PROBLEM SET

RUOBING ZHANG

Contents

1. Basic Riemannian geometry 1
2. Geometry of Riemannian manifolds 2
3. More examples of Riemannian structures 3
4. The space of metric structures 4

1. Basic Riemannian geometry

Exercise 1.1. *Let $\gamma(t) = (t, y(t))$, $t \in [0,1]$ be the curve defined by*

$$y(t) = \begin{cases} t\cos(\frac{\pi}{2t}), & t \neq 0, \\ 0, & t = 0. \end{cases} \tag{1.1}$$

Show that γ is non-rectifiable.

Exercise 1.2. *Let g be a Riemannian structure on U. Show that $d_g : U \times U \to [0,\infty)$ gives a metric structure on U such that (U, g) becomes a metric space.*

Exercise 1.3. *Show that any geodesic in $\mathbb{R}^n$ is a line segment. Then prove that $(\mathbb{R}^n, d_0)$ is a length space. Under what condition of U is the space (U, d_0) a length space?*

Recall the definition of covariant derivative along a curve γ.

Definition 1.4 (Covariant derivative)**.** For any smooth map $\Gamma : U \to (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$ that maps every $x \in U$ to a bilinear map $\Gamma_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the corresponding covariant derivative X along γ is defined by

$$\nabla_\gamma X = \frac{d}{dt}X(t) + \Gamma_{\gamma(t)}(\gamma'(t), X). \tag{1.2}$$

Exercise 1.5. *In (1.2), assume that Γ is chosen such that $\Gamma_x(u, v) \equiv \frac{1}{2}G_x^{-1}DG_x(u)v$ for any $x \in U$. Let X, Y be C^1-vector fields along γ and let $f : U \to \mathbb{R}$ be any C^1-function. Show that*

(1) $\nabla_\gamma(\alpha X + \beta Y) = \alpha\nabla_\gamma X + \beta\nabla_\gamma Y$,
(2) $\nabla_\gamma(fX) = f' \cdot X + f\nabla_\gamma X$,
(3) $\frac{d}{dt}g_\gamma(X, Y) = g_\gamma(\nabla_\gamma X, Y) + (X, \nabla_\gamma Y)$.

Exercise 1.6. *In geodesic normal coordinates centered at $p \in U$, we denote*

$$g_{ij} \equiv g(\partial_i, \partial_j), \quad \nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k. \tag{1.3}$$

Show that $g_{ij}(p) = \delta_{ij}, \quad \Gamma_{ij}^k(p) = 0.$

Exercise 1.7. *We define* $\mathscr{R}(X,Y)Z \equiv \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ *for* $X, Y, Z \in \mathfrak{X}(U)$. *Show that* $\mathscr{R}$ *is tensorial and* $R = \mathscr{R}$.

Exercise 1.8. *Let* $\{x_i\}_{i=1}^n$ *be the geodesic normal coordinate system at* $p \in U$. *Let* $\gamma : [0,1] \to U$ *be a geodesic with* $\gamma(0) = p$ *and* $\gamma'(0) = \sigma$. *Prove the following Taylor expansions:*

(1) *Let* $J(t)$ *be a Jacobi field along* γ *with* $J(0) = 0$ *and* $J'(0) = v$. *Then*

$$\|J(t)\|^2 = t^2 - \frac{1}{3}\sec_p(\sigma, v) \cdot t^4 + O(t^5). \tag{1.4}$$

(2) *Let* $G \equiv \det(g_{ij})$. *Then*

$$\sqrt{G} = 1 - \frac{1}{6}\operatorname{Ric}_{ij}(p)x_i x_j + O(|x|^3). \tag{1.5}$$

(3) *Let* $B_r(p) \equiv \{x \in U | d_g(x,p) \leq p\}$. *Then*

$$\frac{\operatorname{Vol}(B_r(p))}{\omega_n r^n} = 1 - \frac{\operatorname{Sc}(p)}{6(n+2)}r^2 + O(r^3). \tag{1.6}$$

(4) *Let* $S_r(p) \equiv \partial B_r(p)$. *Then*

$$\frac{\operatorname{Area}(S_r(p))}{n\omega_n r^{n-1}} = 1 - \frac{\operatorname{Sc}(p)}{6}r^2 + O(r^3). \tag{1.7}$$

Exercise 1.9. *Consider the following warped product metrics on the* 2*-disc* $\mathbb{D} \subset \mathbb{R}^2$*:*

$$g_+ = dr^2 + \sin^2(r) \cdot d\theta^2, \tag{1.8}$$

$$g_- = dr^2 + \sinh^2(r) \cdot d\theta^2. \tag{1.9}$$

Prove that $\sec_{g+} \equiv 1$ *and* $\sec_{g_-} \equiv -1$.

2. Geometry of Riemannian manifolds

Exercise 2.1. *Let* $\gamma : [0,1] \to U$ *be a curve. Let* ∇ *be a linear connection on* U. *A vector field* X *is said to be parallel along* γ *if* $\nabla_{\gamma'(t)}X(t) \equiv 0$ *for any* $t \in [0,1]$. *Prove that for any* $v \in T_{\gamma(0)}U$, *there exists a unique parallel vector field* X *along* γ *with* $X(0) = v$.

Exercise 2.2. *Let* ∇ *be the Levi-Civita connection on* (U, g). *Prove the following Koszul formula*

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ & + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X). \end{aligned} \tag{2.1}$$

Exercise 2.3. *Let* M^n *be a differentiable* n*-manifold. Then* $\dim(T_pM^n) = n$ *for any* $p \in M^n$.

Exercise 2.4. *Let* $(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}_2, g_{flat})$ *be a flat torus. Write a geodesic* $\gamma \subset \mathbb{T}^2$ *which is dense* $\mathbb{T}^2$.

Exercise 2.5. *Let* $\varphi : (U, g_0) \to (\varphi(U), g_0)$ *be an isometry. Show that there exists an isometry* $\bar{\varphi}$ *on* $\mathbb{R}^n$ *such that* $\varphi = \bar{\varphi}|_U$.

Exercise 2.6. *Show that* $\Delta(fg) = f\Delta g + g\Delta f + 2g(\nabla f, \nabla g)$.

Exercise 2.7. *Let (M^n, g) and (N^k, h) be Riemannian manifolds. Then the Levi-Civita connection ∇ of $(M^n \times N^k, g \oplus h)$ satisfies*

$$\nabla_{Y_1+Y_2}(X_1 + X_2) = \nabla^g_{Y_1} X_1 + \nabla^h_{Y_2} X_2, \quad X_i + Y_i \in TM^n \oplus TN^k, \ i = 1, 2. \tag{2.2}$$

In particular, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a geodesic if γ_1 and γ_2 are geodesics on M^n and N^k, respectively.

Exercise 2.8. *Let $p = (x_1, y_1)$ and $q = (x_2, y_2)$ be any points on the product manifold $(M^n \times N^k, g \oplus h)$. Then*

$$d^2_{g\oplus h}(p, q) = d^2_g(x_1, x_2) + d^2_h(y_1, y_2). \tag{2.3}$$

Exercise 2.9. *Let (M^n, g) be a Riemannian manifold. Show that for any $r > 0$, $\mathrm{Sc}_{\bar{g}_r} = r^{-2}\,\mathrm{Sc}_{g_r}$. Then establish the similar rescaling relation for sectional and Ricci curvatures.*

Exercise 2.10. *Consider the Euclidean metric g_0 on $\mathbb{R}^{n+1}$ in polar coordinates $g_0 = dr^2 + r^2 g_{\mathbb{S}^n}$. Prove that $\sec_{\mathbb{S}^n} \equiv +1$.*

Exercise 2.11. *Let $\eta : \underbrace{TM \times \ldots \times TM}_{r} \to C^\infty(M^n)$ be a tensor multilinear map on M^n, called a $(0, r)$-tensor field. Let $X \in \mathfrak{X}(M^n)$. We define*

$$\mathfrak{L}_X \eta(Y_1, \ldots, Y_r) \equiv X(\eta(Y_1, \ldots, Y_r)) - \sum_{i=1}^{r} \eta(X_1, \ldots, [X, Y_i], \ldots, X_r). \tag{2.4}$$

Exercise 2.12. *Let η and ζ be $(0, r)$ and $(0, s)$ tensor fields on M^n, repsectively. For any $X \in \mathfrak{X}(M^n)$, show that*

$$\mathfrak{L}_X(T \otimes S) = (\mathfrak{L}_X T) \otimes S + T \otimes (\mathfrak{L}_X S). \tag{2.5}$$

As a special case, $\mathfrak{L}_X(fT) = X(f) + f\mathfrak{L}_X T$ for any $f \in C^\infty(M^n)$.

Exercise 2.13. *Prove that in the above warped product metric, $2\,\mathrm{Hess}(r) = \mathfrak{L}_{\partial_r} g$ and $\mathrm{Hess}(r) = (f \cdot \partial_r f)h$.*

Exercise 2.14. *Let $(r, x_1, \ldots, x_{n-1})$ be local coordinates on $(a, b) \times N^{n-1}$. Prove the following curvature identities for the warped product metric $dr^2 + f^2(r)h$:*

1. $R^g_{ijk\ell} = f^2(r)R^h_{ijk\ell} + f^2(r)(f'(r))^2(h_{ik}h_{j\ell} - h_{i\ell}h_{jk})$.
2. $R^g_{ij\ell r} = 0$ *and* $R^g_{rijr} = -f(r) \cdot f''(r)g_{ij}$.
3. $\sec_g(\partial_i, \partial_j) = f^{-2}(r)(\sec_h(\partial_i, \partial_j) - (f'(r))^2)$ *and* $\sec_g(\partial_r, \partial_i) = -f^{-1}(r)f''(r)$.
4. $\mathrm{Ric}^g_{ij} = \mathrm{Ric}^h_{ij} - \Big((n-2)(f'(r))^2 + f(r)f''(r)\Big)g_{ij}$.
5. $\mathrm{Ric}^g_{ir} = 0$ *and* $\mathrm{Ric}^g_{rr} = -(n-1)f^{-1}(r)f''(r)$.

3. More examples of Riemannian structures

Exercise 3.1. *Let $g_C = dr^2 + r^2 \cdot h$ be the cone metric of $C(\Sigma)$. Show that $\mathfrak{L}_{\partial_r} g = \frac{2}{r} g$.*

Exercise 3.2. *Prove that a metric cone $C(\Sigma)$ is smooth everywhere if and only if $C(\Sigma)$ is flat which is equivalent to say the cross-section Σ is isometric to the round sphere of curvature $+1$.*

Exercise 3.3. *Let $(C(\Sigma), g_C, z_*)$ be a metric cone over a compact manifold (Σ, h), where z_* is the cone tip. Prove that away from the cone tip z_*, $\mathrm{Ric}_{g_C} \equiv 0$ iff $\mathrm{Ric}_h \equiv (n-2)h$, and g_C is flat iff $\sec_h \equiv +1$.*

Exercise 3.4. *Let* $Z \equiv \mathrm{Susp}_k(\Sigma)$ *with* $k \in \{-1, 1\}$. *Show that* $\sec_\Sigma \equiv 1$ *if and only if* $\sec_Z \equiv k$.

Exercise 3.5. *Let* M^n *be a differentiable manifold. Show that for any* $X, Y, Z \in \mathfrak{X}(M^n)$, *we have the following identities:*

(1) $[Y, X] = -[X, Y]$.
(2) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

As a result, $\mathfrak{X}(M^n)$ *is a Lie algebra (of infinite dimension).*

Exercise 3.6 (An alternative definition of Lie derivative)**.** *Let* R *be a* $(0, r)$ *tensor field on* M^n. *Let* X *be a vector field with a flow* ϕ_t. *Then we define*

$$(\mathfrak{L}_X \mathcal{R})_p(v_1, \ldots, v_r) = \frac{d}{dt}\Big|_{t=0} (\phi_t^* \mathcal{R}_p(v_1, \ldots, v_r)), \tag{3.1}$$

where $\phi_t^* \mathcal{R}(v_1, \ldots, v_r) \equiv \mathcal{R}(D\phi_t(v_1), \ldots, D\phi_t(v_r))$ *and* $v_1, \ldots, v_r \in T_p M^n$. *Show that this definition coincides with the previous one involving the Lie bracket.*

Exercise 3.7. *Let* $X \in \mathfrak{X}(M)$. *Show that* $\mathfrak{L}_X g = 0$ *if and only if the flow of* X *is an isometric action .*

4. The space of metric structures

Exercise 4.1. *Show that the tangent cone at any point in Riemannian n-manifold is isometric to* $\mathbb{R}^n$.

Exercise 4.2. *Let* (Σ, h) *be any closed Riemannian manifold with* $\mathrm{diam}_h(\Sigma) \leq \pi$. *Let* $(C(\Sigma), d_C)$ *be the metric cone over* Σ *with a vertex* z_*. *Show that the tangent cone of* $C(\Sigma)$ *at* $p \neq z_*$ *is isometric to* $\mathbb{R}^n$, *and the tangent cone of* $C(\Sigma)$ *at* z_* *is isometric to itself.*

Exercise 4.3. *Let* (Σ, h) *be any closed Riemannian manifold with* $\mathrm{diam}_h(\Sigma) \leq \pi$. *Let* $(\mathrm{Susp}_{+1}(\Sigma), d_C)$ *be the spherical suspension over* Σ *with vertices* z_* *and* w_*. *Show that the tangent cone of* $\mathrm{Susp}_{+1}(\Sigma)$ *at any vertex is a metric cone.*

Exercise 4.4. *Show that the asymptotic cone of a complete non-compact metric space* (X, d) *is independent of the choice of the reference point* p.

Exercise 4.5. *Let* (X, d) *be a compact metric space. Show that for any* $\epsilon > 0$, *there is a finite* ϵ*-net* $X(\epsilon) \subset X$.

Exercise 4.6. *Read Chapter 5 of Petersen's textbook. Understand the concepts: conjugate point, injectivity radius, and segment domain.*