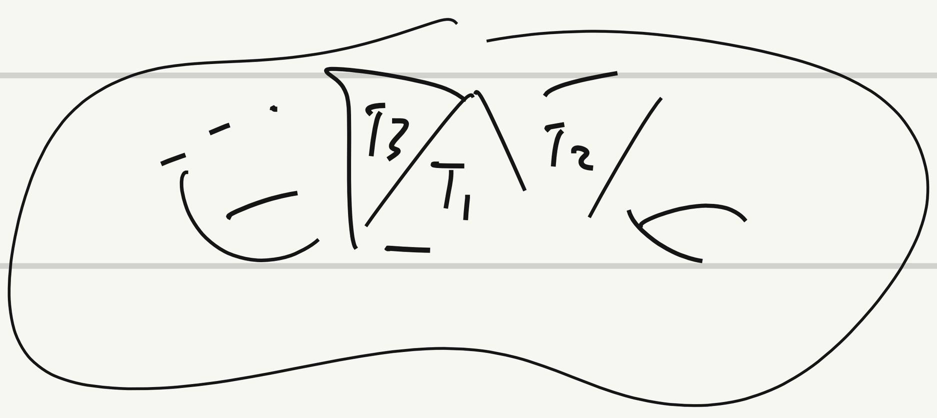
IX Identities on Hyperbolic Surfaces

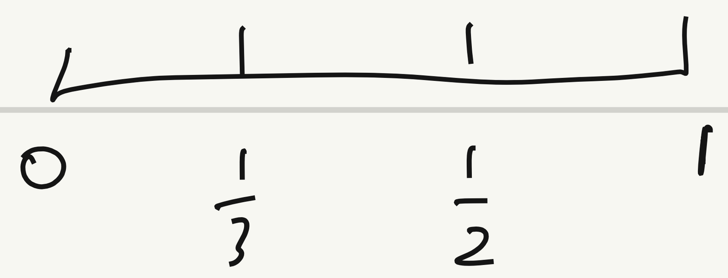
1.



$$\underline{\sum_{n=1}^{N} A_{Hr}(T_n) = A_{Hr}(S)}$$

Baby example:

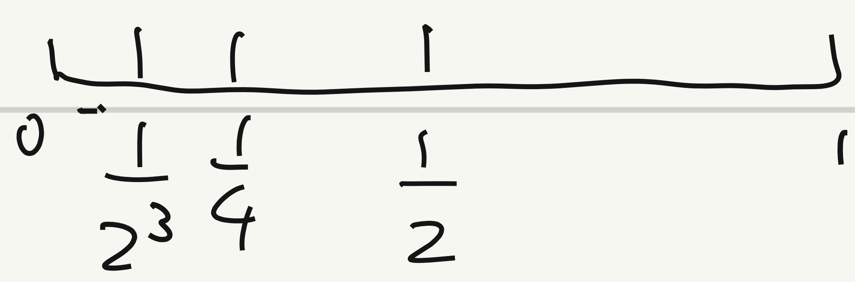
①



$$[0,1] = \underbrace{\{0, \tfrac{1}{3}, \tfrac{1}{3}, 1\}}_{Z} \cup \left(\underset{I_1}{(0,\tfrac{1}{3})} \cup \underset{I_2}{(\tfrac{1}{3},\tfrac{2}{3})} \cup \underset{I_3}{(\tfrac{2}{3},1)}\right)$$

$$\underbrace{\sum_{j=1}^{3} \mathrm{Leb}(I_j) = \mathrm{Leb}([0,1])}_{1} - \underbrace{\mathrm{Leb}(Z)}_{=0}$$

②

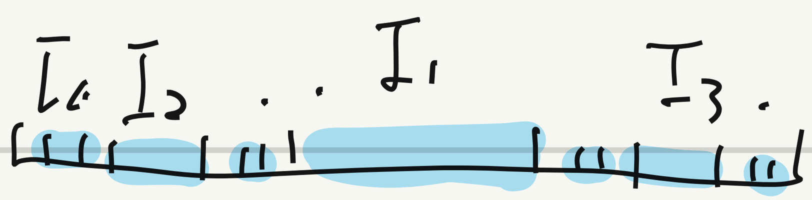


$$[0,1] = \underbrace{\left\{\tfrac{1}{2^n} \,\middle|\, n \in \mathbb{N}\right\} \cup \{0\}}_{Z} \cup \left(\underbrace{\bigsqcup \left(\tfrac{1}{2^{n+1}}, \tfrac{1}{2^n}\right)}\right)$$

countable

$$\underbrace{\sum_{j=0}^{\infty} \mathrm{Leb}\left(\tfrac{1}{2^{n+1}}, \tfrac{1}{2^n}\right)} + \underbrace{\mathrm{Leb}(Z)}_{0} = \mathrm{Leb}([0,1]) = 1$$

③ uncountable



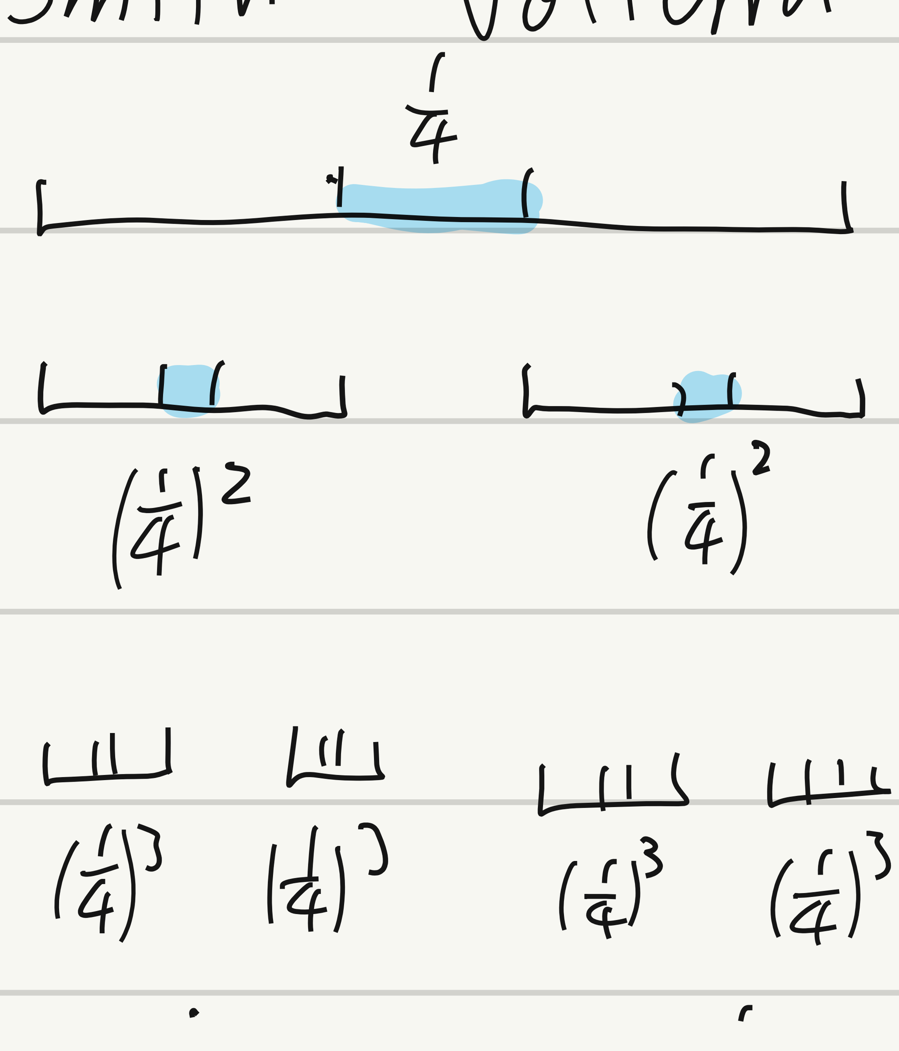
$\frac{1}{3}$ Cantor set $= C(\frac{1}{3})$

$$[0,1] = \underbrace{C(\tfrac{1}{3})}_{Z} \cup \left(\bigsqcup_{j=1}^{\infty} I_j\right)$$

$$\mathrm{Leb}\left(\bigsqcup_{j=1}^{\infty} I_j\right) = \sum_{j=1}^{\infty} \mathrm{Leb}(I_j) = \frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\left(\frac{2}{3}\right)^2 + \cdots$$
$$= 1$$

$\underline{\mathrm{Leb}(Z) = 0}$ $\quad \mathrm{Leb}\left(\bigsqcup_{j=1}^{\infty} I_j\right) = \mathrm{Leb}([0,1])$

④ Smith – Volterra – Cantor set.



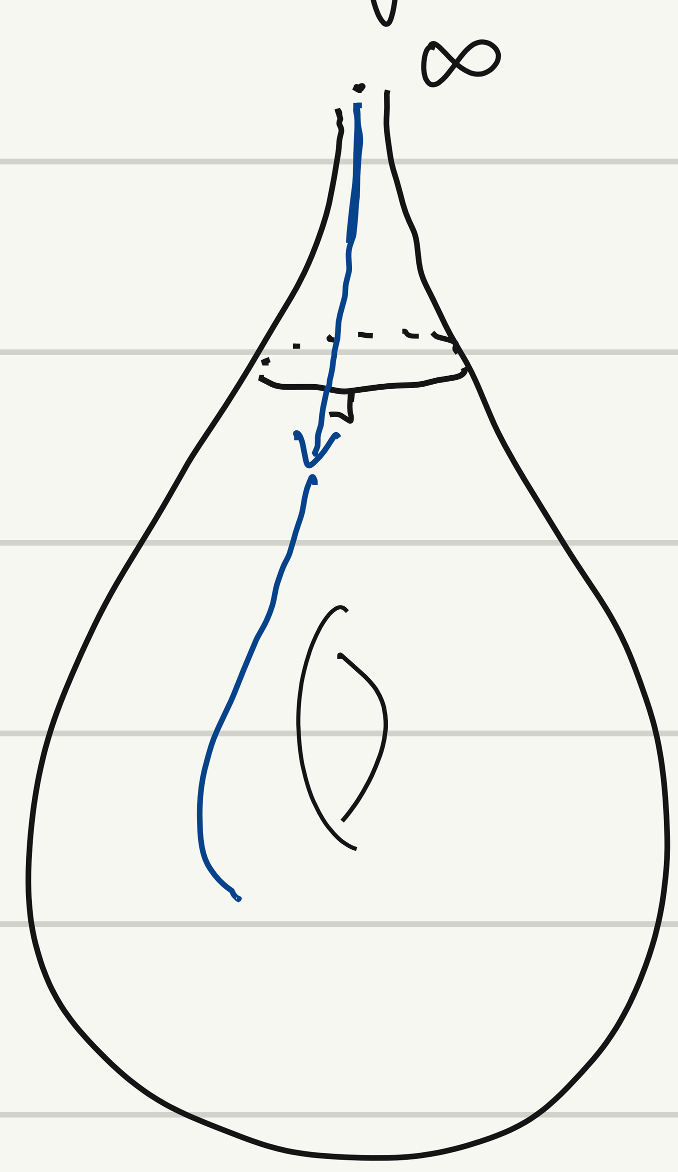
$$[0,1] = \underset{\underset{C}{\shortparallel}}{Z} \cup \left(\bigsqcup_{j=1}^{\infty} I_j\right)$$

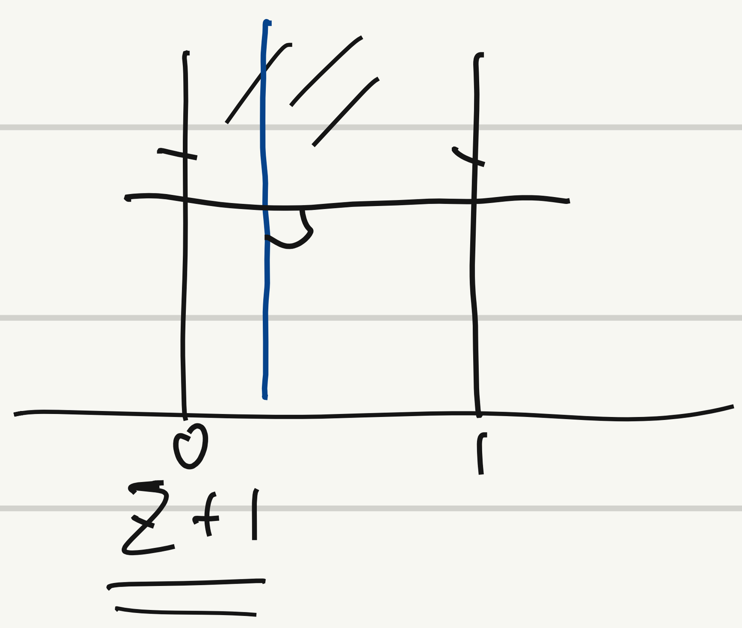
$$\mathrm{Leb}\left(\bigsqcup_{j=1}^{\infty} I_j\right) = \sum_{j=0}^{\infty} \frac{2^j}{2^{2(j+1)}} = \left(\sum_{j=0}^{\infty} 2^{-j}\right)\frac{1}{4}$$
$$= \frac{1}{2}$$

$$\mathrm{Leb}(C) = 1 - \frac{1}{2} = \frac{1}{2}$$

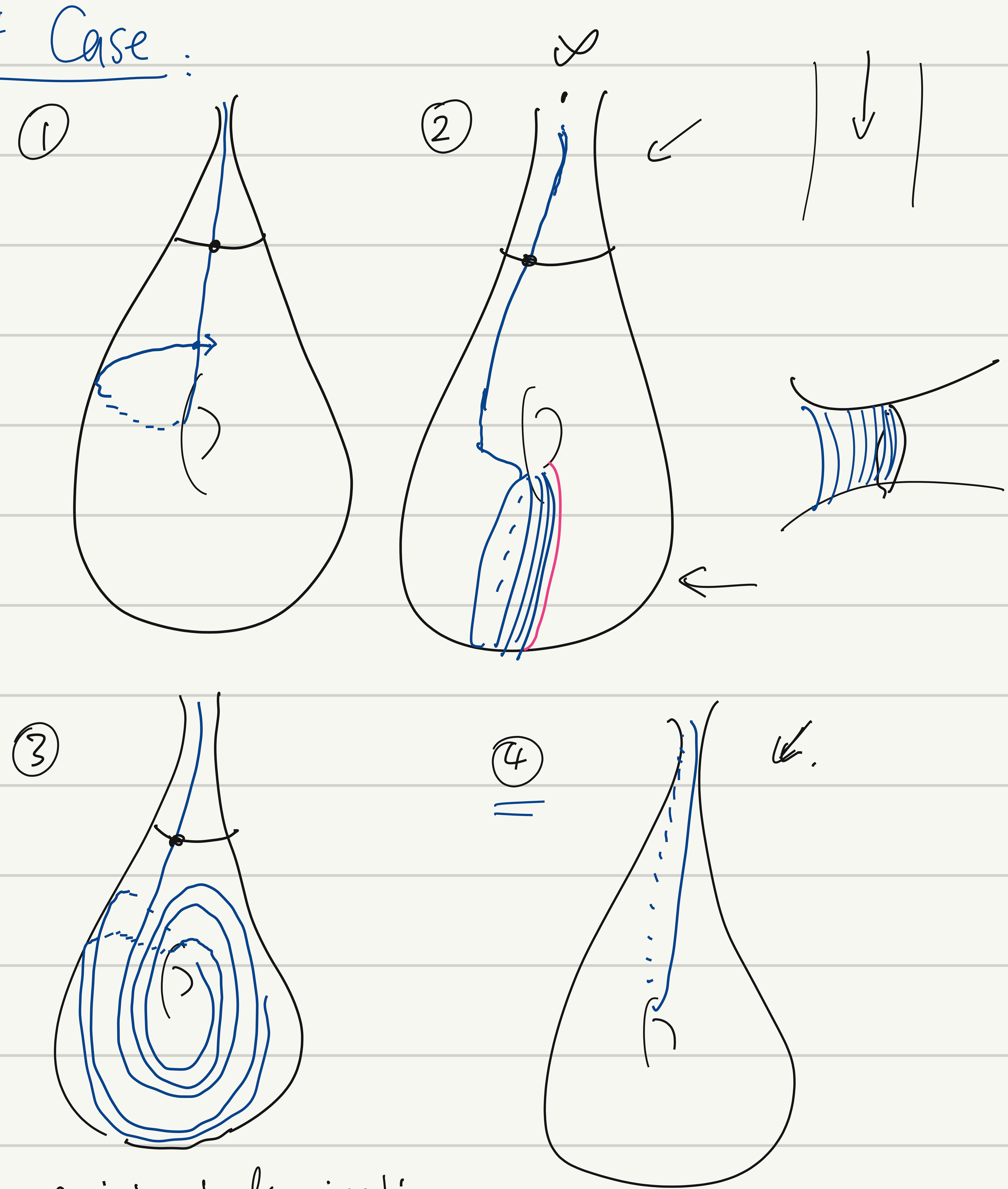
2. McShane identity : $\Sigma_{1,1}$ genus 1 , 1 cusp $S_{1,1}=S$

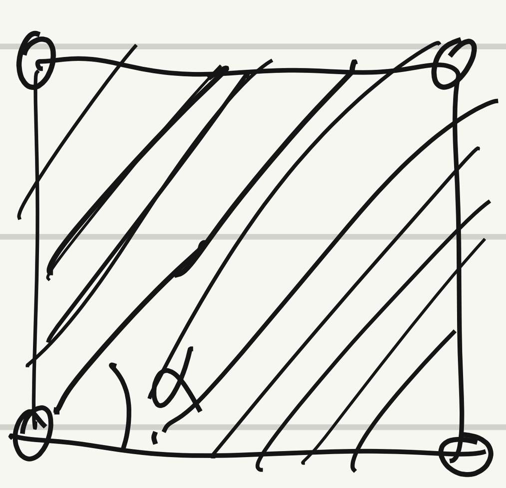
$[0,1] \rightsquigarrow$ Horocycle





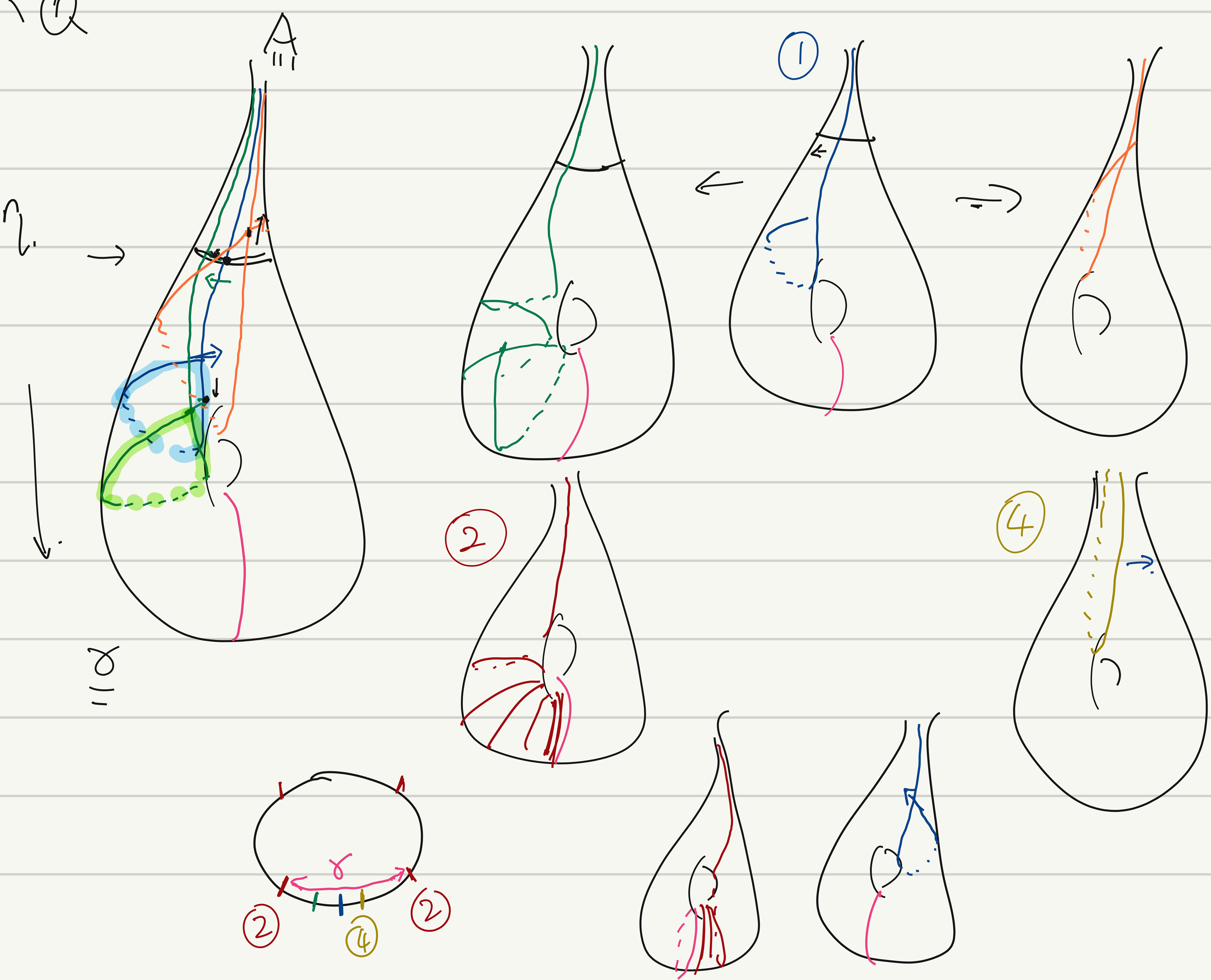
4 Case :

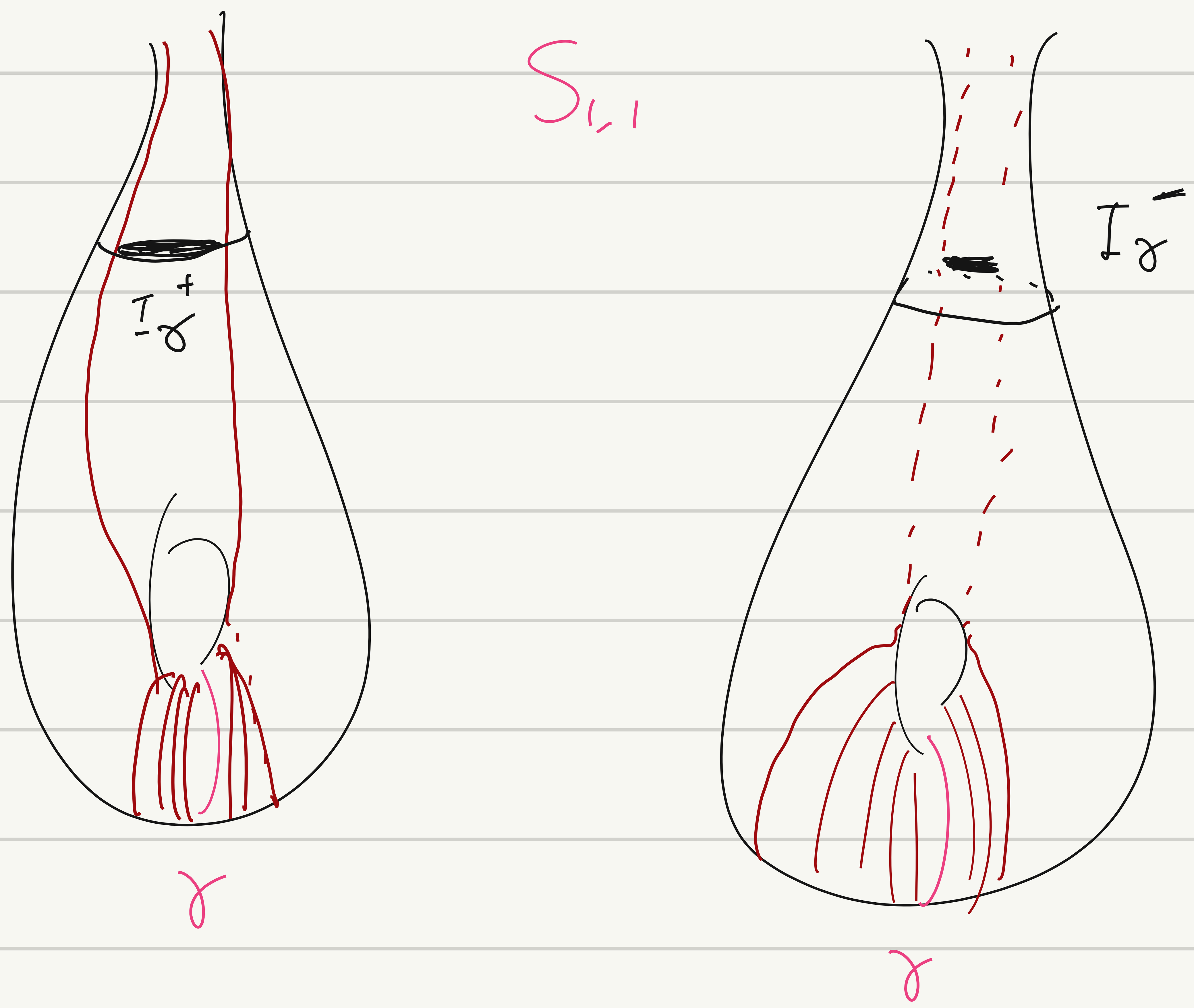




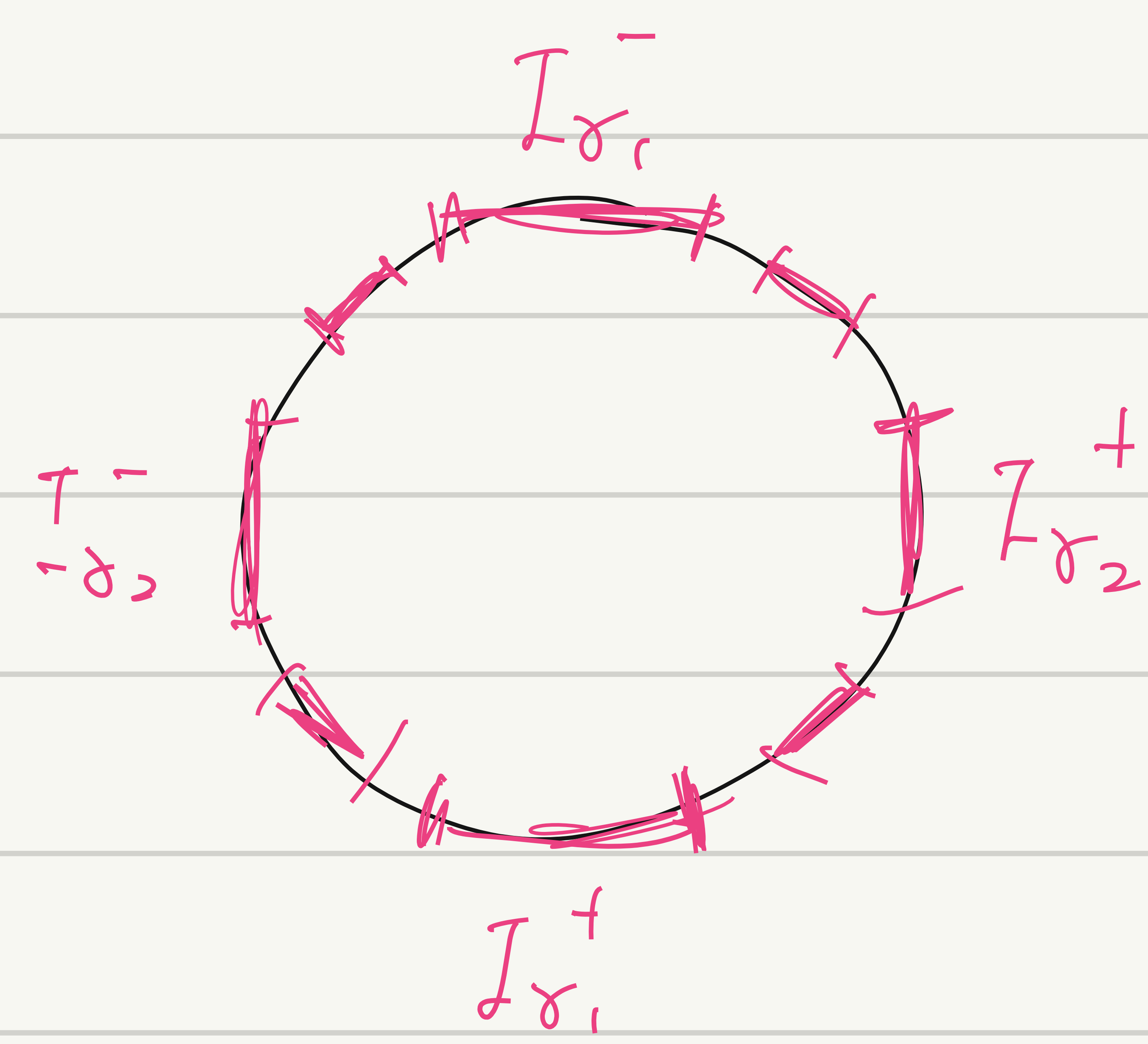
$\alpha \in \mathbb{R}\setminus\mathbb{Q}$

$\rightsquigarrow$ minimal lamination





$$\mathrm{Leb}(I_\gamma^+) = \mathrm{Cap}^+(\gamma)$$
$$\mathrm{Leb}(I_\gamma^-) = \mathrm{Cap}^-(\gamma)$$



H : horocycle.

$$H = Z \sqcup \Big(\bigsqcup_{\gamma \text{ s.c.g.}} (I_\gamma^+ \cup I_\gamma^-) \Big)$$

simple closed geodesic.

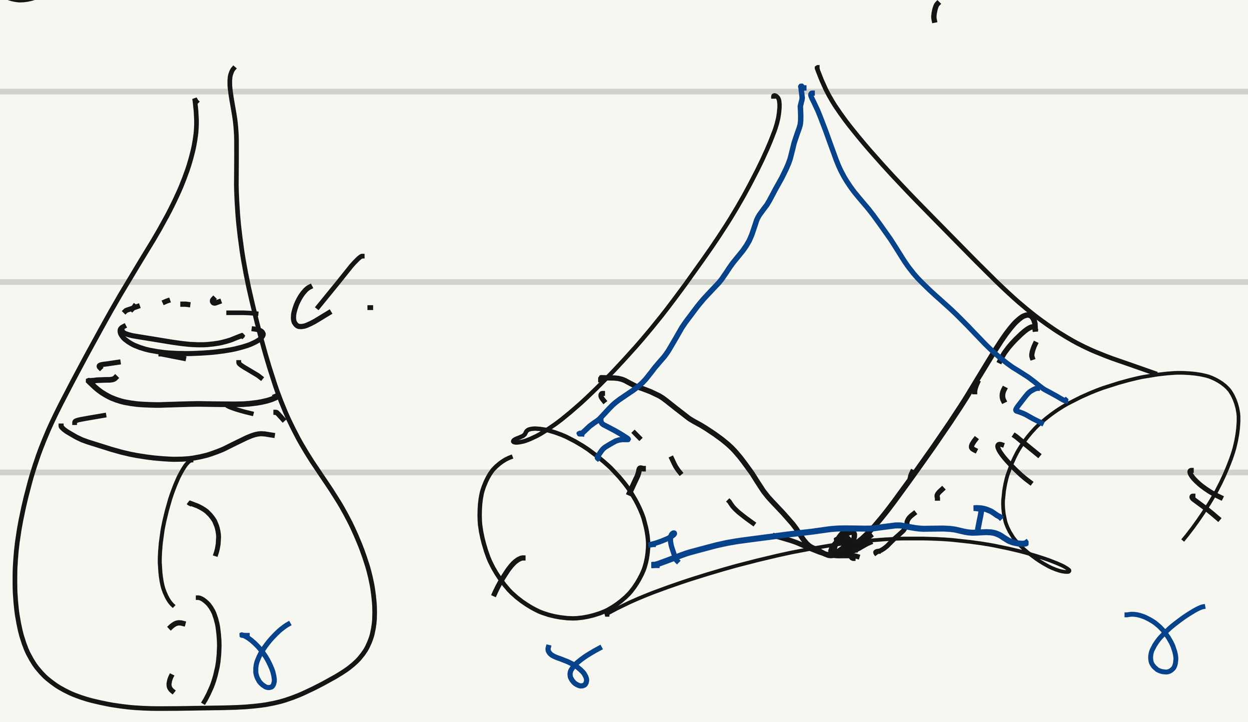
$\eta \cap H \in Z$

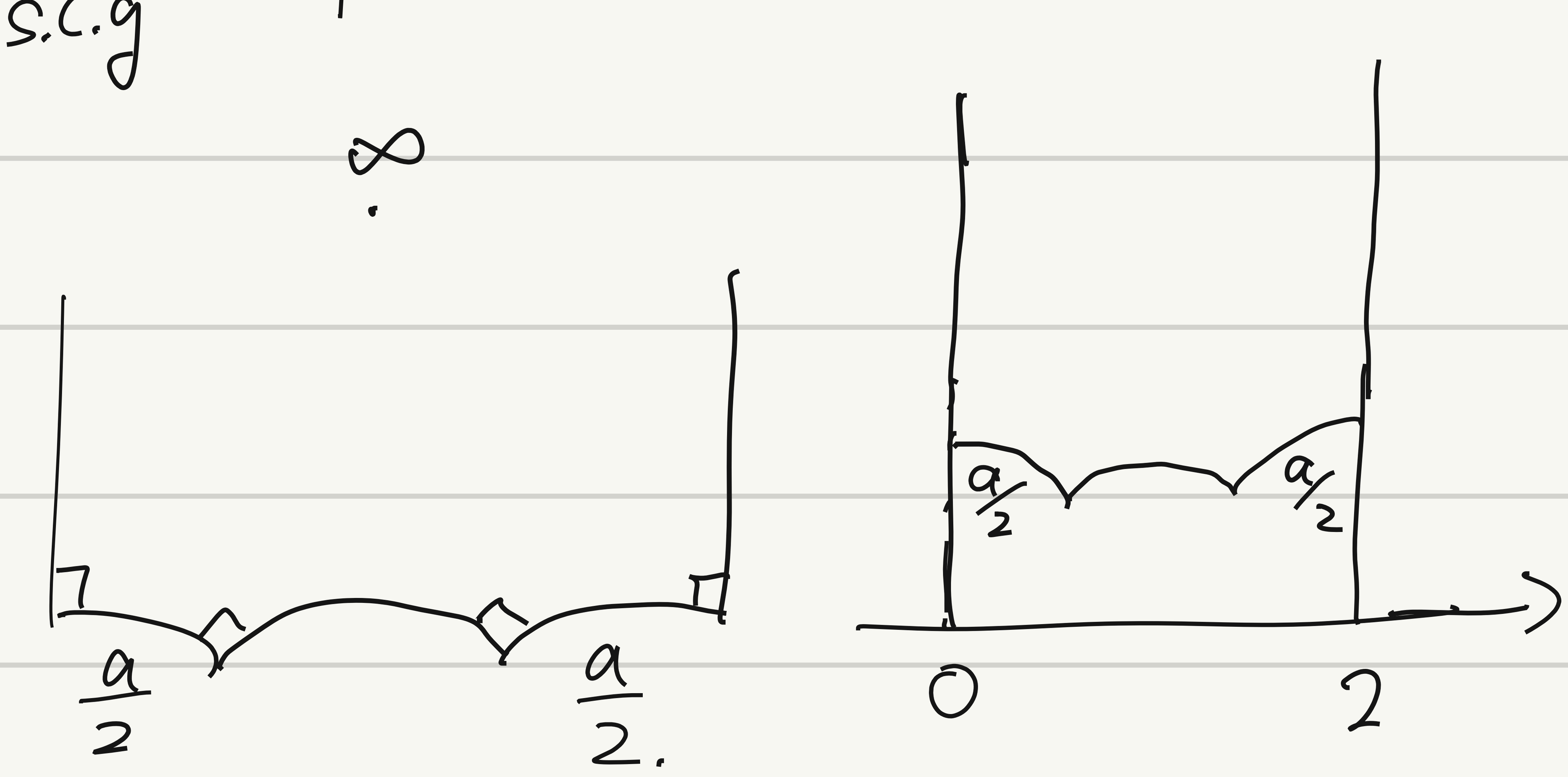
$\Rightarrow \eta$ simple $\quad \eta \in \partial I_\gamma^\pm \Rightarrow \eta.$ ②

$\eta \notin \partial I_\gamma^\pm \Rightarrow \eta$ ③

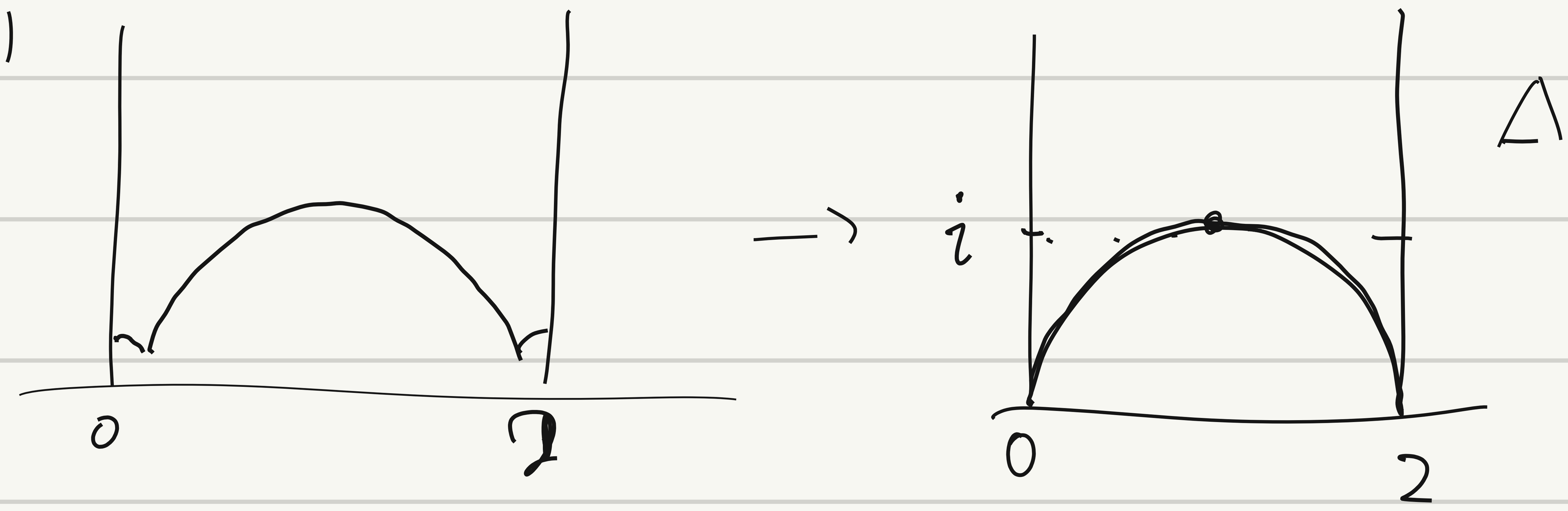
$$\underbrace{\mathrm{Leb}(H)}_{①} = \underbrace{\mathrm{Leb}(Z)}_{②} + \sum_{\gamma \text{ s.c.g}} \mathrm{Cap}(\gamma) \quad = \mathrm{Cap}^+(\gamma) + \mathrm{Cap}^-(\gamma)$$

Q: ① $= ?$ $\quad$ ② $\overset{?}{=} 0$





limit (1)



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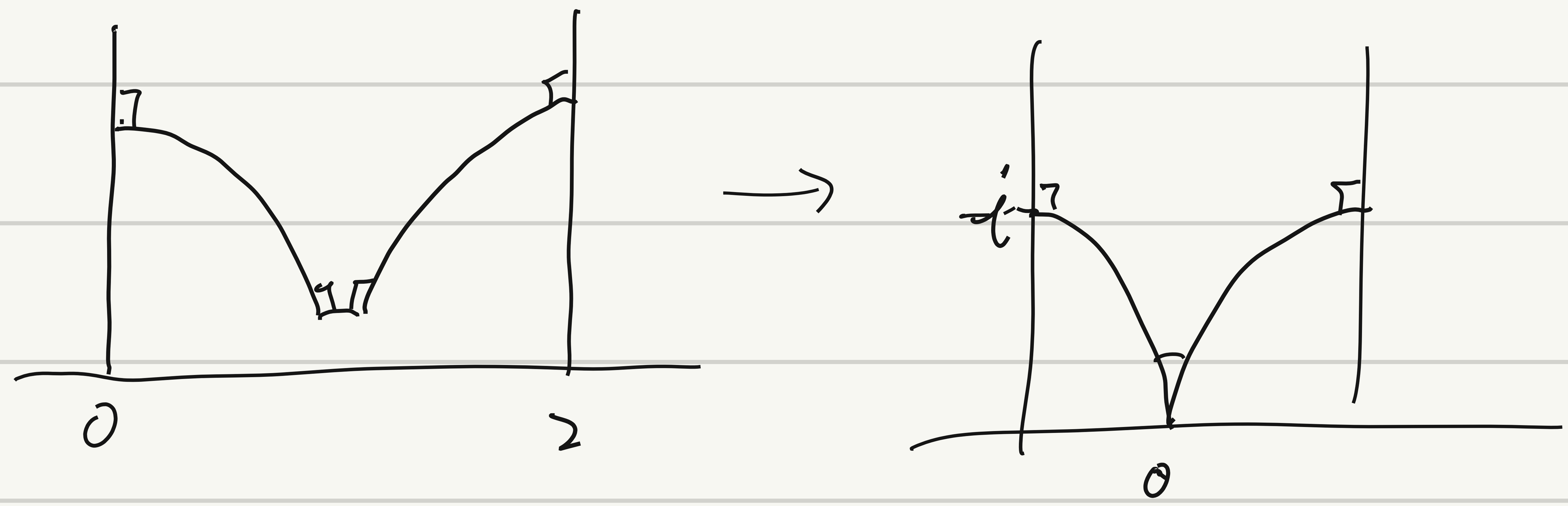
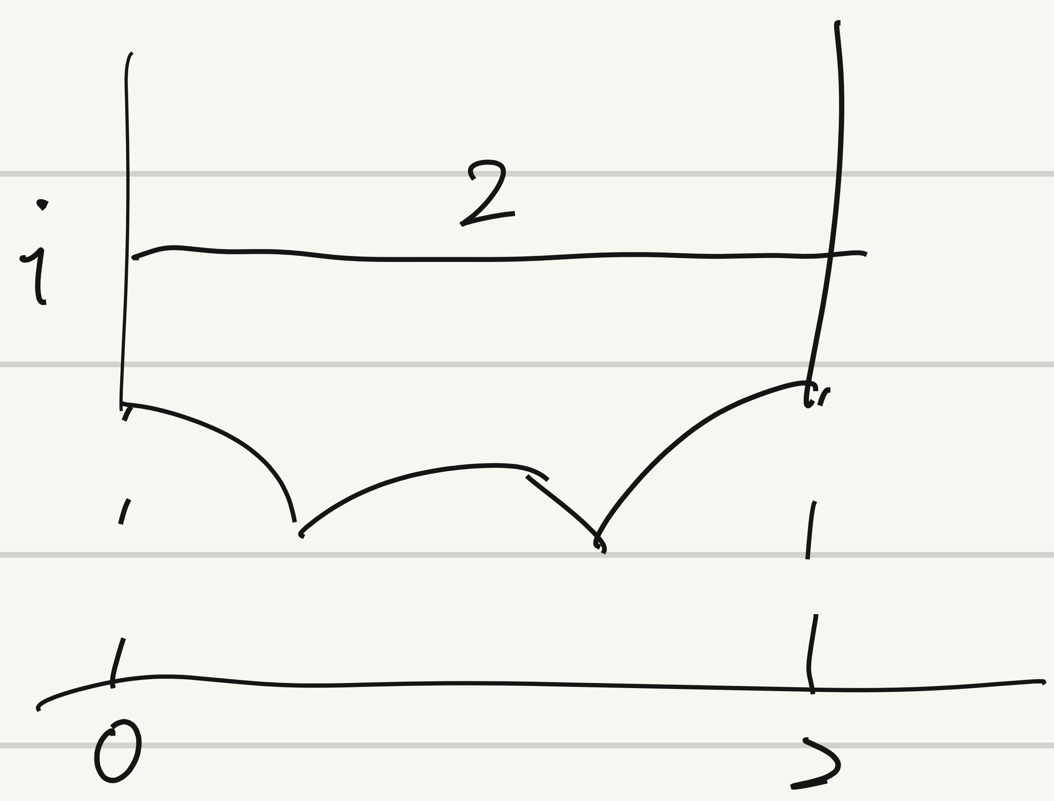
limit (2)





$\ell(H) = 4 \qquad Leb(H) = 4$

$$\underline{\underline{Z}} \subset \bigcup \{ \gamma \text{ simple geod in } S_{1,1} \} \cap H$$
$$\quad \text{//}$$
$$BS_0(S_{1,1})$$

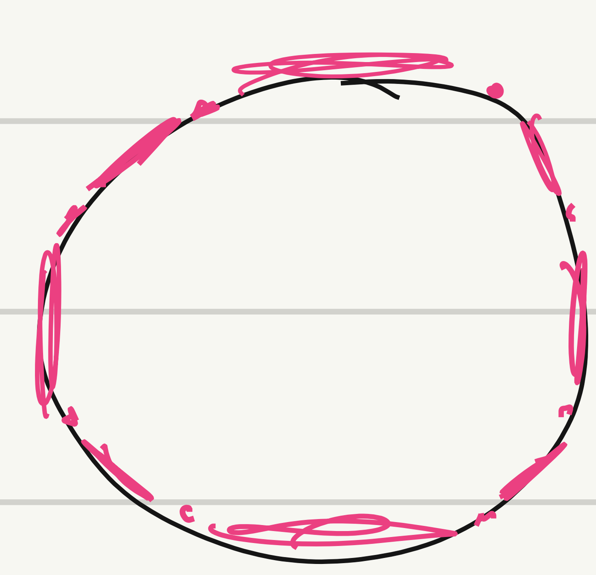
<u>Birman - Series</u>: $BS_0(S_{1,1})$ has Hausdorff dimension 1

$BS_0(S_{1,1}) \cap K$ has Hausdorff dim 0.

K: geod seg transvers to $BS_0(S_{1,1})$

$\Rightarrow Leb(BS_0(S_{1,1}) \cap K) = 0.$

<u>Coro</u>: $Leb(Z) = 0$



$$4 = Leb(H) = \sum_{\gamma \text{ s.c.g}} \underline{Gap(\gamma)}$$

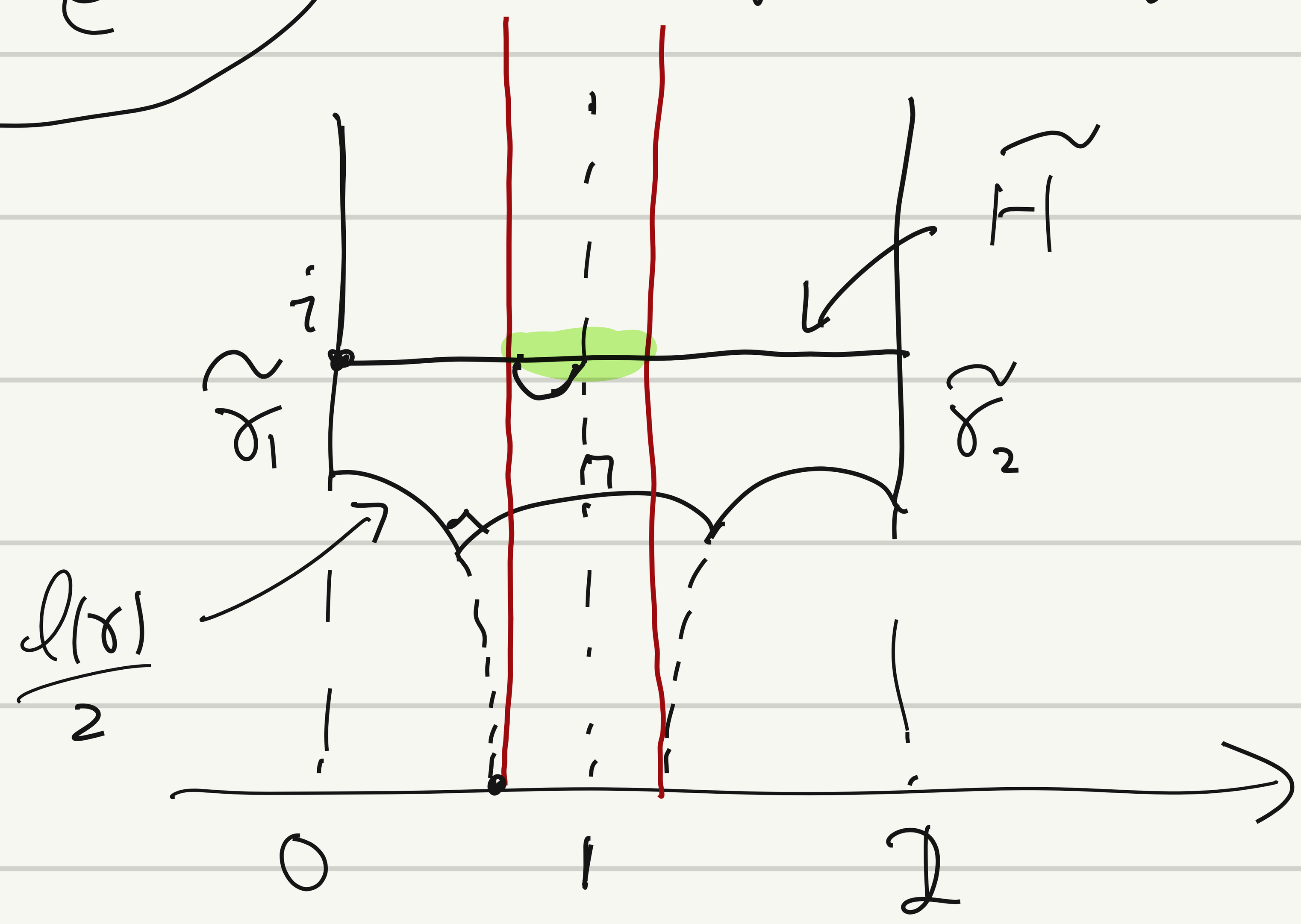
$\sim \ell(\gamma)$ geometrically

<u>Prop</u>: $\frac{1}{4} Gap(\gamma) = \frac{2}{1+e^{\ell(\gamma)}}$

$$4 = \mathrm{Leb}(H) = \sum_{\gamma \text{ s.c.g}} \underline{\mathrm{Gap}(\gamma)}$$

$\sim l(\gamma)$ geometrically

Prop: $\boxed{\frac{1}{4}\mathrm{Gap}(\gamma) = \frac{2}{1+e^{l(\gamma)}}}$



McShane identity: $\underbrace{\sum_{\gamma \text{ s.c.g}} \underbrace{\frac{2}{1+e^{l(\gamma)}}}}_{\text{depend on hyp str on } \Sigma_{1,1}} = \underline{1}$

$\uparrow$ indep. of hyp str on $\Sigma_{1,1}$

$$f(x) = \frac{2}{1+e^{x}}.$$

$S \in \mathcal{T}(\Sigma_{1,1})$.

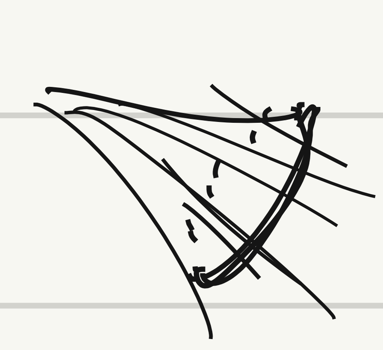
$$F(S) = \sum_{\gamma \text{ s.c.g}} f(l_S(\gamma)) : \mathcal{T}(\Sigma_{1,1}) \longrightarrow \mathbb{R}.$$

$l_S(\gamma)$ = S-length of γ.

$F \equiv 1$

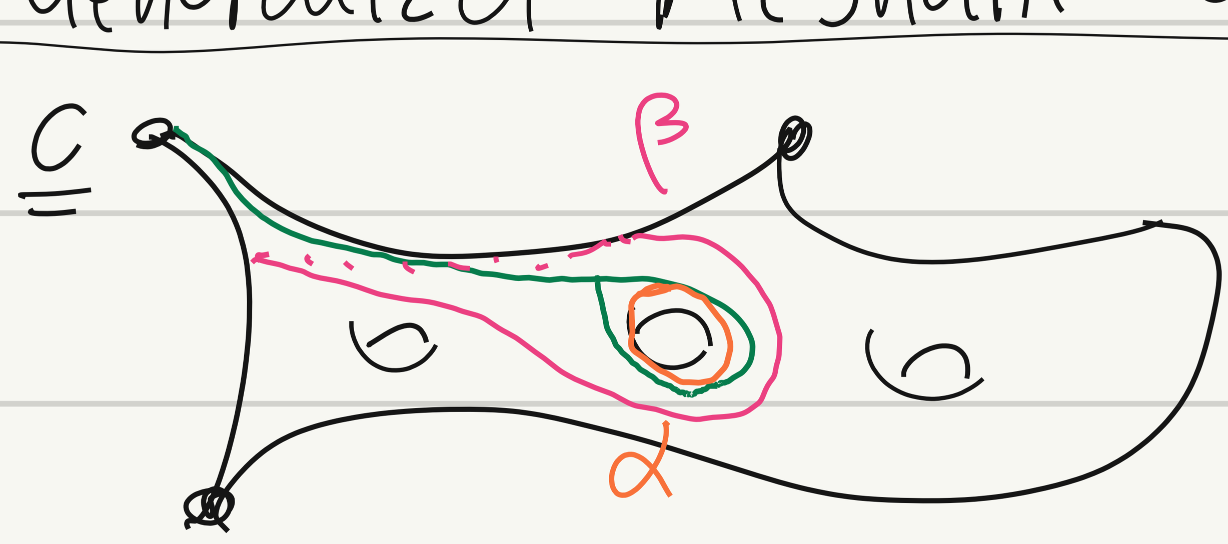
$$f'(x) = -\frac{2e^{x}}{(1+e^{x})^{-2}}$$

Cor: $\not\exists\, S_1\ S_2 \in \mathcal{T}(\Sigma_{1,1})$ s.t. $\forall \gamma\ l_{S_1}(\gamma) > l_{S_2}(\gamma)$

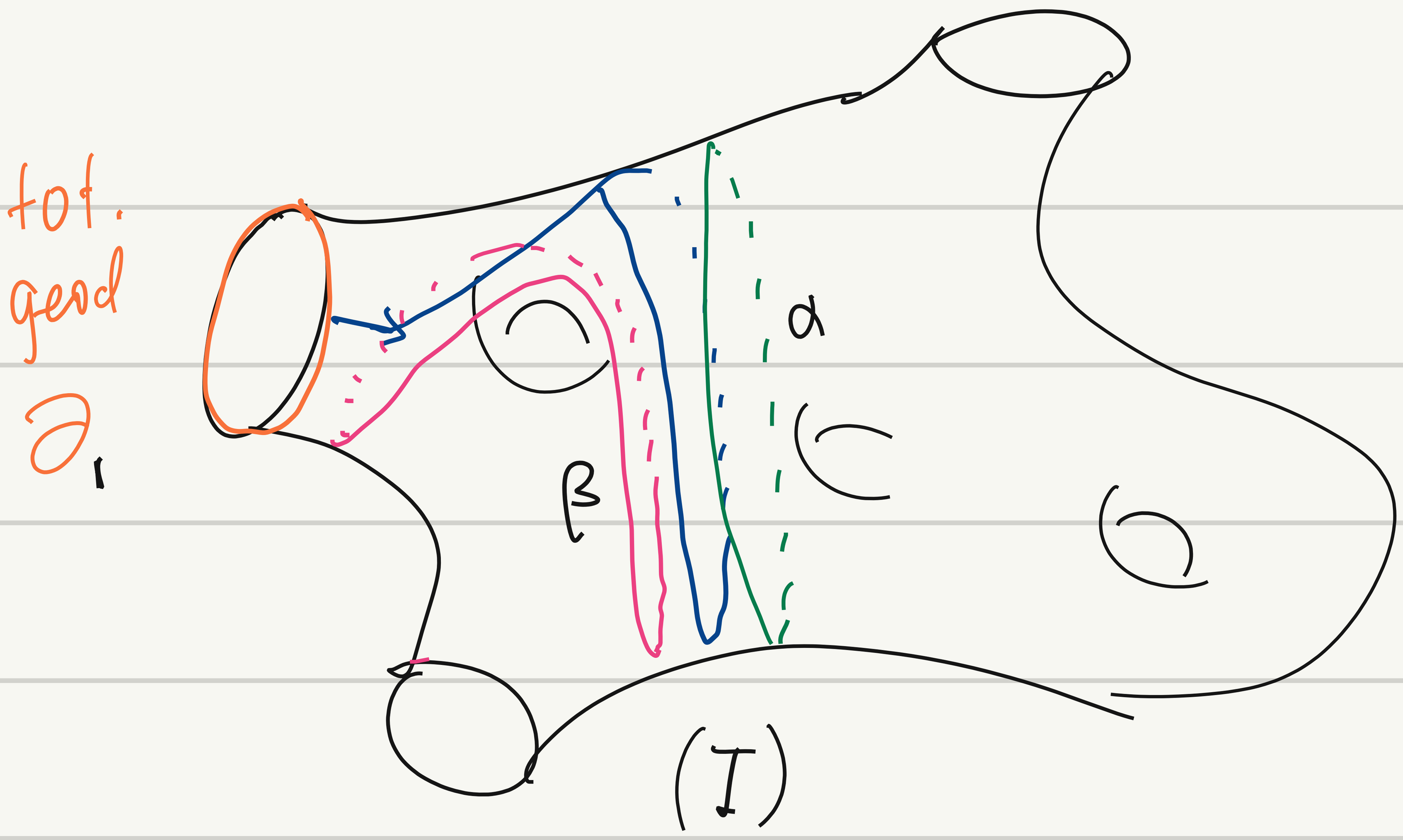


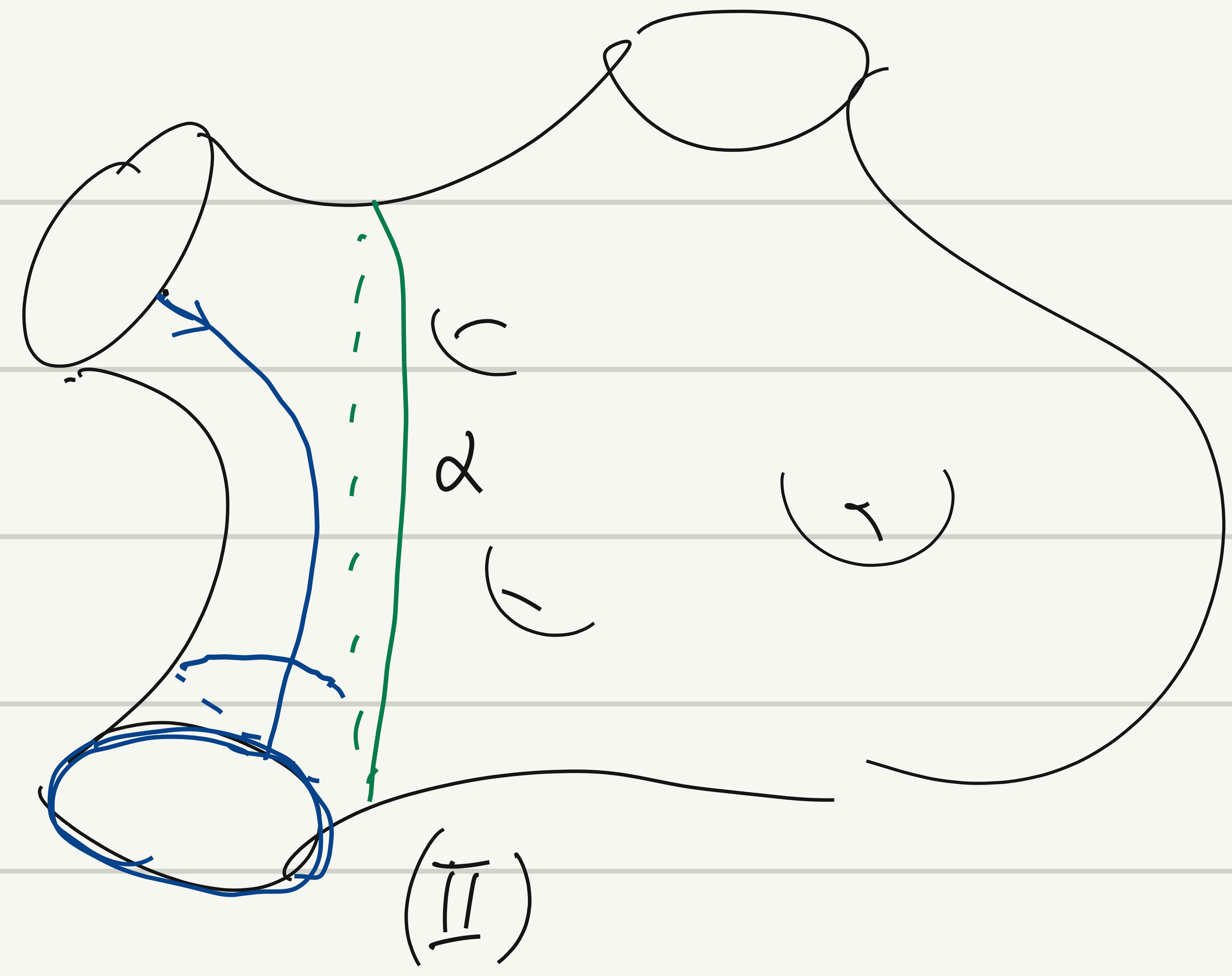
3. More identies:

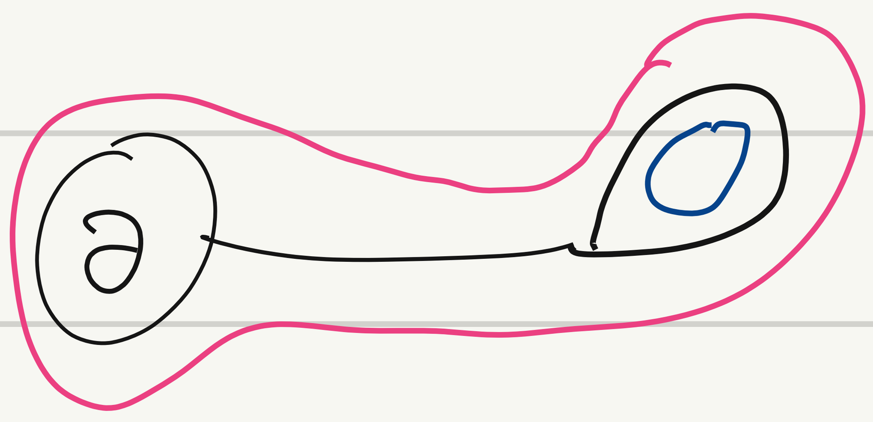
① Generalized McShane id.



(McShane): $\sum_{(\alpha,\beta)} \frac{1}{1+e^{\frac{l(\alpha)+l(\beta)}{2}}} = \frac{1}{2}.$







(Mirzakhani)

$$\sum_{\alpha,\beta}^{\overbrace{\partial_1\ \partial_2 \cdots \partial_n}} D(L_1, \ell_S(\alpha), \ell_S(\beta)) + \sum_{j=2}^{n}\sum_{\alpha} R(L_1, L_j, \ell_S(\alpha)) = L_1$$

For ∂_1,

⋮

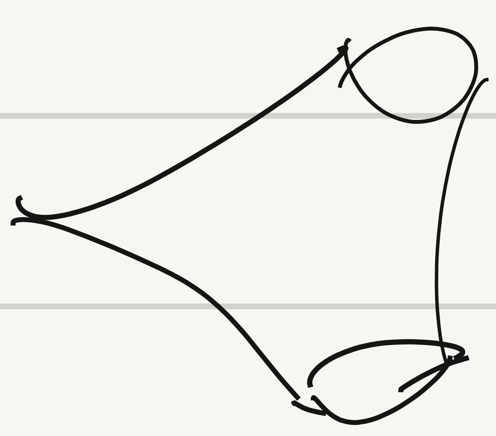
∂_j,

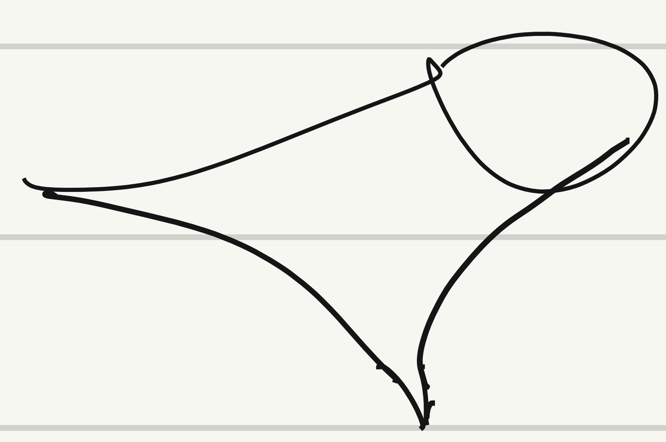
(I) $\quad L_1 = \ell_S(\partial_1)$ $\quad$ $\ell_S(\alpha)$: s-length of α $\quad$ $\ell_S(\beta)$: s-length of β

$$\overset{L_1}{D}(x, y, z) = 2\log \frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{-\frac{x}{2}} + e^{\frac{y+z}{2}}}$$

$$\overset{L_1\ L_j}{R}(x, y, z) = x - \log \frac{\cosh\frac{y}{2} + \cosh\frac{x+z}{2}}{\cosh\frac{y}{2} + \cosh\frac{x-z}{2}}.$$

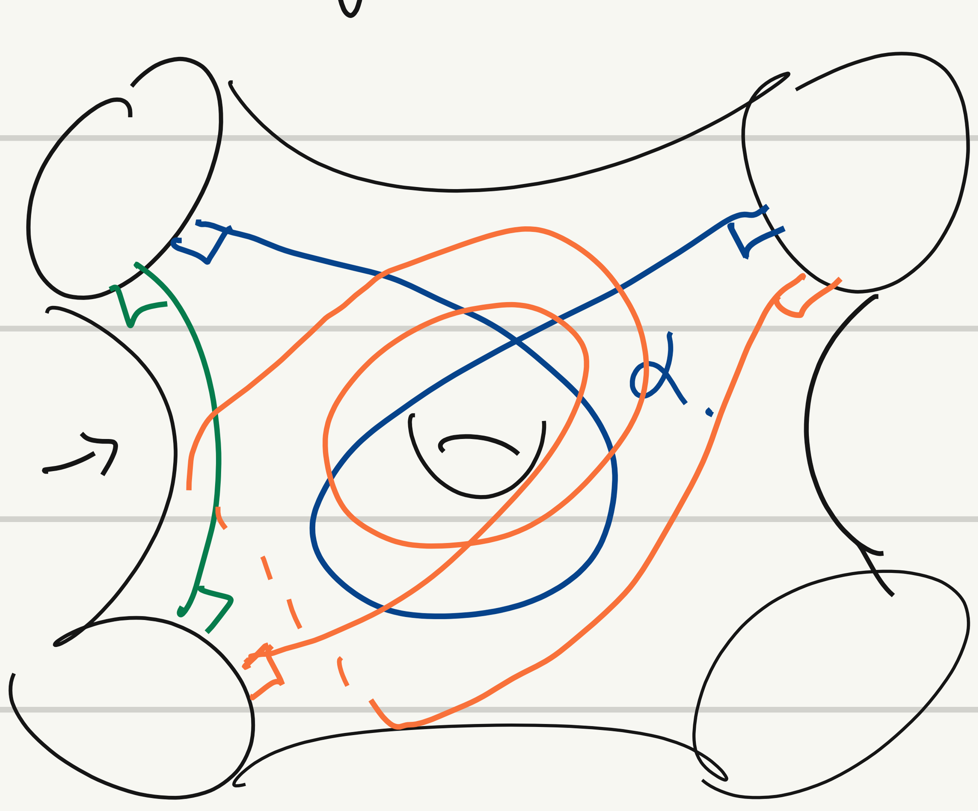
$$\left.\frac{\partial D(x,y,z)}{\partial x}\right|_{x=0} = \frac{2}{1+e^{\frac{y+z}{2}}} \qquad \left.\frac{\partial R(x,0,z)}{\partial x}\right|_{x=0} = \frac{2}{1+\frac{z+0}{2}}$$





② Basmajian id. $S_{g,n}$.

orthogeodesic. $\mathcal{O} = \{$ orthogeodesics on $S\}$



$$\sum_{\alpha\in\mathcal{O}} 2\log\left(\coth\frac{\ell(\alpha)}{2}\right) = \ell(\partial)$$

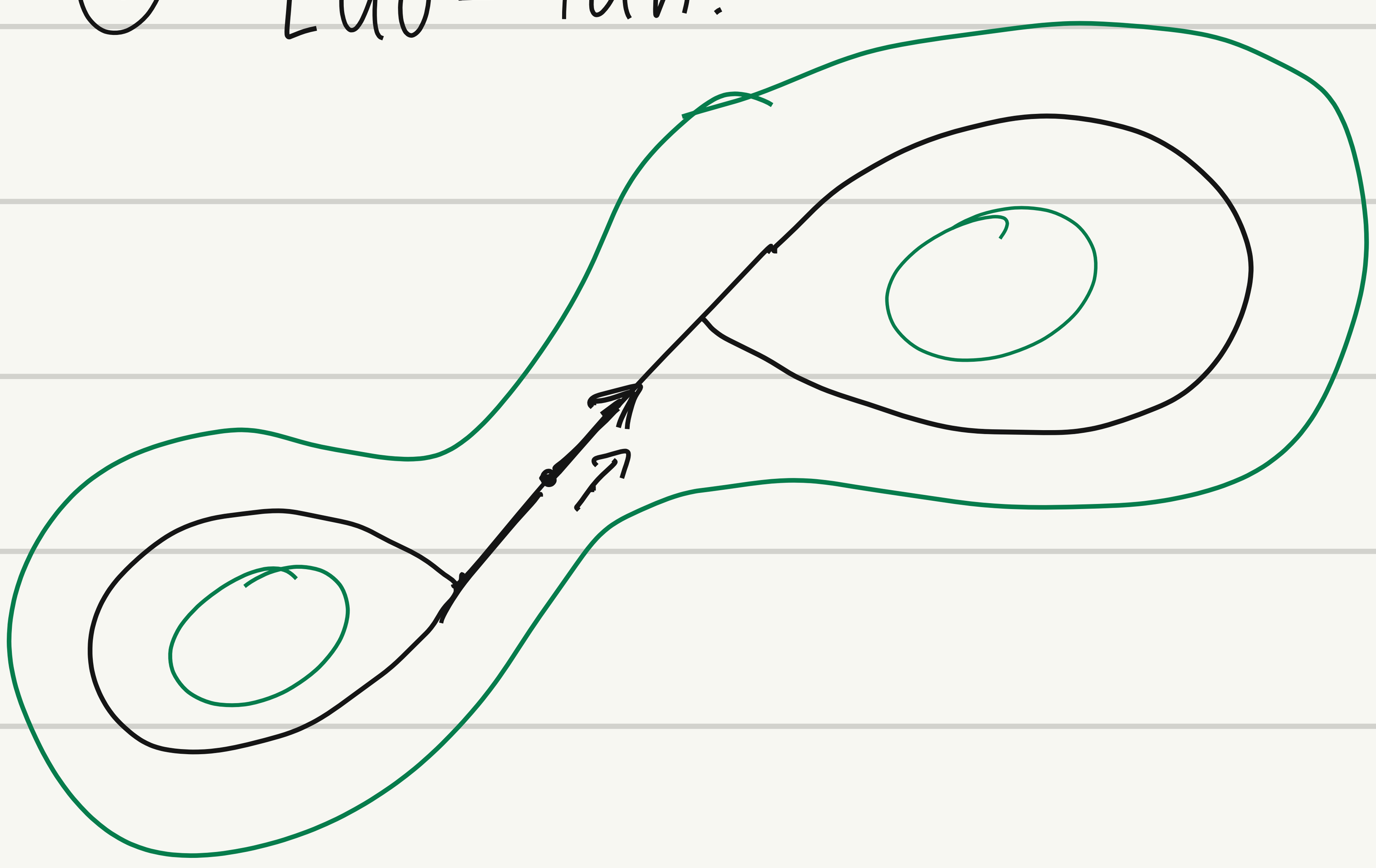
$= \ell(\partial_1) + \ell(\partial_2) + \cdots + \ell(\partial_n)$

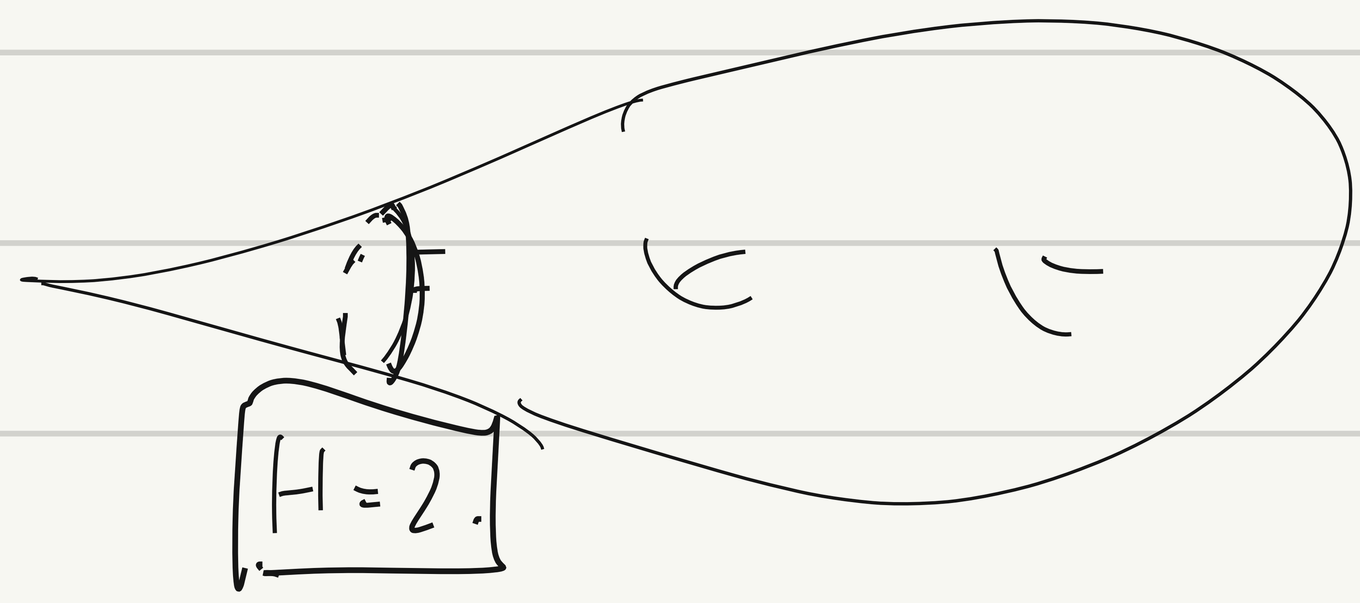
③ Bridgeman

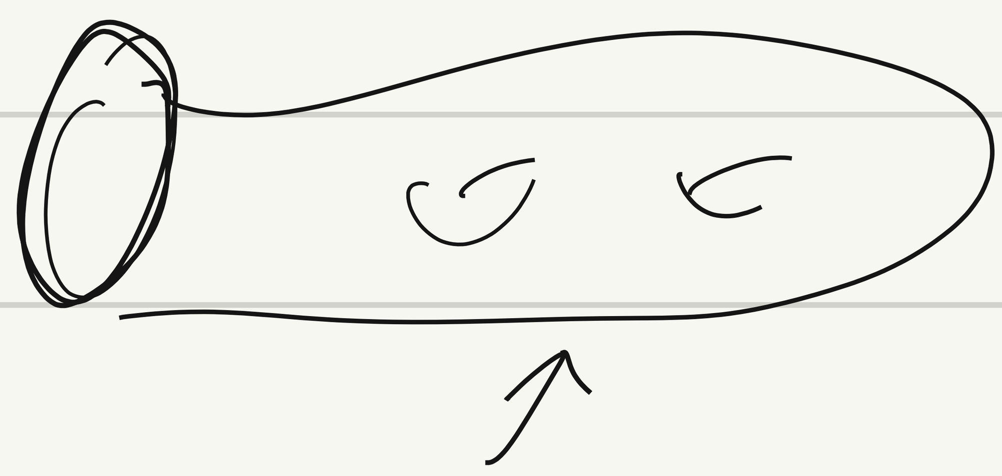
④ Luo-Tan.

$$\mathrm{Vol}(T_1 S) = 2\pi \times 2\pi |\chi(S)|$$

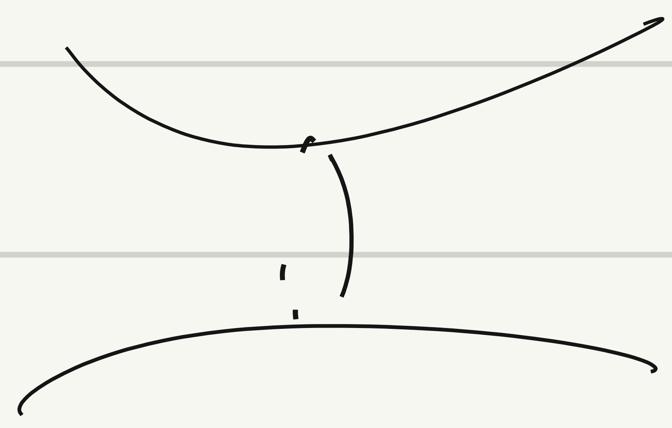
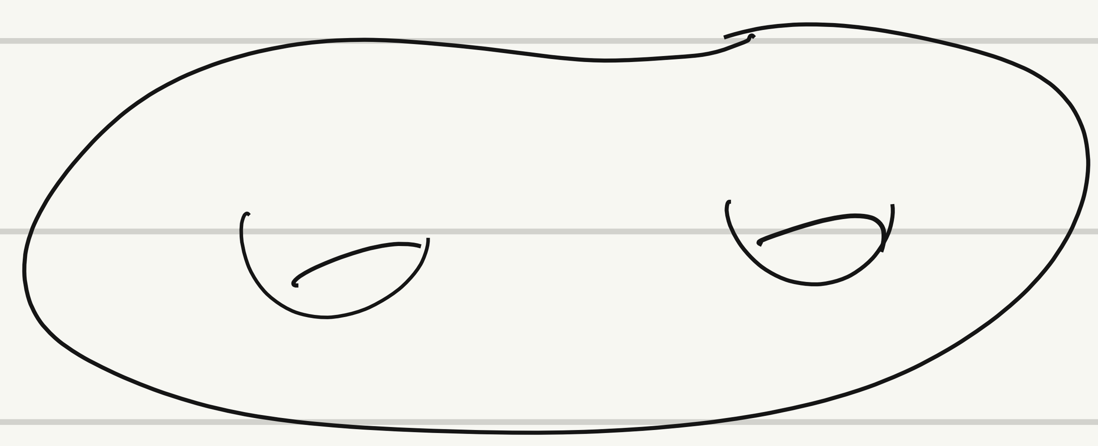
unit tangent boundle.







X Counting Geodesics on Hyperbolic Surface.

$\forall L > 0.$

$\# \left\{ \begin{array}{l} \gamma \text{ closed geod} \\ \text{on } S \\ \ell(\gamma) < L \end{array} \right\} < +\infty$

$$\underline{N_S(L)} := \# \{ \gamma \text{ closed geod on } S \ \ \ell(\gamma) < L \}.$$

$N_S(L) \nearrow \quad L \nearrow \qquad N_S(L) \sim f(L) \quad L \to \infty$

$$\lim_{L \to \infty} \frac{N_S(L)}{f(L)} = 1$$

$e^x + x^2 \quad x \to \infty$

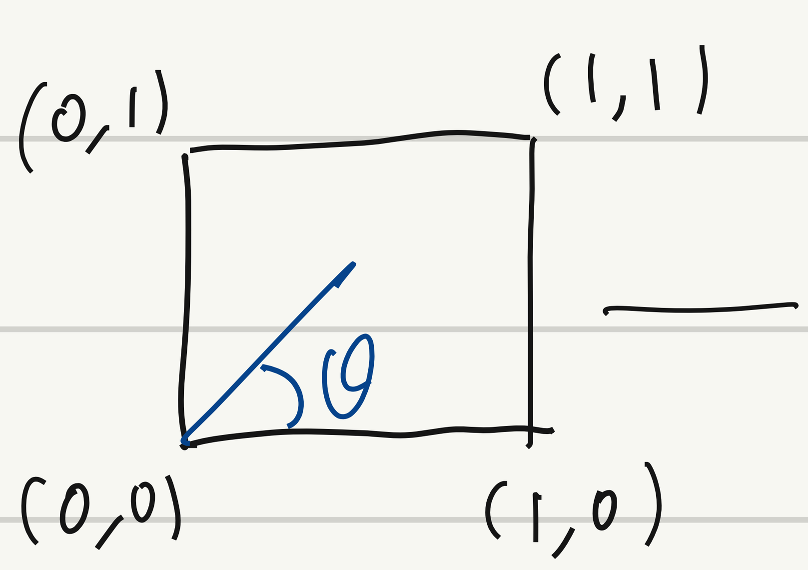
$e^x \sim e^x + x^2, \ x \to \infty$

$A(x) \asymp B(x)$, as $x \to a$.

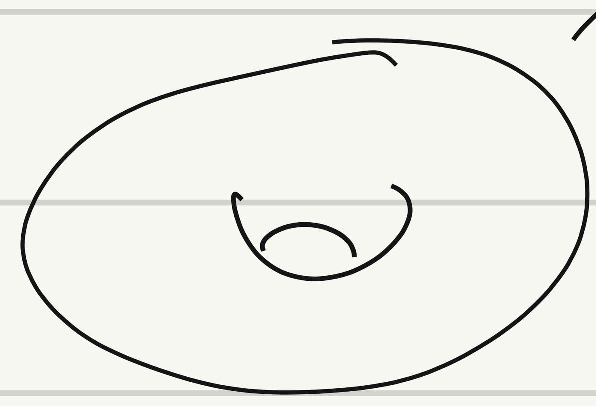
$\exists\, \varepsilon > 0 \quad \forall x \in]a-\varepsilon, a+\varepsilon[\quad \exists\, C_1\ C_2 > 0$

$$C_1 < \frac{A(x)}{B(x)} < C_2$$

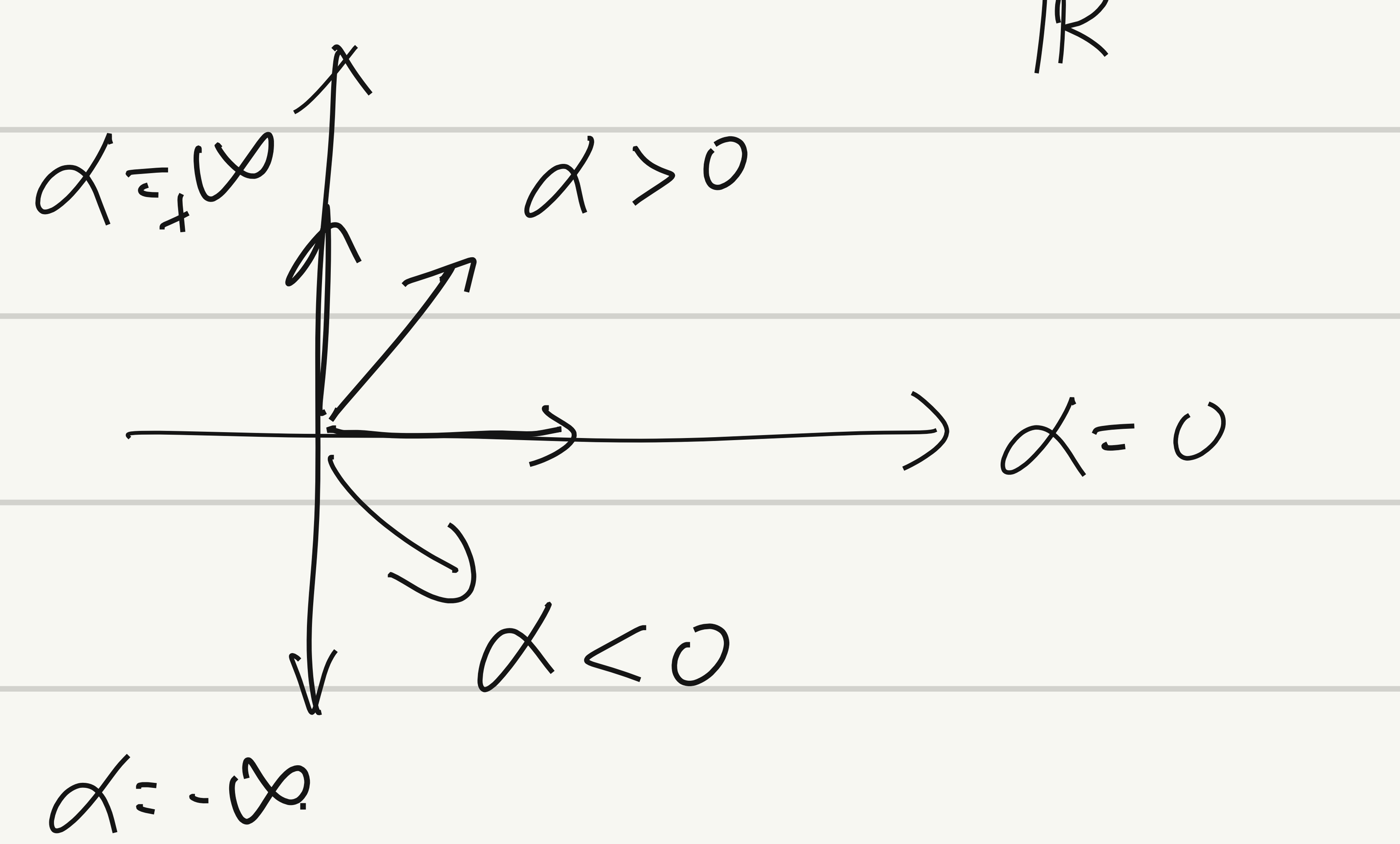
1. Flat torus:



$= T$



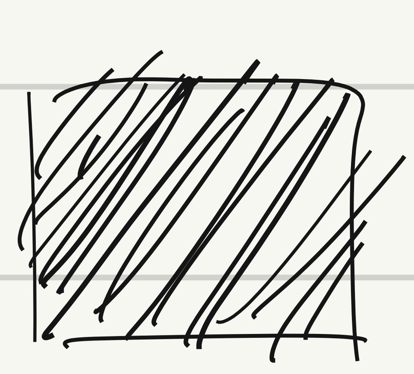
$\mathbb{R}^2$

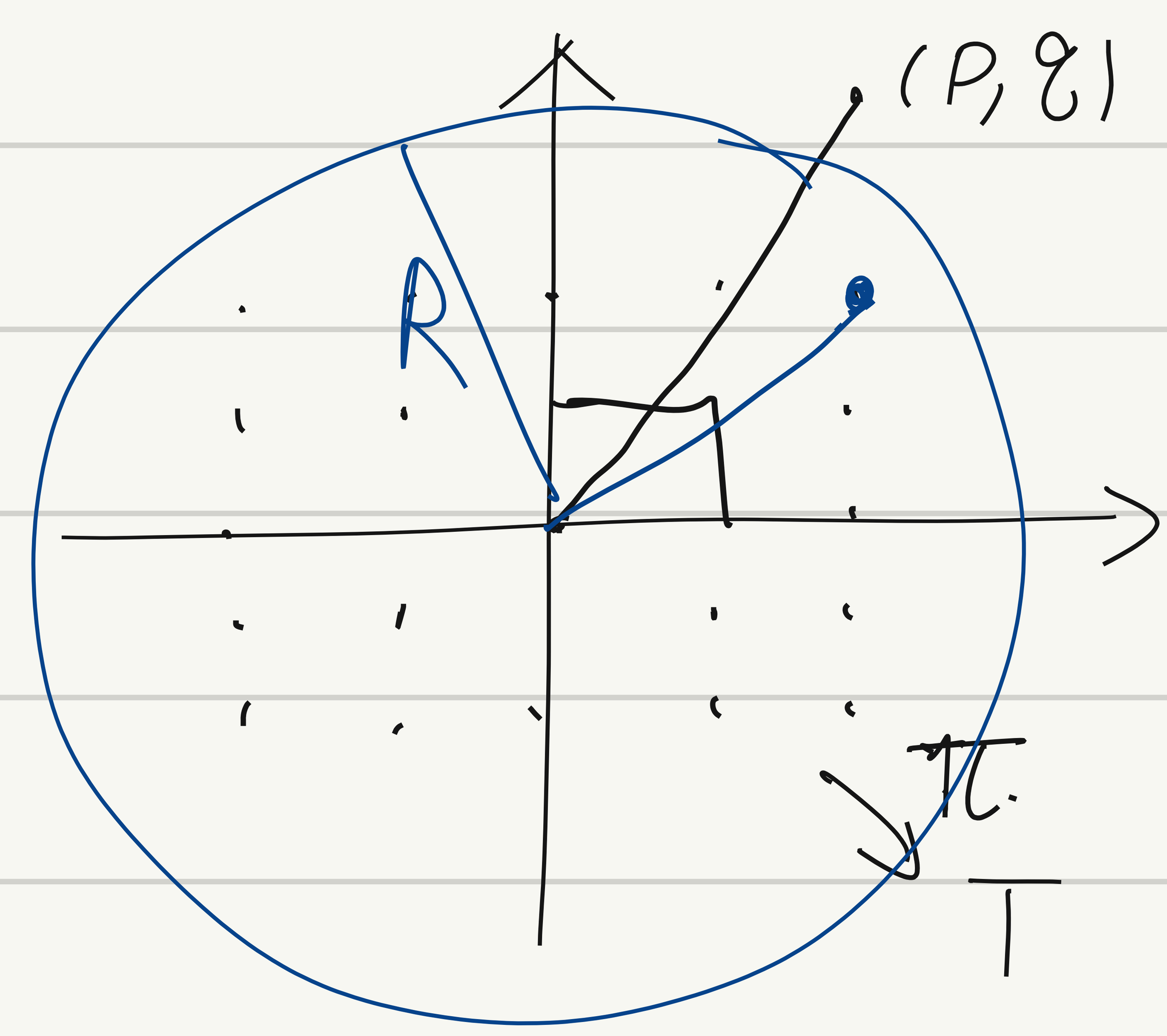


slope $\alpha = \tan\theta \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

$\alpha \in \mathbb{R}$ | γ_α geod in T along α-direction.
γ_α closed iff $\alpha \in \mathbb{Q}$

if $\alpha \notin \mathbb{Q}$





① All geod are simple. ② γ_α closed iff $\alpha \in \mathbb{Q}$.

$\gamma(p,q) = \pi([(0,0),(p,q)])$ ③ $\gamma(p,q)$. primitive iff $\gcd(p,q) = 1$

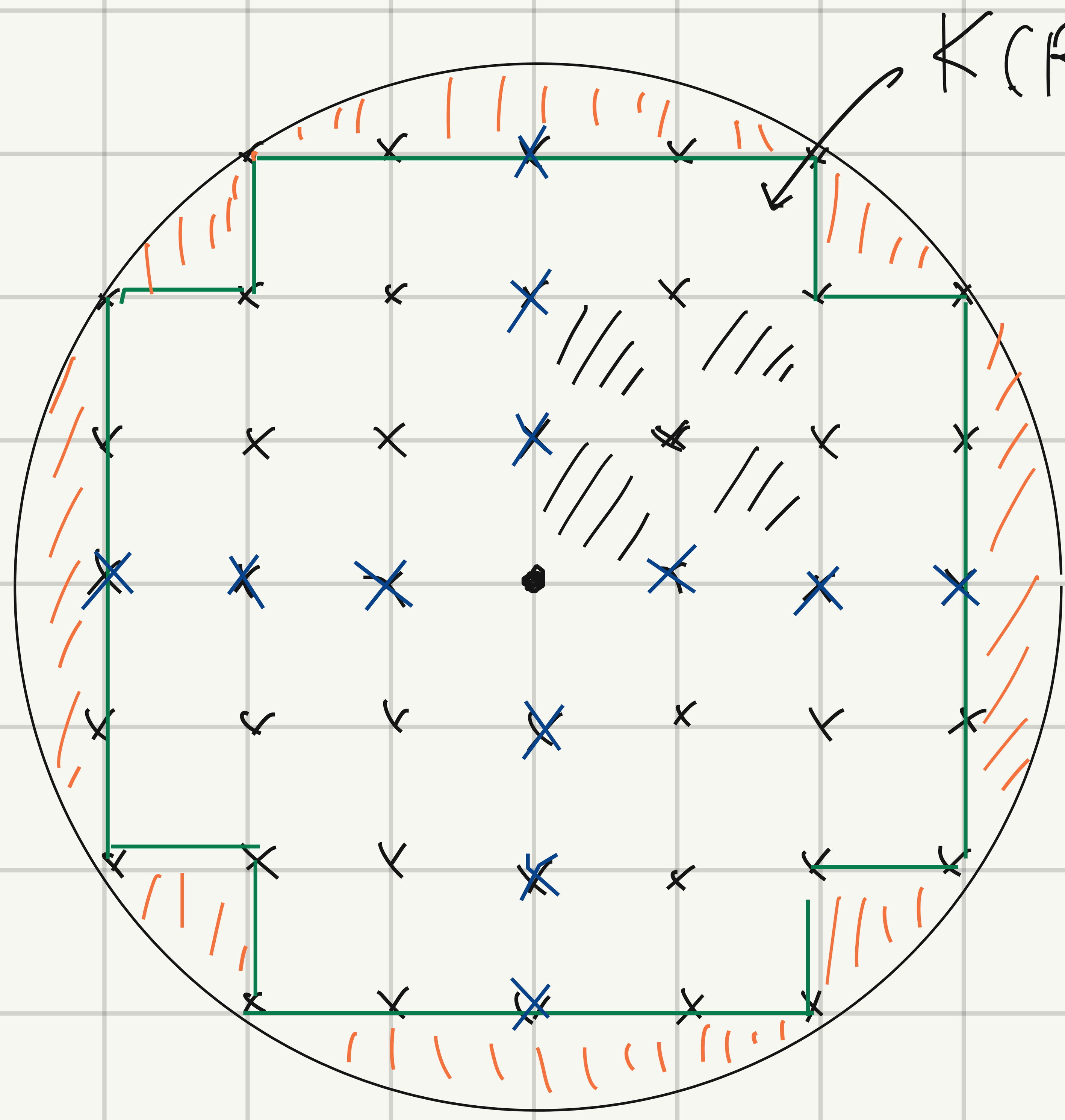
$\mathcal{G}(T) = \{\gamma \text{ closed primitive geodesic (unoriented) on } T\}$

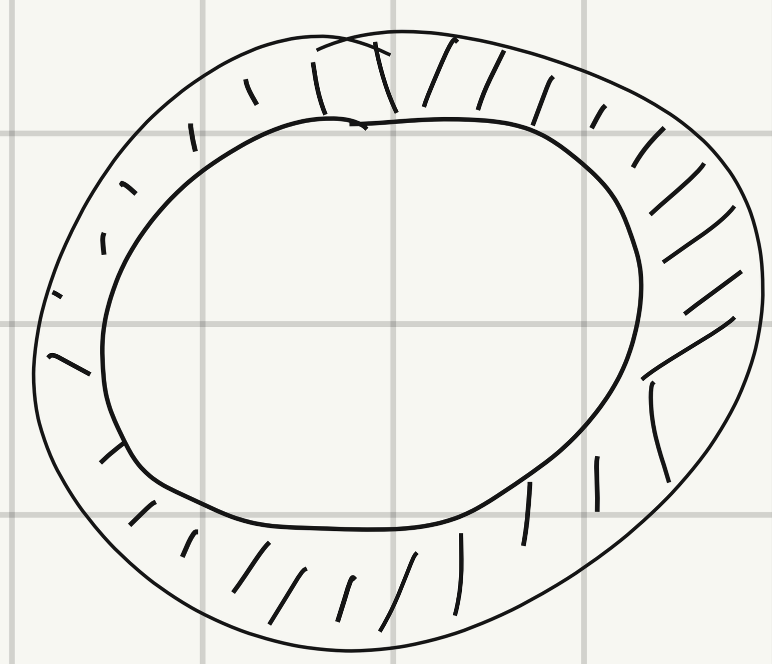
$\forall L > 0, \quad C_T(L) = \#\{\gamma \in \mathcal{G}(T) \mid \ell(\gamma) < L\}$

Q : $C_T(L) \sim \underline{\underline{f(L)}} \quad (L \to \infty)$

K(R)

D(R) $\to$





$$\underline{A_E(K(R))} + 4\lfloor R \rfloor$$

$$\sim A_E(D(R)) + 4\underline{\underline{R}}, \qquad R \to \infty$$

diam $\square = \sqrt{2} \Rightarrow A_E(D(R)) - \sqrt{2}R \le A_E(K(R)) \le A_E(D(R))$

$$A_E(K(R)) \sim A_E(D(R)) = \pi R^2$$

$$\pi R^2 + R \sim \pi R^2 \qquad R \to \infty$$

$C_T(L) = ?$ $\qquad \gcd(p,q) = 1$

Rmk: $\dfrac{\{-N, -N+1, \ldots, N-1, N\} \cap p\mathbb{Z}}{\{-N, \ldots, N\}} \to \dfrac{1}{p}$ as $N \to \infty$

(p, q). $k \mid \gcd(p,q) \iff k \mid p \wedge k \mid q$.

$(p,q) \in k\mathbb{Z} \times k\mathbb{Z}$.

$$\frac{\left(D(R) \cap (k\mathbb{Z})^2\right)}{\left(D(R) \cap \mathbb{Z}^2\right)} \to \frac{1}{k^2}$$

$6 = 2 \times 3 \qquad (6\mathbb{Z})^2 \subseteq (2\mathbb{Z})^2 \cap (3\mathbb{Z})^2$

Let $\{p_1, p_2, \ldots, p_n, \ldots\} = \mathbb{P}$ prime number...
$\quad\;\; 2 \quad 3 \;\ldots$ indexed by their order on $\mathbb{R}$.

$$\frac{\# D(R) \cap \{(m,n) \in \mathbb{Z}^2 \mid p_1 \nmid \gcd(m,n), \ldots, p_k \nmid \gcd(m,n)\}}{\# D(R) \cap \mathbb{Z}^2}$$

$$\to \prod_{j=1}^{k} \left(1 - \frac{1}{p_j^2}\right), \qquad R \to \infty$$

$$\frac{\# D(R) \cap \{(m,n) \in \mathbb{Z}^2 \; \gcd(m,n) = 1\}}{\# D(R) \cap \mathbb{Z}^2}$$

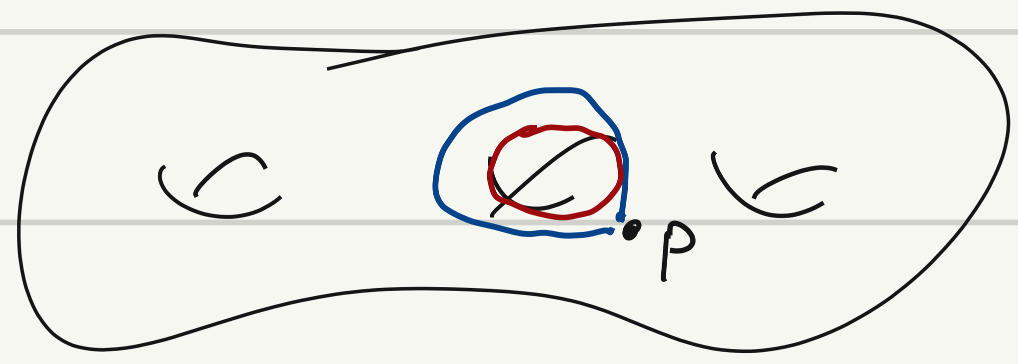
$$\to \prod_{j=1}^{\infty} \left(1 - \frac{1}{p_j^2}\right) = \zeta(2)^{-1} = \frac{6}{\pi^2}.$$

$(R \to \infty)$

Hence. $C_T(R) \sim \frac{6}{\pi^2} \cdot (\pi R^2) = \frac{6}{\pi} R^2$ as $R \to +\infty$.

$C_T(L) \sim \frac{6}{\pi} L^2$, as $L \to +\infty$.

2. Hyperbolic surface S_g.

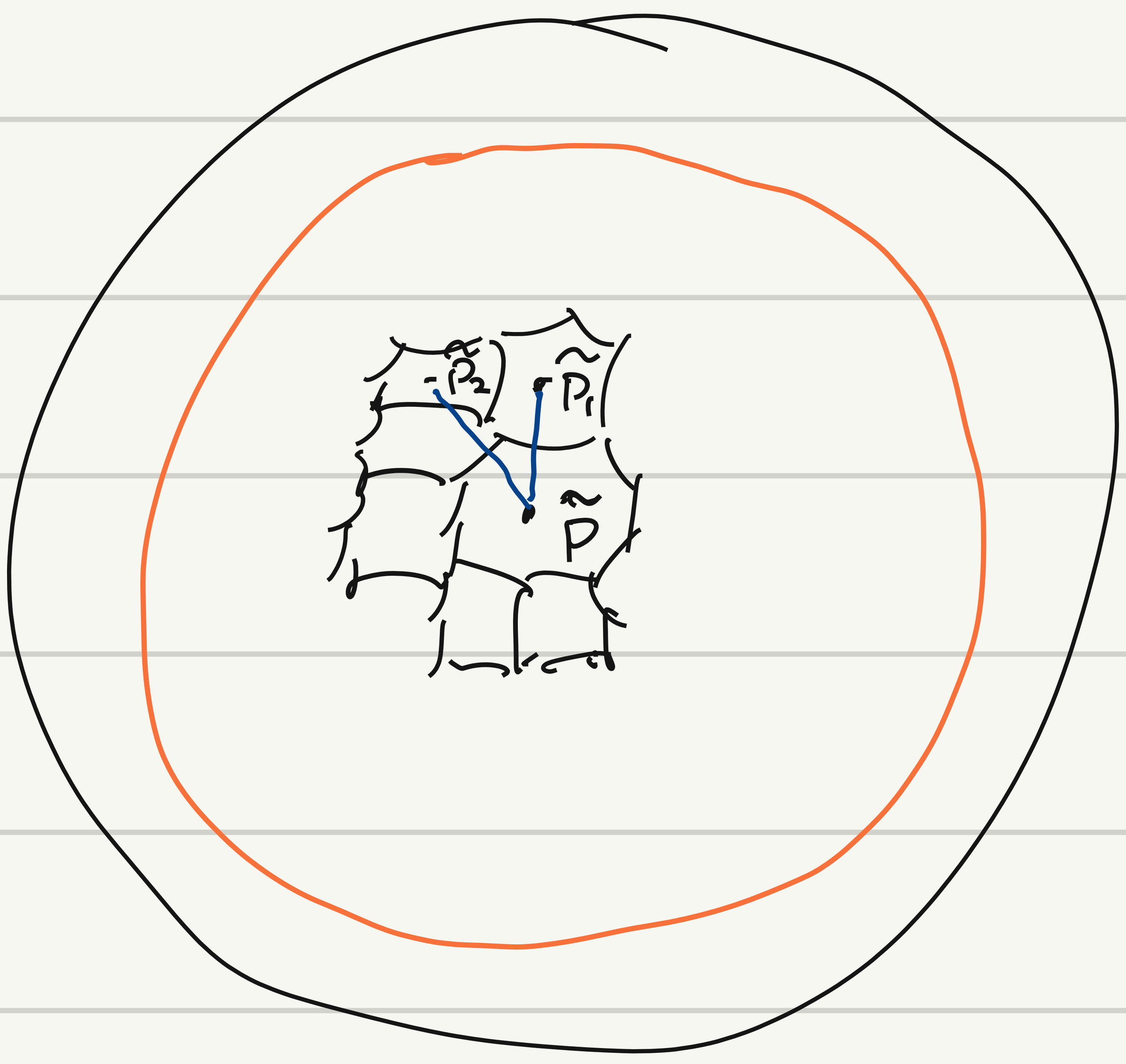


$$\#\{\underline{A(D)} \mid A \in \Gamma,\ A(D) \subset D(\hat{p}, R)\}$$

$$\sim \frac{A_{H^1}(D(\hat{p}, R))}{A_{H^1}(D)} = \frac{2\pi(\operatorname{ch}R - 1)}{2\pi(2g-2)}$$

$$\sim C \cdot \underline{e^R}, \quad R \to \infty$$

constant.



$\boxed{C \cdot R^2}$

Problem : $\exists$ overcounting.

① $\exists\ W \sim W'$ conj.

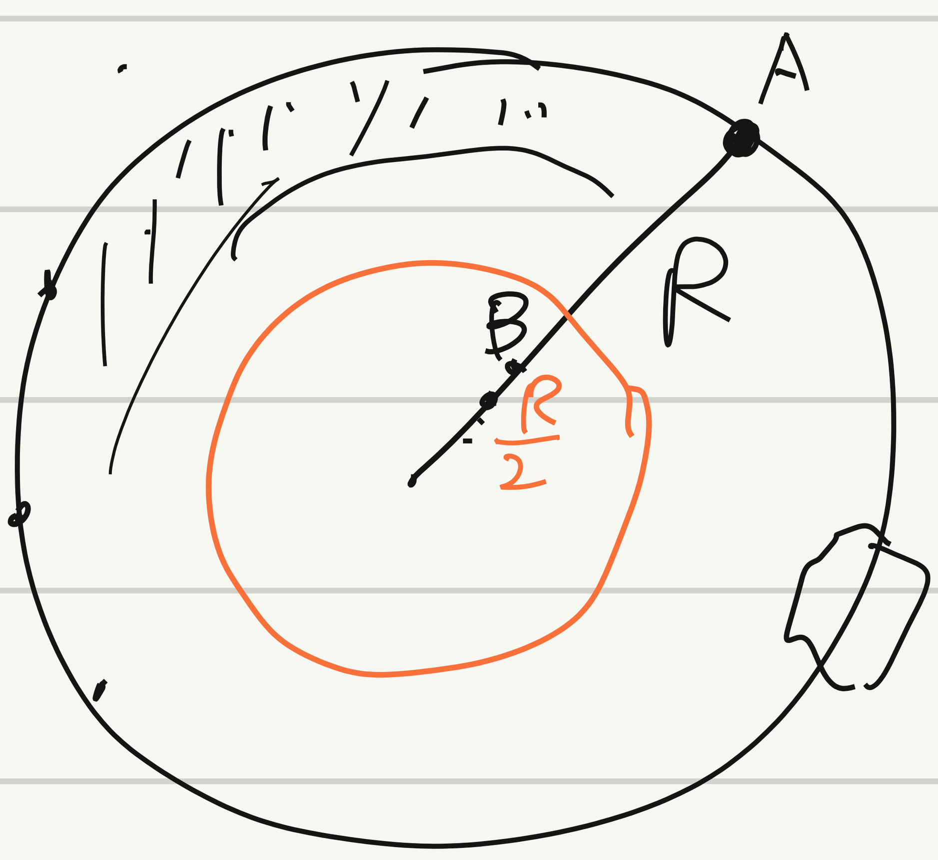
② $\exists$ non-primitive.

$W(A_1, \dots, A_n) \in \Gamma$.

γ.

$\underline{AWA^{-1}} \sim \underline{\gamma}$

②



$2\pi \sinh R$

$2\pi(\cosh R - 1)$

$A = B^2$.

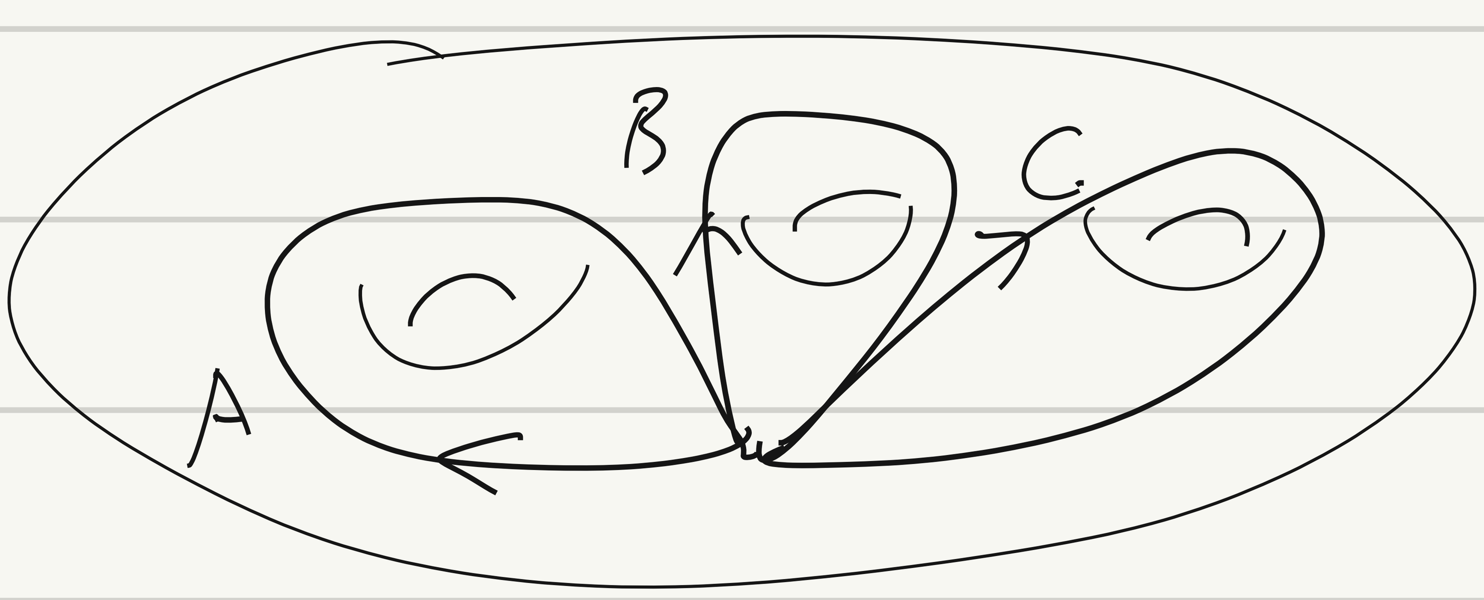
$e^R - e^{\frac{R}{2}} \sim e^R$, as $R \to R$.

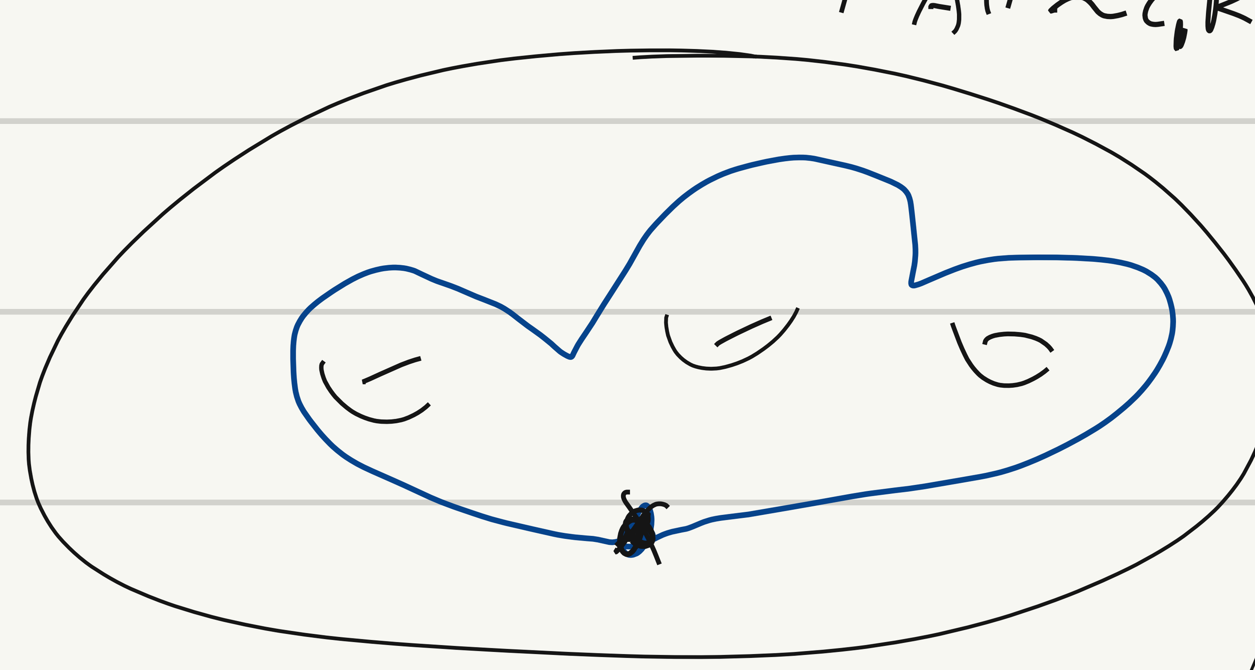
① $\underline{ABC} \to \underline{BCA} \to \underline{CAB}$

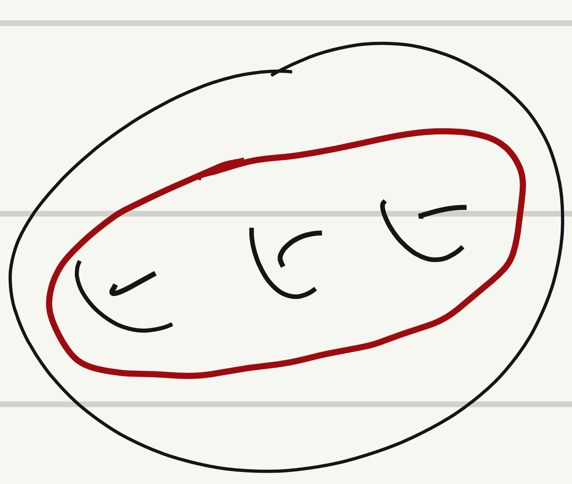
$A(D) \cap C(\hat{p}, R) \neq \phi$

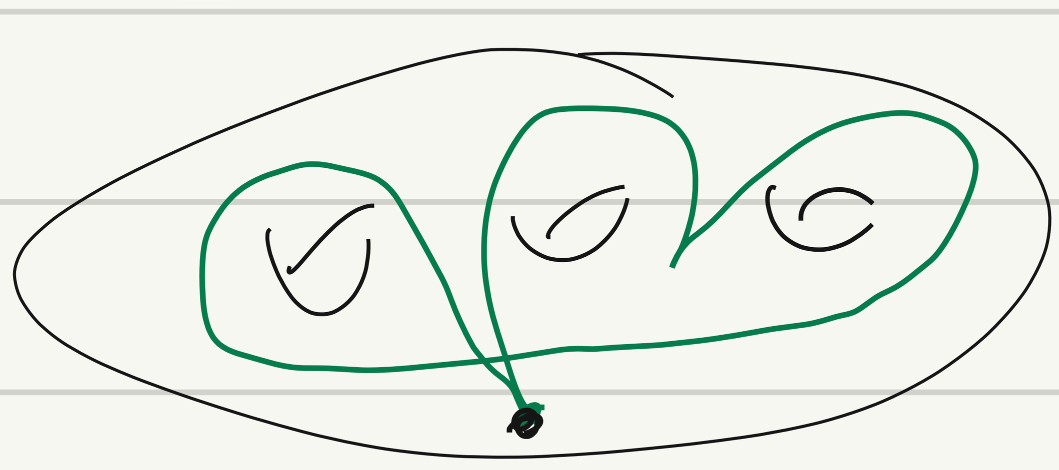
$A = W(A_1 .. A_n)$

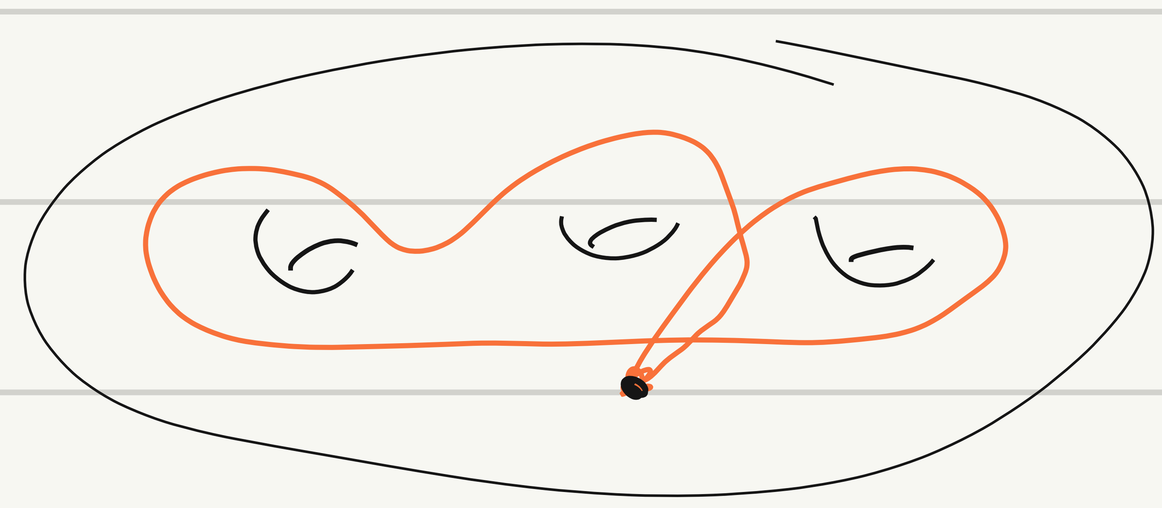
$\|A\| \sim_c R$











$\underline{\dfrac{e^R}{R}}$.

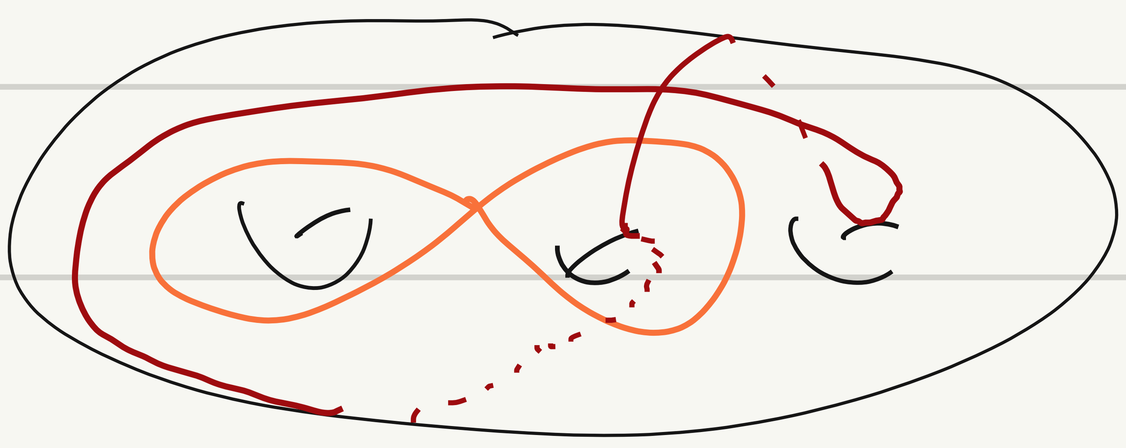
Thm: (Prime geodesic thm)

$$C_{S_g}(L) = \#\{\gamma \text{ closed primitive geod in } S_g \;\; \ell(\gamma)\le L\}$$

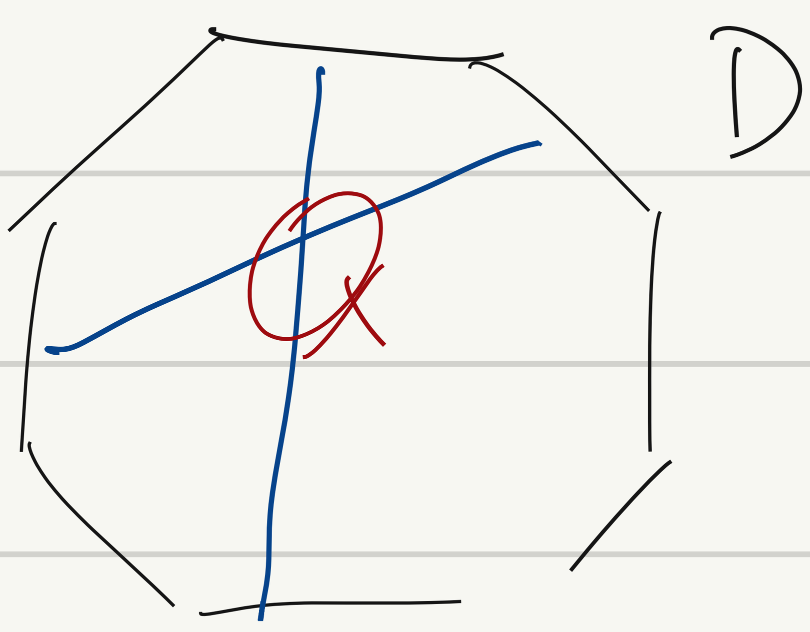
$$\sim \frac{e^R}{R}, \quad \text{as } R\to\infty$$

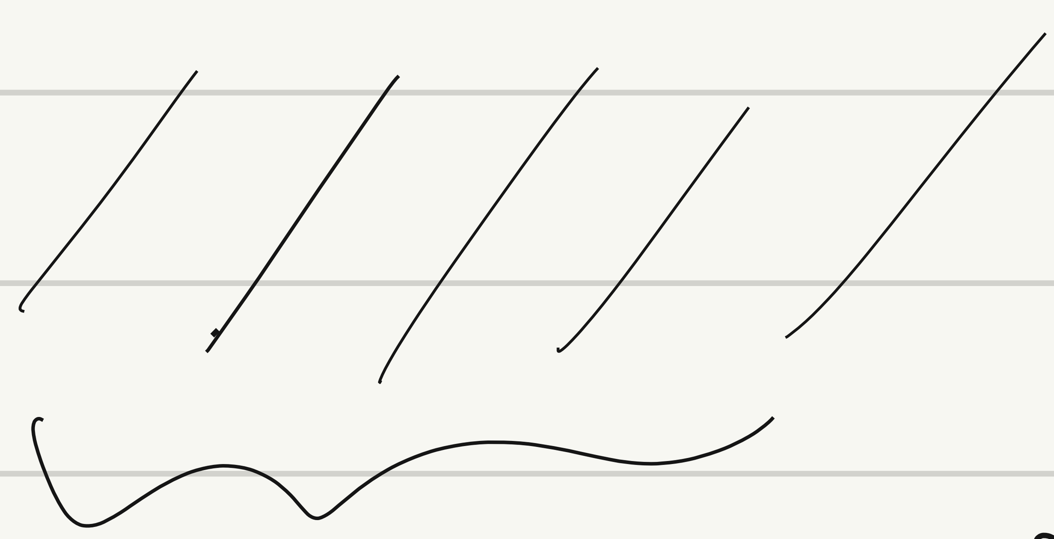
Rmk: (Prime number thm): $\pi(x)\sim \frac{x}{\log x}$ as $x\to\infty$.

- Huber.
- Margulis mixing.
- Bowen-Ruelle.
- Parry-Pollicott.

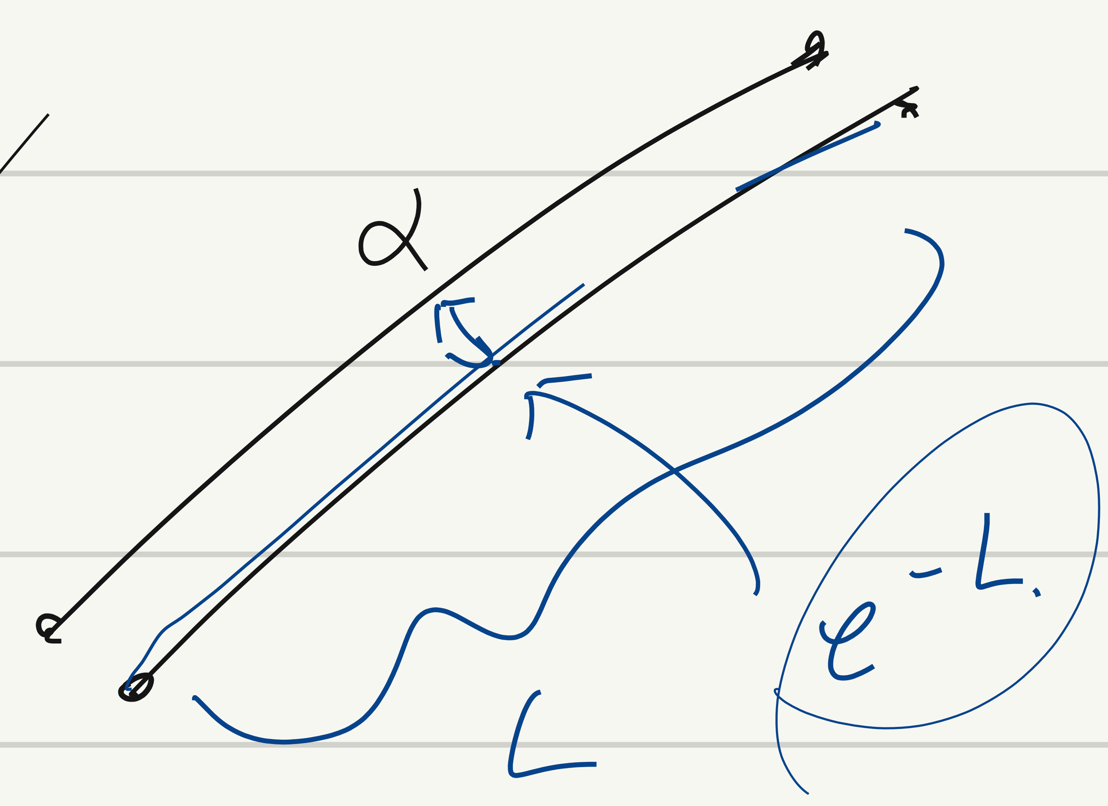


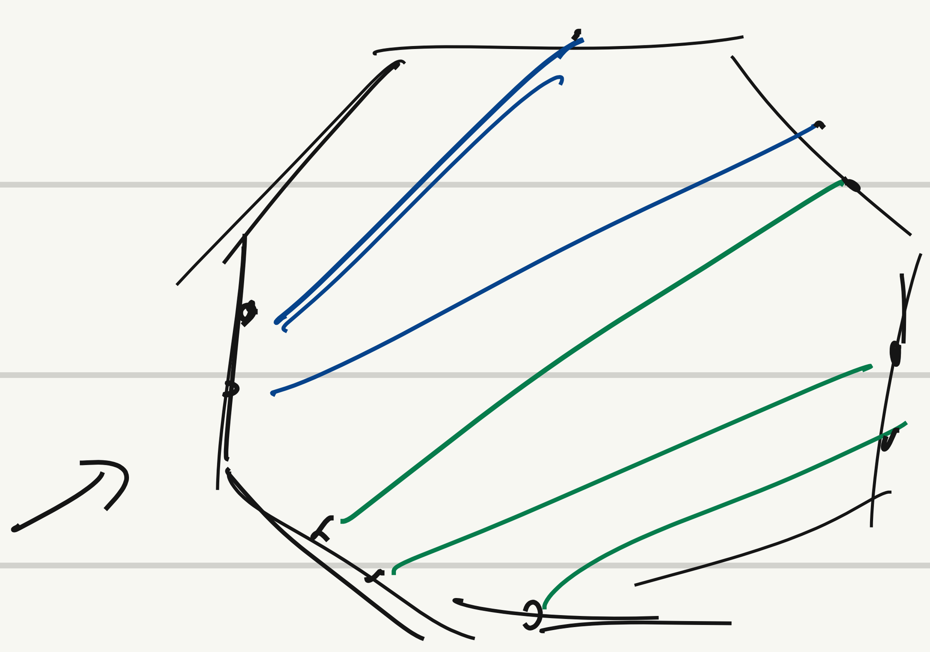
3. Simple geodesic on S_g.

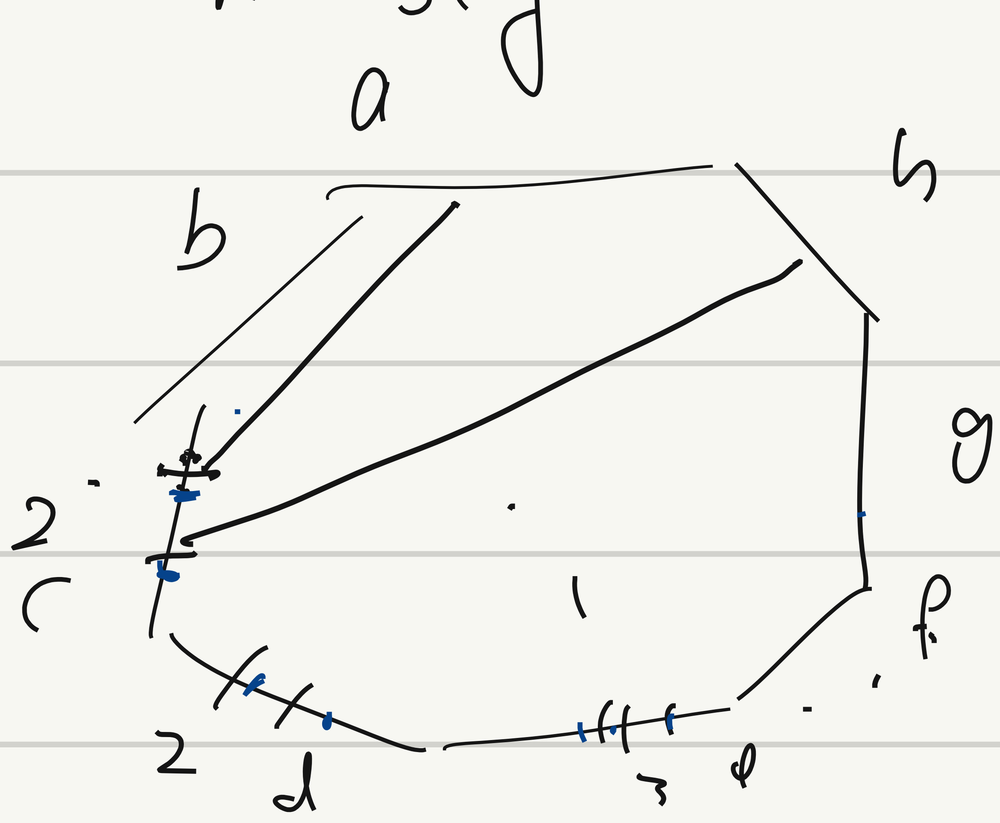




k seg.







$$\#\{\gamma, \#\text{seg} < L\} \le P(L).$$

Thm (Mirzakhani) For any γ simple closed geod on S_g.

$$N_{S_g}(L,\gamma) := \{\eta \in \mathrm{Mod}(S_g).\gamma,\; \ell_S(\eta) < L\}$$

$$\lim_{L\to\infty} \frac{N_{S_g}(L,\gamma)}{L^{6g-6}} = \frac{c(\gamma)\, B(S_g)}{b_g}$$

Thm (Mirzakhani) For any γ simple closed geod on S_g

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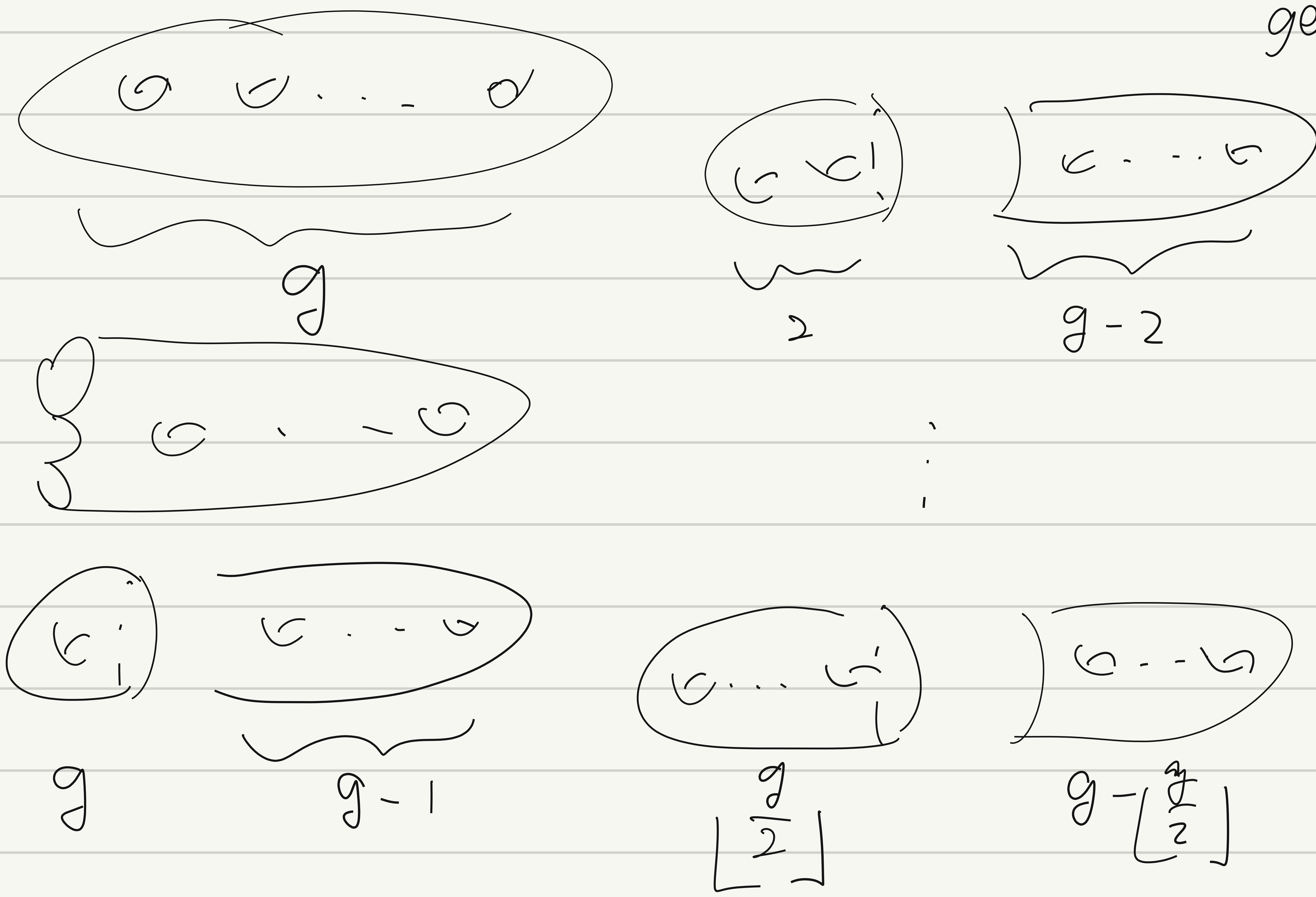
$c(\gamma) \in \mathbb{Q}^+$. $\quad b_g = \int_{\mathcal{M}_g} B(S_g)\, d\mathrm{Vol}_{wp}$

$B: \mathcal{T}(S_g) \to \mathbb{R}^+$ primitive

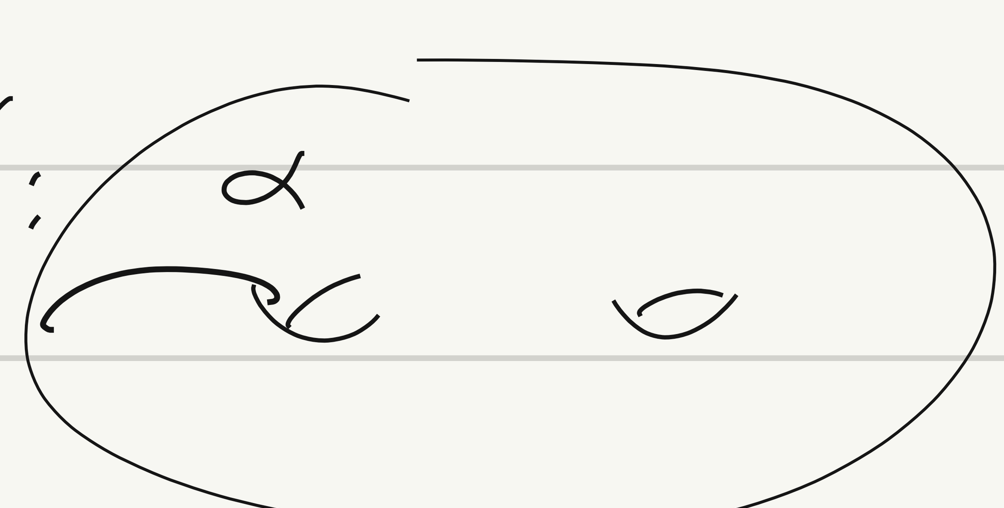
Rmk: Up to $\mathrm{Mod}(S_g)$, only finitely simple closed geodesic.



$$C^{prim}_{S_g}(L) \sim c\cdot L^{6g-6}, \quad (L\to\infty)$$

$$C_{S_g}(L) \sim \frac{e^L}{L}.$$

Ex:



$$\lim_{L\to\infty} \frac{N_{S_g}(L,\alpha)}{N_{S_g}(L,\beta)} = 6 \in \mathbb{Q}.$$