{\cong}$ horodisk in $\mathbb{H}^2$

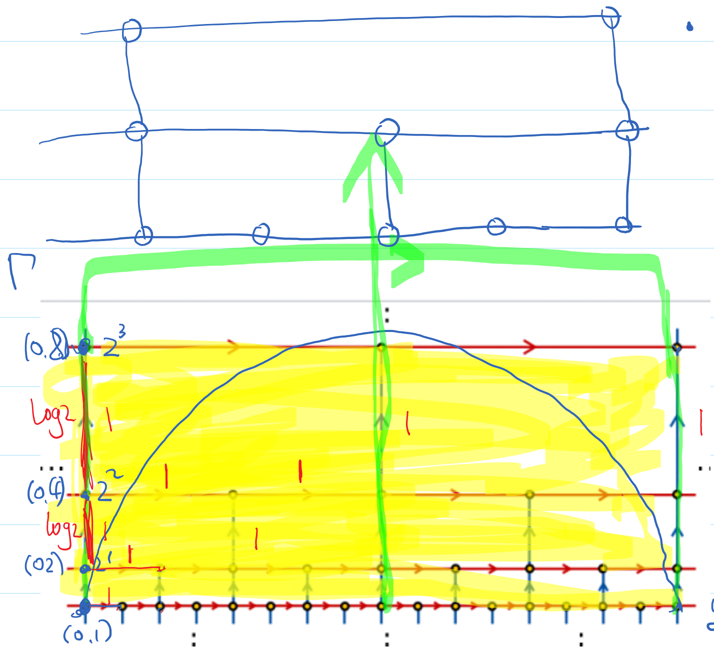
$\cup$ Γ is the horocycle

Γ is 2-dim Grid ($\mathbb{Z}^2$)

then $\mathcal{H}(\Gamma) \overset{Q.I.}{\cong}$ horoball in $\mathbb{H}^3$

$\cup$ Γ is the horosphere

. . .



$\subseteq \mathbb{H}^2$

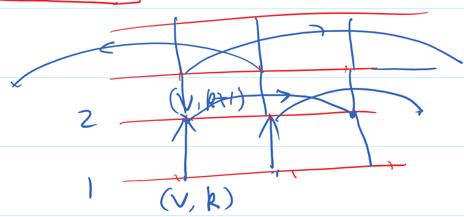
$y=1$

Fact: subgraph above level $y=1$ is Q.I. horodisk in $\mathbb{H}^2$

Def: (Combinatorial Horoball with horosphere Γ)

$\mathcal{H}(\Gamma) = \Gamma \times (\mathbb{N} \cup \{0\})$ with the following new added set of edges:

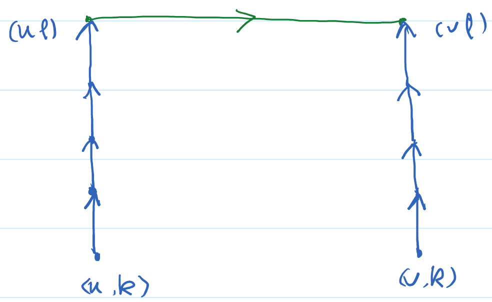
1) vertical edge $(v,k+1) \updownarrow (v,k)$ $\forall v \in \Gamma,\ k \geq 0$



2. horozontal edge

$(u,k) \longleftrightarrow (v,k)$ iff $d_\Gamma(u,v) \leq 2^k$

Def: Special paths in $\mathcal{H}(\Gamma)$:



when $d(u,v) \leq 2^\ell$

Lemma 14.12. *The graph $\mathcal{H}(\Gamma)$ is a δ-hyperbolic space where δ is a universal constant, where special paths are uniformly quasi-geodesics. The Gromov boundary of $\mathcal{H}(\Gamma)$ consists of only one point.*

Cor: Any f.g. group acts properly on a proper δ-hyp. space !

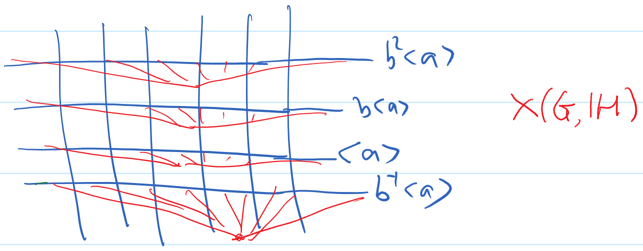
相对双曲群

2021年7月29日 16:28

Let G be f.g. group $\mathbb{H}$ a finite set of subgroups

$\mathrm{Cay}(G, S) \hookleftarrow \{gH : g \in G,\ H \in \mathbb{H}\}$

$(\mathbb{Z}^2, \{\mathbb{Z} = \langle a \rangle\})$



Along the $\boxed{\text{Cayley graph of } gH}$ we attach a combinatorial horoball.

This defines $\boxed{\text{Augmented space of } G \text{ along } \mathbb{H}}$. denoted by $X(G, \mathbb{H})$ local finite

Def: G is called hyperbolic relative to $\mathbb{H}$ if $X(G, \mathbb{H})$ is δ-hyperbolic space.

Rmk: If $\mathbb{H} = \{1\}$ then a relative hyperbolic pair $(G, \mathbb{H})$ $\Rightarrow$ G is hyperbolic group (relative to $\boxed{\text{trivial subgps}}$).

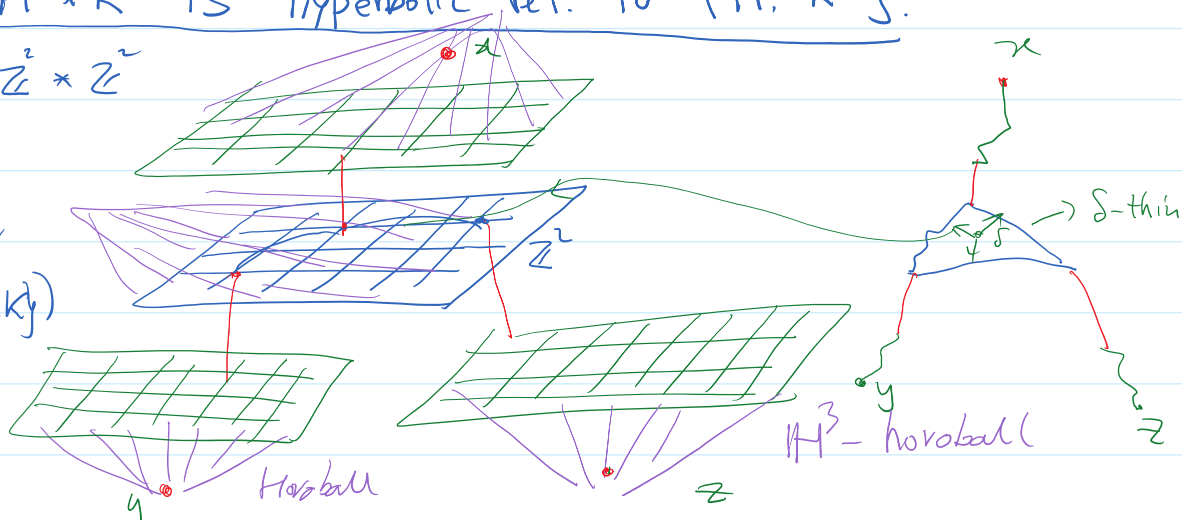
Ex: $(\mathbb{Z}^2, \{\mathbb{Z}\})$ is NOT RH!

Examples:

- $H * K$ is hyperbolic rel. to $\{H, K\}$.

$\mathbb{Z}^2 * \mathbb{Z}^2$

$H * K \circlearrowright X(G, \{H, K\})$



- $PSL(2, \mathbb{Z})$ is hyperbolic rel. to $\{\mathbb{Z} \cong \pi_1(\text{Cusp})\}$.

$(X(PSL(2, \mathbb{Z}), \mathbb{Z}) \cong \mathbb{H}^2)$

($X(\underline{PSL(2,\mathbb{Z})}, \mathbb{Z}) \cong \mathbb{H}^2$)

- π_1(n-dim hyp manifold with finite volume)
  is hyp. rel. to $\{\pi_1(\text{Cusp}) \cong \underline{\mathbb{Z}^{n-1}} : \underline{\text{all cusps}}\}$

- π_1($\underline{\text{Mixed 3-mfd}}$) is $\underline{\text{hyp rel}}$ to
  $\begin{cases} \pi_1(\underline{\text{JSJ torus}}) \cong \mathbb{Z}^2 \\ \pi_1(\underline{\text{maximal graph manifold}}) \end{cases}$

- Groups with ∞-ends are Rel. hyp. !