几何群论参考书目

2021年7月12日 9:50

The basic references to this course are the following

(1) Bowditch, **A course on geometric group theory**, Mathematical Society of Japan, Tokyo, 2006.
http://www.warwick.ac.uk/~masgak/papers/bhb-ggtcourse.pdf
(2) CF Miller III, **Combinatorial group theory**,
http://www.ms.unimelb.edu.au/~cfm/notes/cgt-notes.pdf.

A comprehensive treatment to hyperbolic groups among many other things:

- M. Bridson and A. Haefliger, **Metric spaces of non-positive curvature**, vol. 319, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1999.

A undergraduate textbook discussing many interesting examples:

- J.Meier, **Groups, graphs and trees: An introduction to the geometry of infinite groups**, Cambridge University Press, Cambridge, 2008.

Here are more references you might be interested in.
The original monograph of M. Gromov introducing hyperbolic groups:

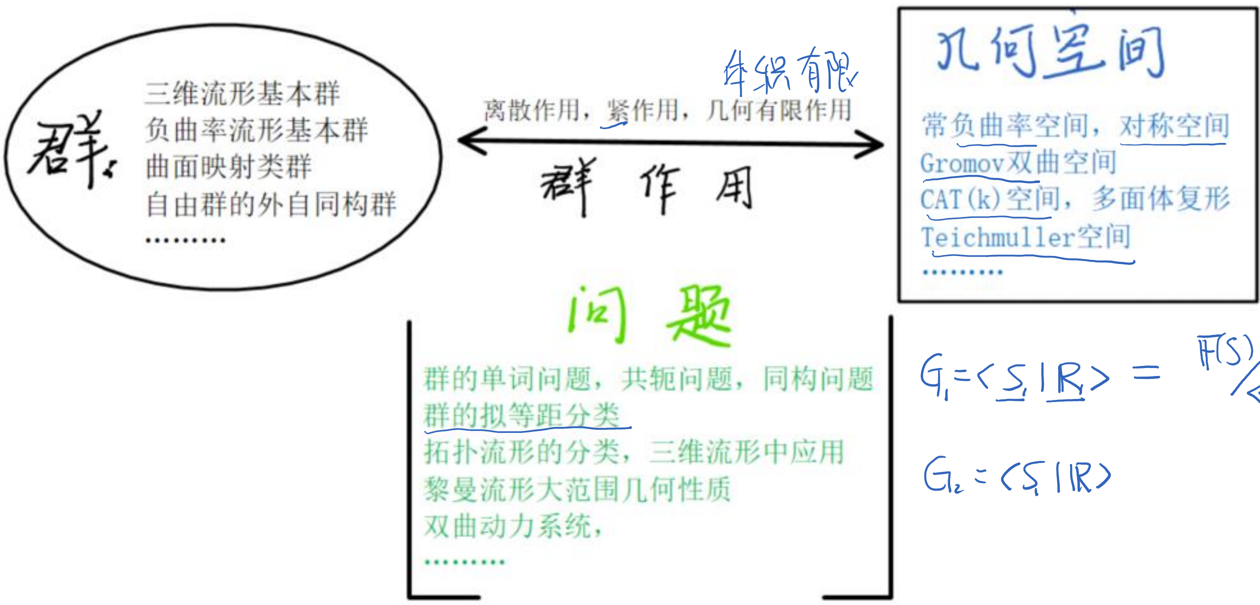
- M. Gromov, **Hyperbolic groups**, in "Essays in Group Theory" (G. M. Gersten, ed.), MSRI Publ. 8, 1987, pp. 75-263

Three early collaborative works to understand the above Gromov 1987 monograph:

(1) H. Short (editor), **Notes on hyperbolic groups**, Group theory from a geometrical viewpoint, World Scientific Publishing Co., Inc., 1991.
https://www.i2m.univ-amu.fr/~short/Papers/MSRInotes2004.pdf
(2) É. Ghys and P. de la Harpe (editors), **Sur les groupes hyperboliques d'aprs Mikhael Gromov**. Progress in Mathematics, 83. Birkhuser Boston, Inc., Boston, MA, 1990. xii+285 pp. ISBN 0-8176-3508-4
(3) M. Coornaert, T. Delzant and A. Papadopoulos, **Géométrie et théorie des groupes: les groupes hyperboliques de Gromov**, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, Berlin, 1990, x+165 pp.

几何群论简介

2021年7月15日 12:36



$G_1 = \langle S \mid R \rangle = F(S)/\langle\langle R \rangle\rangle$

$G_2 = \langle S \mid R \rangle$

Max. Dehn: (1878-1952)

1911: Dehn Problem:
- word problem,
- conjugacy problem,
- isomorphism problem

1912: Dehn's algorithm 双曲几何

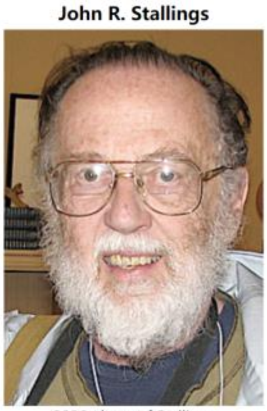
Max Dehn

John R. Stallings (1935-2008)

1968: Stallings' theorem about ends of groups:

a finitely generated group *G* has more than one end if and only if the group *G* admits a nontrivial decomposition as an amalgamated free product or an HNN extension over a finite subgroup.

John R. Stallings



2006 photo of Stallings

Mikhael Gromov (1943-)

1981: Gromov's theorem on groups of polynomial growth:

A finitely generated group has polynomial growth if and only if it has a nilpotent subgroup that is of finite index.

1987: 《Hyperbolic groups》

Mikhael Leonidovich Gromov

Mikhail Gromov in 2009

凯莱图

2021年7月15日 11:26

<< Tree >> Serre

Definition 5.1. A *graph* $\mathcal{G}$ consists of a set V of vertices and a set E of directed edges. For each directed edge $e \in E$, we associate to e the *initial point* $e_- \in V$ and *terminal point* $e_+ \in V$. There is an orientation-reversing map

$$^{-}: E \overset{1:1}{\to} E, e \to \bar{e}$$

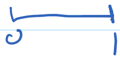
such that $e \neq \bar{e}$, $e = \bar{\bar{e}}$ and $e_- = (\bar{e})_+, e_+ = (\bar{e})_-$.

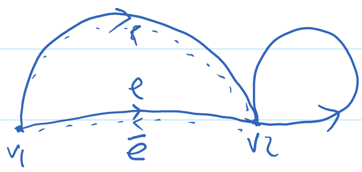
1-dim CW-复形

metric graph.

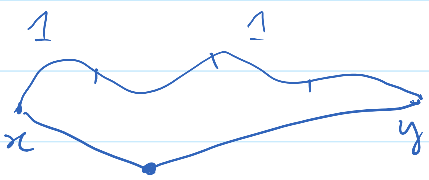
every edge has edge 1

$\cong$





$\rightsquigarrow$ graph metric



Definition 5.2. (right) Let G be a group and S be a symmetric generating set without identity. The Cayley graph $\mathscr{G}(G, S)$ of G with respect to S is a graph with the vertex set G and edge set $\boxed{G \times S}$. Define $(g,s)_- = g, (g,s)_+ = gs$, and the map

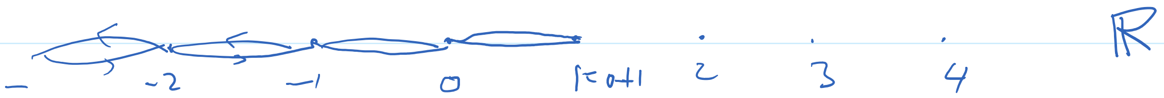
$$^{-}: G \times S \to G \times S, (g,s) \to (gs, s^{-1}).$$

It is clear to see that $^{-}$ satisfies the conditions in definition of a graph.

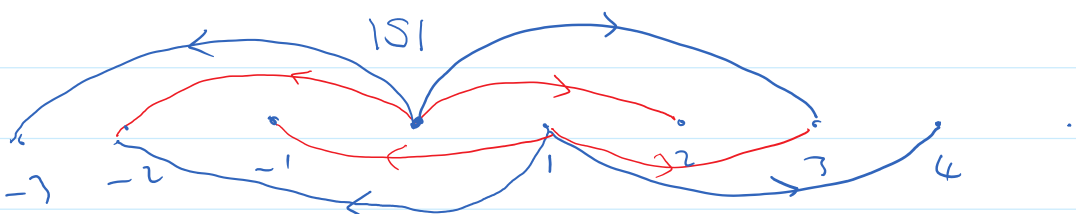
$1 \notin S = S^{-1}$

$e_+ = gs$ (右乘)

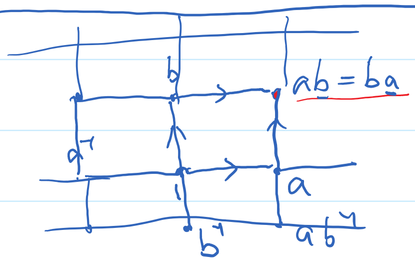
$\mathbb{Z} = G$ $S = \{1, -1\} \not\ni 0$ $S = \{\pm 2, \pm 3\}$



$|S|$



$\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z} = \langle a \rangle \oplus \langle b \rangle$



$S = \{a, b, a^{-1}, b^{-1}\}$ 字母表

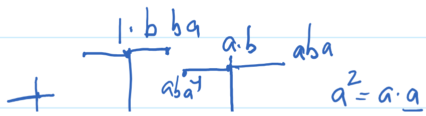
$\mathbb{F}(S) \triangleq \{$所有由S上字母形成既约单词$\}$

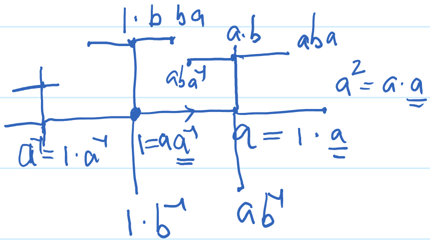
不允许 aa^{-1}, $a^{-1}a$, bb^{-1} 与 $b^{-1}b$ 出现

$w_1 \cdot w_2 \triangleq w_1 w_2$ 的既约形式

$aba^{-1} \cdot ab^{-1}a \triangleq ab\underline{a^{-1}ab^{-1}}a = a^2$

Ex: $(\mathbb{F}(S), \cdot)$ 是群. 空单词是恒等元素. $ba \neq ab \in \mathbb{F}S$





$Cay(\mathbb{F}_2, S)$ = 四叉树

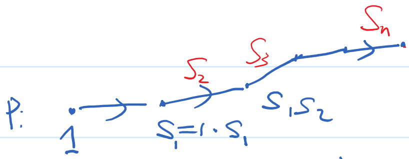
Definition 5.5. The *word norm* $|g|_S$ of an element $g \in G$ is the shortest length of a word w such that $w =_G g$. Formally,

$$|g|_S = \inf\{|w| : \phi(w) = g, \phi : F(S) \to G\}.$$

We define the *word metric* d_S on G: $d_S(g,h) = |g^{-1}h|_S$ for any $g, h \in G$. $g^{-1}h = s_1 \cdots s_n$

Fact: $Cay(G, S)$ is a connected graph of degree $|S|$

$G \curvearrowright$ left multiplication



$g = s_1 \cdots s_n$ $s_i \in S$

$Lab(p) \triangleq s_1 \cdots s_n$

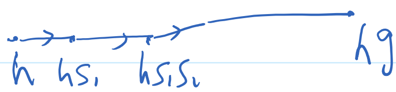
Lab: $\mathbb{E} = \mathbb{E}(Cay(G,S)) \longrightarrow S$ 定义为所有Cayley graph上的路径集合

$\qquad \| $

$\qquad G \times S$

$\qquad (g, s) \longmapsto s$

$\qquad (gs, s^{-1}) \longmapsto s^{-1}$



Lab: $\{\text{Paths on } Cay(G,S)\} \longrightarrow \{S \text{ 上 单词集合}\} = W(S)$

Lab is bijective between: {paths from identity} and $W(S)$

Lemma 5.7. *Let S, T be two finite generating sets of G. Then thee exists a constant $C \geq 1$ such that*

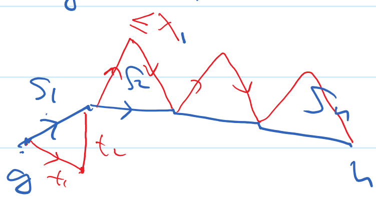
$$C^{-1} d_T(.,.) \leq d_S(.,.) \leq C d_T(.,.).$$

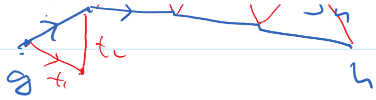
proof: $\forall g, h \in G$: $\lambda_1 d_S(g,h) \geq d_T(g,h)$

$\qquad \|$

The shortest length of path from g to h in $Cay(G,S)$

$C = \max\{\lambda_1, \lambda_2\}$





Define $\lambda_1 \triangleq \max\{ d_T(1, s) : s \in S \}$

$\lambda_2 \triangleq \max\{ d_S(1, t) : t \in T \}$

长度空间

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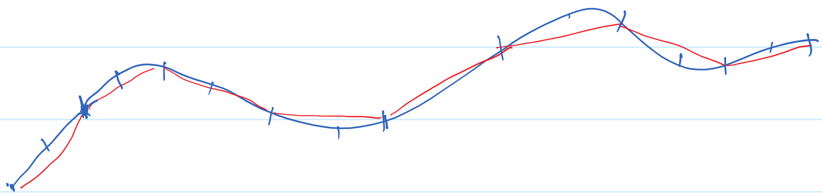
6.1. **Length metric spaces.** Let (X, d) be a metric space. Let $p : [a, b] \to X$ be a parameterized continuous path, where $a, b \geq 0$. It is called *rectifiable* if

$$\sup \sum_{0 \leq i \leq n} d(p(t_i), p(t_{i+1})) < \infty \tag{7}$$

over all finite partitions $\{t_0 = a, ..., t_n = b\}$ of $[a, b]$. The *length* $\mathrm{Len}(p)$ of p is defined to be the supremum of the above sum (7) over all possible partitions of $[a, b]$.



Definition 6.1. Let (X, d) be a metric space. We define an induced metric $\bar{d}$ called *length metric* as follows. Let $x, y \in X$ be two points. Then $\bar{d}(x, y)$ is the infimum of lengths of all possible rectifiable paths between x, y.

If $d = \bar{d}$, then (X, d) is called a *length metric* space.

Definition 6.2. A metric space (X, d) is called *proper* if any closed ball $\bar{B}(x, r)$ at $x \in X$ with radius $r \geq 0$ is compact.

Fix $x \in X$

$$\begin{aligned} X &\longrightarrow \mathbb{R} \\ y &\longmapsto d(y, x) \end{aligned}$$ is proper $\Longleftrightarrow$ X is proper

Theorem 6.3 (Hopf-Rinow). *Let (X, d) be a length metric space. Then (X, d) is proper if and only if it is a locally compact and complete space.* In particular, (X,d) is geodesic space

Proof: $\Rightarrow$ ✓

$\Leftarrow$: Fix $o \in X$. Let $R \triangleq \sup\{r : \overline{B(o,r)} \text{ is compact}\} \subseteq [0, +\infty]$

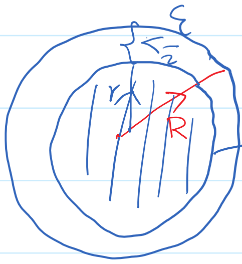
locally compact $\Rightarrow R > 0$

• $\overline{B(o,R)}$ is compact. $(\Rightarrow R = \infty)$

criterion of compactness: compact $\Leftrightarrow$ complete & totally bounded

$\forall \varepsilon > 0\ \exists F \subseteq \overline{B(o,R)}$

$\# F < \infty \quad N_\varepsilon(F) \supseteq \overline{B(o,R)}$

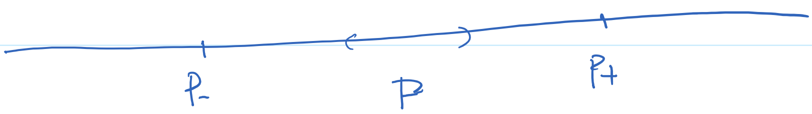


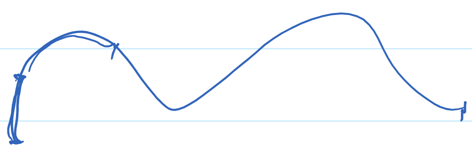
$\to$ compact

segment

Definition 6.4. A path p is called a *geodesic* if $\text{Len}(p) = d(p_-, p_+)$. Equivalently, a path is a geodesic if its length parametrization $\bar{p} : [0, \text{Len}(p)] \to X$ is an isometric map.

A metric space is called a *geodesic* metric space if there exists a geodesic between any two points.





(Ascoli–Arzela 引理)

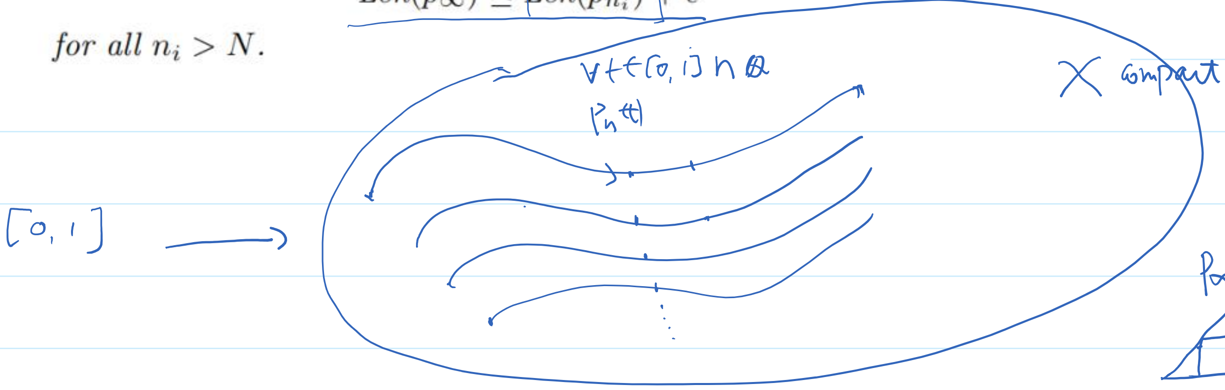
Lemma 6.5. *Let (X, d) be a compact metric space, and $p_n : [0, 1] \to X$ a sequence of linearly parameterized paths with uniformly bounded length. Then*

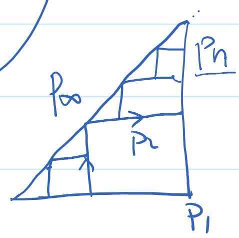
(1) *There exists a subsequence p_{n_i} of p_n which uniformly converges to a path $p_\infty : [0, 1] \to X$.*

(2) *For any $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that*

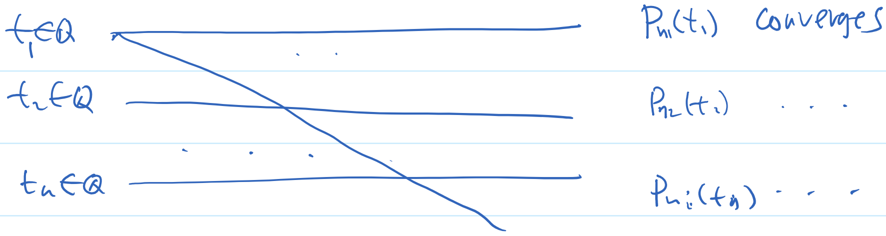
$$Len(p_\infty) \le \boxed{Len(p_{n_i})} + \epsilon$$

for all $n_i > N$.





Cantor对角线 $\Longrightarrow$



$p_{n_i}(t) \Longrightarrow p_\infty(t)$: $\quad Len(p_\infty) \le \liminf_{n_i \to \infty} Len(p_{n_i})$

Theorem 6.6. *A proper $\boxed{\text{length metric}}$ space is a geodesic metric space.*

proof: $\quad d(x, y) \triangleq \inf Len(p_n) \qquad p_n : x \to y$

By lemma, $p_n \Rightarrow p_\infty$ & $Len(p_\infty) = \lim_{n \to \infty} Len(p_n) = d(x, y)$

$\exists K \subset X$ compact

$GK = X$

loc. compact & complete

Lemma 6.21 (Svarc-Milnor Lemma). *Suppose G acts properly and co-compactly on a proper length space (X, d). Then*

(1) *G is finitely generated by a set S.*

(2) *Fix a basepoint $o \in X$. Then the map*

$$(G, d_S) \to (\overset{\phi}{G}o, d),\ g \to go,$$

is a G-equivariant quasi-isometric map.

$G \curvearrowright X \xrightarrow{\phi} G \curvearrowright Y$

$\phi(g.x) = g.\phi(x)$

Definition 6.17. The action of G on X is called a *proper* action if the following set

$$\{g : gK \cap K \neq \emptyset\}$$

is finite for any compact set K in X. We also say that G acts properly on X.

Cor: · $G_x = \{g : gx = x\}$ is finite $\forall$ · Gx is discrete in X

Definition 6.9. Let $\phi : (X, d_X) \to (Y, d_Y)$ be a map between two metric spaces. Given constants $\lambda \geq 1, c > 0$, ϕ is called a (λ, c)-*quasi-isometric embedding map* if the following inequality holds

$$\lambda^{-1} d_X(x, x') - c \leq d_Y(\phi(x), \phi(x')) \leq \lambda d_X(x, x') + c, \tag{12}$$

for all $x, x' \in X$.

If, in addition, there exists $R > 0$ such that $Y \subset N_R(\phi(X))$, then ϕ is called a (λ, c)-*quasi-isometry*. In this case, we also say that X is *quasi-isometric* to Y.

Rmk: · Quasi-isometric embedding may not be injective but injective on large scales:

$$d_X(x, x') > \lambda c \implies \phi(x) \neq \phi(x')$$

· Ex: $\mathbb{Z} \hookrightarrow \mathbb{R}$

Def: ψ is a quasi-isometry from X to Y

ϕ is a quasi-inverse of ψ if $\forall x \in X: d(\phi \circ \psi x, x) \le R$ $\exists R > 0$

$X \xrightarrow{\psi} Y$

$x \longmapsto \psi x$

Lemma 6.12. (1) *Let $\phi : X \to Y$ and $\psi : Y \to Z$ be two quasi-isometric embeddings. Then $\psi \cdot \phi : X \to Z$ is a quasi-isometric embedding.*

(2) *Let $\phi : X \to Y$ be a quasi-isometry. Then any $\boxed{\text{quasi-inverse}}$ of ϕ is a quasi-isometry $\psi : Y \to X$. So a quasi-isometric embedding is a quasi-isometry if and only if it has a quasi-inverse.*

Proof: $\phi: X \to Y$ Q.I. $\rightsquigarrow$ $\boxed{\text{quasi-inverse}}$ $\varphi: Y \to X$

$N_R(\phi(X)) \supseteq Y$: $\forall y \longmapsto \varphi(y) \in X$

$\Downarrow$

$\exists x \in X$ s.t. $d_Y(\phi x, y) \le R$

Define $\varphi(y) \triangleq x$.

Lemma 6.21 (Svarc-Milnor Lemma). *Suppose G acts properly and co-compactly on a proper length space (X, d). Then* geometric action

(1) *G is finitely generated by a set S.*

(2) *Fix a basepoint $o \in X$. Then the map*

$$(G, d_S) \to (Go, d), \ g \to go,$$

is a G-equivariant quasi-isometric map. (In fact, quasi-isometry)

Corollary 6.23. *Let G be a finitely generated group. Then any finite index subgroup is finitely generated and quasi-isometric to G.*

Exercise 6.24. *Let*

$$1 \to N \to G \to \Gamma \to 1$$

be a group extension, where G is finitely generated. Assume that N is finite. Then G is quasi-isometric to Γ.

Examples 6.25. (1) If $n \neq m$, then $\mathbb{R}^n$ is not quasi-isometric to $\mathbb{R}^m$.
(2) All free groups of finite rank at least two are quasi-isometric.

Bridson's universe

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