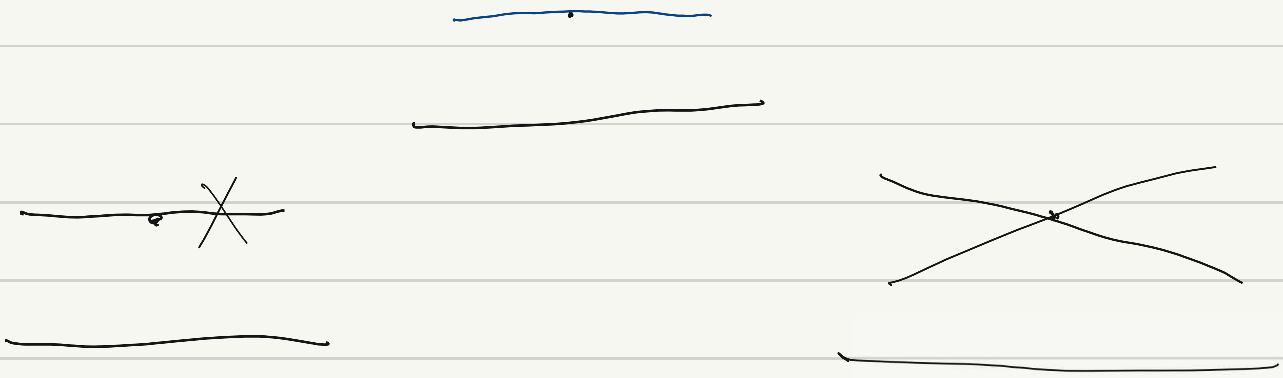
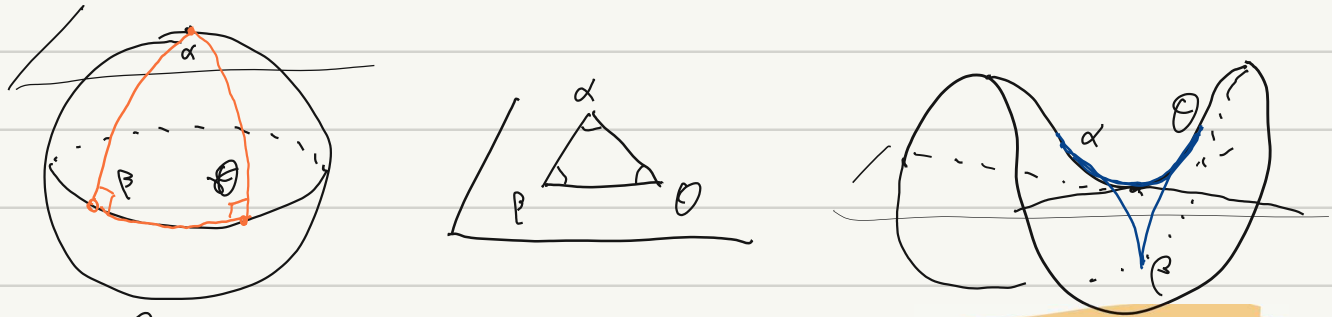
Introduction to hyperbolic surfaces I

0 Introduction.

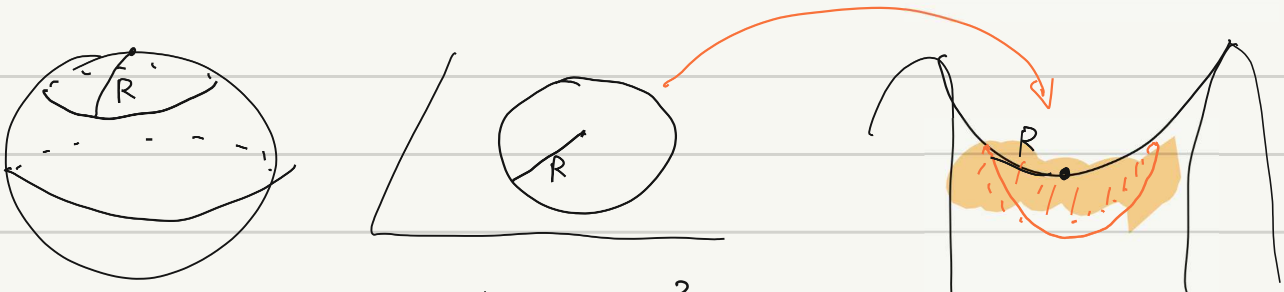


Spherical geom   hyperbolic geom.

$K>0$ $K=0$ $K<0$



$\alpha+\beta+\theta>\pi$ $\alpha+\beta+\theta=\pi$ $\alpha+\beta+\theta<\pi$



$A_S<\pi R^2$ $A_E=\pi R^2$ $A_H>\pi R^2$

Plan:

I 1. Upper half plane model $\mathbb{H}$:

metric, distance, geodesic, 2h circle, horocycle, hypercycle, $\partial\mathbb{H}$ 2h

isometry, Möbius transformation 2h

2. Poincaré disk model $\mathbb{D}$:

Cayley transformation $\mathbb{H} \to \mathbb{D}$ / Stereographic projection.

3. Convex subset of $\mathbb{H}$

4. Polygone : triangles, ideal triangles, trigonometry formulas, polygones. 2h

II 1. Discrete subgroups of $\mathrm{Isom}(\mathbb{H})$

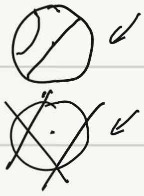
2. Poincaré polygone theorem. 2h

I Hyperbolic plane:

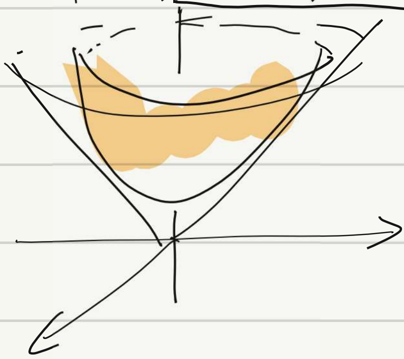
1. Models:

4 used a lot:

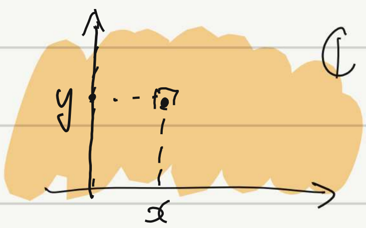
- upper half plane model
- Poincaré disk model $\mathbb{C}$
- Kleinian model $\mathbb{RP}^2$
- Minkowski model.



$\mathbb{R}^{2,1}$ $\boxed{\mathbb{R}^3 \quad \langle x,x\rangle = x_1^2 + x_2^2 - x_3^2}$



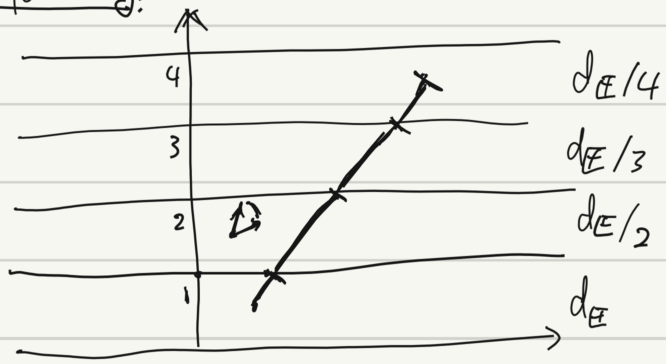
2. Upper half plane $\mathbb{H}$:

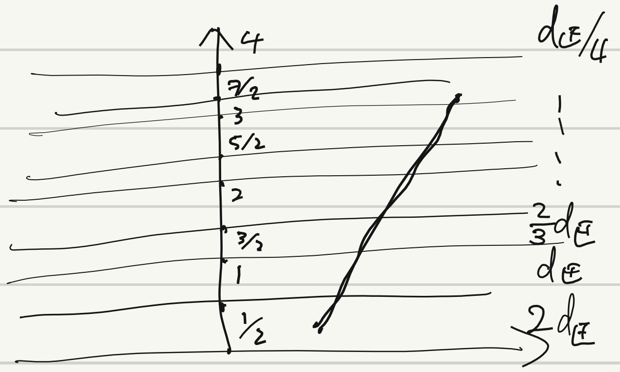


$$\mathbb{H} := \{ z = x + iy \in \mathbb{C} \mid y > 0 \}$$

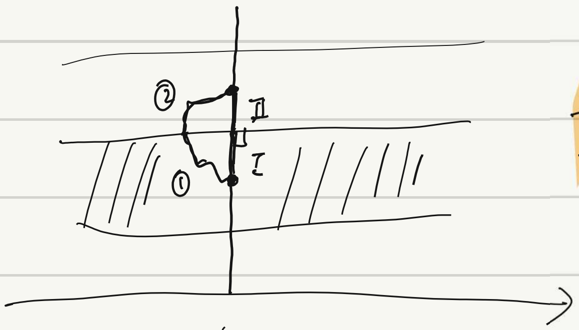
$$ds := \frac{\sqrt{dx^2 + dy^2}}{y} \qquad d\,\mathrm{Vol} := \frac{dx\,dy}{y^2}$$

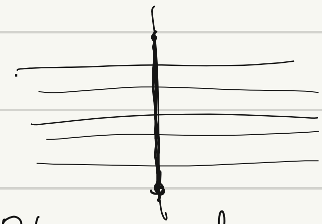
Informally:



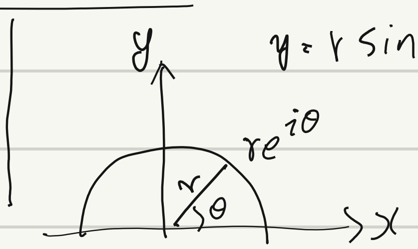


- Shortest paths ?





- Polar coord.



$x = r\cos\theta$

$y = r\sin\theta$

$$ds = \frac{\sqrt{dr^2 + r^2 d\theta^2}}{r\sin\theta}$$

$$dv = \frac{r\,dr\,d\theta}{r^2\sin^2\theta}$$

3. Distance :

Def: A (parametrized) path in $\mathbb{H}$ is the image $\gamma([a,b])$ of a continuous map $\gamma : [a,b] \to \mathbb{H}$

parametrization.

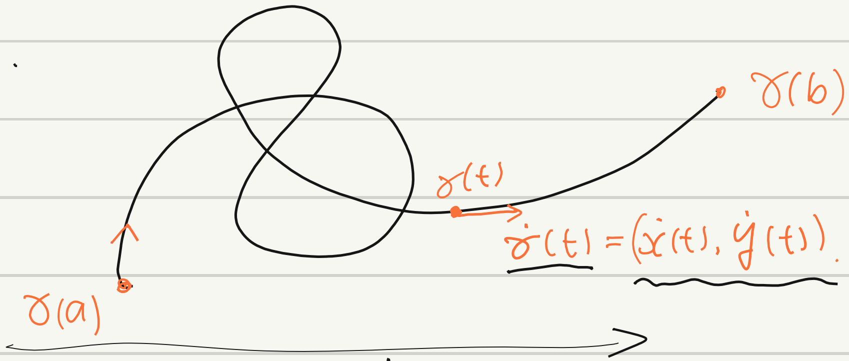
$\forall t \in [a,b]$, $\gamma(t) = x(t) + i\,y(t)$. $\quad x : [a,b] \to \mathbb{R}$ continuous.

$y : [a,b] \to \mathbb{R}_+$

Def. A path γ is regular (i.e. of class C^1), if both $x : [a,b] \to \mathbb{R}$ $\quad$ $y : [a,b] \to \mathbb{R}_+$. are of class C^1

($\exists$ $\dot{x} : [a,b] \to \mathbb{R}$ $\quad$ $\dot{y} : [a,b] \to \mathbb{R}_+$, continuous.)

Def: The tangent vector at $\gamma(t)$, for $t \in [a,b]$, is given by $\dot\gamma(t) = (\dot x(t), \dot y(t))$.



$$\|\dot\gamma(t)\|_{E} = \sqrt{\dot x(t)^2 + \dot y(t)^2} \quad \text{Euclidean norm}$$

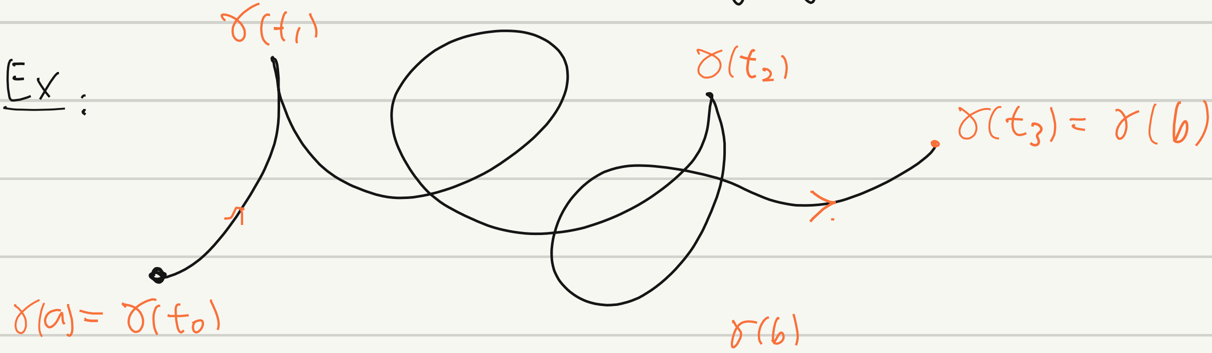
$$\|\dot\gamma(t)\|_{H} = \frac{\sqrt{\dot x(t)^2 + \dot y(t)^2}}{y(t)} \quad \text{hyperbolic norm.}$$

Def. The (hyperbolic) length of a regular path $\gamma : [a,b] \to H$ is:

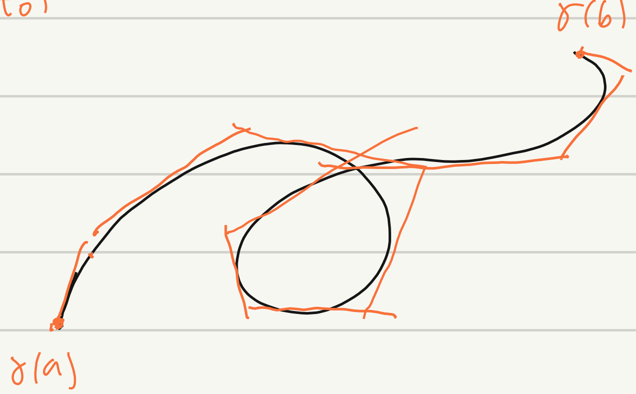
$$\ell_{H}(\gamma) := \int_a^b \|\dot\gamma(t)\|_{H} \, dt.$$

Def: A path $\gamma : [a,b] \to H$ is piecewise regular if $\exists\ a = t_0 < t_1 < \cdots < t_n = b$ finite partition of $[a,b]$ s.t. $\forall\ j = 0, \ldots, n-1$, $\gamma|_{[t_j, t_{j+1}]}$ is regular.

Ex:



Informal.



Ob: $\exists$ more than 1 parametrization of a path.

Prop: If $\gamma: [a,b] \to \mathbb{H}$ is a regular path,

$\phi: [c,d] \to [a,b]$ is of class C^1 with $\dot{\phi}(s) > 0$ $\forall s \in [c,d]$

$s \longmapsto t$

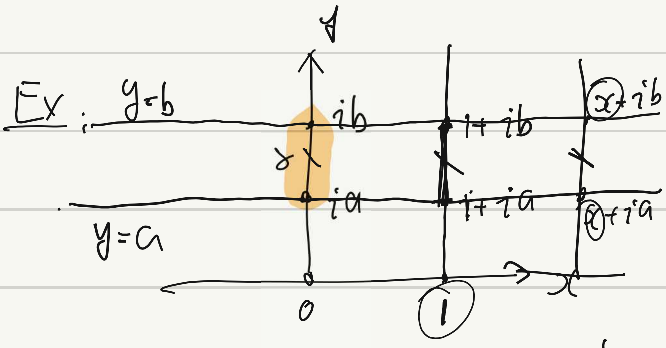
$\eta(s) = (\gamma \circ \phi)(s): [c,d] \to \mathbb{H}$.

Then $\ell_{\mathbb{H}}(\eta) = \ell_{\mathbb{H}}(\gamma)$

Proof: $\ell_{\mathbb{H}}(\gamma) = \int_{a = \phi(c)}^{b = \phi(d)} \|\dot{\gamma}(t)\|_{\mathbb{H}}\, dt = \int_c^d \|\dot{\gamma}(\phi(s))\|_{\mathbb{H}} \dot{\phi}(s)\, ds$

$= \int_c^d \|\dot{\eta}(s)\|_{\mathbb{H}}\, ds = \ell_{\mathbb{H}}(\eta)$ ∎

Ex:



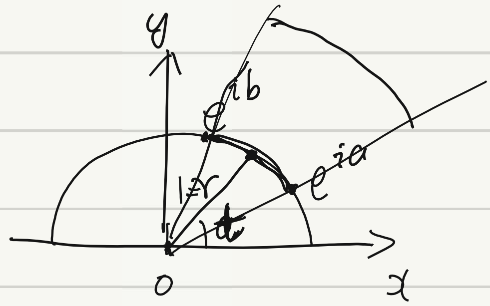
$\gamma(t) = it, \quad t \in [a,b]$

$\gamma(t) = (0, t)$

$\dot{\gamma}(t) = (0,1)$ $\quad \|\dot{\gamma}(t)\|_{\mathbb{H}} = \frac{\|\dot{\gamma}(t)\|_E}{t} = \frac{1}{t}$.

$\ell_{\mathbb{H}}(\gamma) = \int_a^b \frac{1}{t}\, dt = \log \frac{b}{a}$

Ex:



$\gamma(t) = e^{it} \quad t \in [a,b]$

$\gamma(t) = (1, t)$

$\dot{\gamma}(t) = (0, 1)$

$r(t) = (r, t)$

$\|\dot{\gamma}(t)\|_{\mathbb{H}} = \frac{\sqrt{\dot{r}(t)^2 + r(t)^2 \dot{\theta}(t)^2}}{r(t) \sin\theta(t)}$

$= \frac{\sqrt{0+1}}{\sin t} = \frac{1}{\sin t}$

$\ell_{\mathbb{H}}(\gamma) = \int_a^b \frac{1}{\sin t}\, dt = \log \frac{\sin b}{\cos b + 1} - \log \frac{\sin a}{\cos a + 1}$ ∎

Def: Let $w, z \in \mathbb{H}$,

$$d_{\mathbb{H}}(w,z) := \inf \left\{ \ell_{\mathbb{H}}(\gamma) \;\middle|\; \begin{array}{l} \gamma : [a,b] \to \mathbb{H} \quad \gamma(a) = w \\ \text{p.w. regular} \qquad \gamma(b) = z \end{array} \right\}$$

Prop: $d_{\mathbb{H}}$ is a distance on $\mathbb{H}$.

① $d_{\mathbb{H}}(z,z') \geq 0$ "$=0$" iff $z = z'$

② $d_{\mathbb{H}}(z,z') = d_{\mathbb{H}}(z',z)$ ✓

③ $d_{\mathbb{H}}(z,z'') \leq d_{\mathbb{H}}(z,z') + d_{\mathbb{H}}(z,z'')$. ✓

Def: $K, K' \subseteq \mathbb{H}$, distance between K and K':

$$d_{\mathbb{H}}(K,K') := \inf \{ d_{\mathbb{H}}(z,z') \mid z \in K, \ z' \in K' \}$$

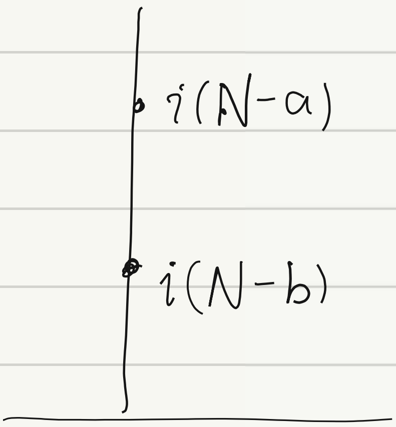
Ex:

$d_{\mathbb{H}}(K,K') = 0$

no pairs realize $d_{\mathbb{H}}(K,K')$.

If $z_0 \in K$, $z_0' \in K'$ s.t. $d_{\mathbb{H}}(z_0, z_0') = d_{\mathbb{H}}(K,K')$

then (z_0, z_0') realizes $d_{\mathbb{H}}(K,K')$.



$\gamma(t) = i(N-t) \quad t \in [a,b]$

$\dot{\gamma}(t) = (0, -1)$.

$$\|\dot{\gamma}(t)\|_{\mathbb{H}} = \frac{\sqrt{0+1}}{\operatorname{Im}\gamma(t)} = \frac{1}{N-t}$$

$$\ell_{\mathbb{H}}(\gamma) = \int_a^b \frac{1}{N-t}\, dt = \log \frac{N-a}{N-b}$$