Introduction to hyperbolic surfaces

Exercises IX

Let D denote the open set in $\mathbb{H}$ bounded by three geodesics $V_{1/2}$, $V_{-1/2}$ and $C(0,1)$:

$$D = \{z \in \mathbb{H} \mid \operatorname{Re} z \in \left(-\frac{1}{2}, \frac{1}{2}\right),\ |z| > 1\}.$$

We would like to show that D is a fundamental domain for $\mathrm{PSL}(2,\mathbb{Z})$ action on $\mathbb{H}$. For our convenience, we use $\mathrm{SL}(2,\mathbb{Z})$ during the proof.

1. (Normal) Consider a matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
in $\mathrm{SL}(2,\mathbb{Z})$. Compute the imaginary part of $A(z)$.

2. (Normal) Show that for $z \in D$, we have $|cz + d| \geq 1$.

3. (Normal) Show that for any $A \in \mathrm{SL}(2,\mathbb{Z})$ and any $z \in D$, if $A(z) \in D$, then $A = \pm \mathrm{I}_2$.

4. (Easy) Show that any point $z \in \mathbb{H}$ can always be sent to the region between $V_{1/2}$ and $V_{-1/2}$, i.e. $\operatorname{Re} z \in [-1/2, 1/2]$.

5. (Easy) Use 4. and the matrix
$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
to show that any point $z \in \mathbb{H}$ with $\operatorname{Im} z \geq \sqrt{3}/2$ can always be sent $\overline{D}$ by an element in $\mathrm{SL}(2,\mathbb{Z})$.

6. (Normal) Show that for any point $z \in \mathbb{H}$ with $\operatorname{Re} z \in [-1/2, 1/2]$ and $\operatorname{Im} z < \sqrt{3}/2$, we have
$$\operatorname{Im} B(z) > \operatorname{Im} z.$$

7. (Normal) Show that there exists a constant $\epsilon > 0$, such that for any point z with
$$\operatorname{Re} z \in [-1/2, -1/2] \quad \text{and} \quad \frac{\sqrt{3}}{2} - \epsilon < \operatorname{Im} z < \frac{\sqrt{3}}{2},$$
we have
$$\operatorname{Im} B(z) > \frac{\sqrt{3}}{2}.$$

8. (Hard) Let z be any point in $\mathbb{H}$. We construct a sequence of points in $\mathbb{H}$ in the following way.

a) Check if $z \in \overline{D}$, if yes, stop; otherwise, apply

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix},$$

for some $n \in \mathbb{Z}$ on z, so that $\operatorname{Re} z \in [-1/2, 1/2]$. We denote the new point by z_1. If z_1 is in $\overline{D}$, stop; otherwise go to step b).

b) Apply B and we get $z_2 = B(z_1)$. Check if z_2 is in $\overline{D}$. If yes, stop; otherwise back to a).

Show that this process will stop in finite time and we will get a point in $\overline{D}$.

(Hint: Prove it by contradiction.)

9. (Easy) Conclude that D is a fundamental domain for $\mathrm{PSL}(2, \mathbb{Z})$-action on $\mathbb{H}$.