PROBLEM SET
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1. BASIC RIEMANNIAN GEOMETRY

Exercise 1.1. Let $\gamma(t) = (t, y(t))$, $t \in [0, 1]$ be the curve defined by

$$y(t) = \begin{cases} t \cos\left(\frac{\pi}{2}t\right), & t \neq 0, \\ 0, & t = 0. \end{cases}$$  \hspace{1cm} (1.1)

Show that $\gamma$ is non-rectifiable.

Exercise 1.2. Let $g$ be a Riemannian structure on $U$. Show that $d_g : U \times U \to [0, \infty)$ gives a metric structure on $U$ such that $(U, g)$ becomes a metric space.

Exercise 1.3. Show that any geodesic in $\mathbb{R}^n$ is a line segment. Then prove that $(\mathbb{R}^n, d_0)$ is a length space. Under what condition of $U$ is the space $(U, d_0)$ a length space?

Recall the definition of covariant derivative along a curve $\gamma$.

Definition 1.4 (Covariant derivative). For any smooth map $\Gamma : U \to (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$ that maps every $x \in U$ to a bilinear map $\Gamma_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the corresponding covariant derivative $X$ along $\gamma$ is defined by

$$\nabla_\gamma X = \frac{d}{dt}X(t) + \Gamma_{\gamma(t)}(\gamma'(t), X).$$  \hspace{1cm} (1.2)

Exercise 1.5. In (1.2), assume that $\Gamma$ is chosen such that $\Gamma_x(u, v) \equiv \frac{1}{2}G^{-1}_{x}DG_{x}(u)v$ for any $x \in U$. Let $X, Y$ be $C^1$-vector fields along $\gamma$ and let $f : U \to \mathbb{R}$ be any $C^1$-function. Show that

1. $\nabla_\gamma(\alpha X + \beta Y) = \alpha \nabla_\gamma X + \beta \nabla_\gamma Y$,
2. $\nabla_\gamma(fX) = f' \cdot X + f\nabla_\gamma X$,
3. $\frac{d}{dt}g_{\gamma}(X, Y) = g_{\gamma}(\nabla_\gamma X, Y) + (X, \nabla_\gamma Y)$.

Exercise 1.6. In geodesic normal coordinates centered at $p \in U$, we denote

$$g_{ij} \equiv g(\partial_i, \partial_j), \quad \nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k.$$  \hspace{1cm} (1.3)

Show that $g_{ij}(p) = \delta_{ij}$, $\Gamma^k_{ij}(p) = 0$. 

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Exercise 1.7. We define \( \mathcal{R}(X, Y)Z \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \) for \( X, Y, Z \in \mathfrak{X}(U) \). Show that \( \mathcal{R} \) is tensorial and \( R = \mathcal{R} \).

Exercise 1.8. Let \( \{x_i\}_{i=1}^n \) be the geodesic normal coordinate system at \( p \in U \). Let \( \gamma : [0, 1] \to U \) be a geodesic with \( \gamma(0) = p \) and \( \gamma'(0) = \sigma \). Prove the following Taylor expansions:

1. Let \( J(t) \) be a Jacobi field along \( \gamma \) with \( J(0) = 0 \) and \( J'(0) = v \). Then
   \[
   ||J(t)||^2 = t^2 - \frac{1}{3} \sec(\sigma, v) \cdot t^4 + O(t^5). \tag{1.4}
   \]

2. Let \( G \equiv \det(g_{ij}) \). Then
   \[
   \sqrt{G} = 1 - \frac{1}{6} \text{Ric}_j(p)x_i x_j + O(|x|^3). \tag{1.5}
   \]

3. Let \( B_r(p) \equiv \{ x \in U | d_g(x, p) \leq r \} \). Then
   \[
   \frac{\text{Vol}(B_r(p))}{\omega_n r^n} = 1 - \frac{\text{Sc}(p)}{6(n + 2)} r^2 + O(r^3). \tag{1.6}
   \]

4. Let \( S_r(p) \equiv \partial B_r(p) \). Then
   \[
   \frac{\text{Area}(S_r(p))}{n \omega_n r^{n-1}} = 1 - \frac{\text{Sc}(p)}{6} r^2 + O(r^3). \tag{1.7}
   \]

Exercise 1.9. Consider the following warped product metrics on the 2-disc \( \mathbb{D} \subset \mathbb{R}^2 \):

\[
\begin{align*}
g_+ &= dr^2 + \sin^2(r) \cdot d\theta^2, \quad \tag{1.8} \\
g_- &= dr^2 + \sinh^2(r) \cdot d\theta^2. \quad \tag{1.9}
\end{align*}
\]

Prove that \( \sec g_+ \equiv 1 \) and \( \sec g_- \equiv -1 \).

2. Geometry of Riemannian manifolds

Exercise 2.1. Let \( \gamma : [0, 1] \to U \) be a curve. Let \( \nabla \) be a linear connection on \( U \). A vector field \( X \) is said to be parallel along \( \gamma \) if \( \nabla_{\gamma'(t)}X(t) \equiv 0 \) for any \( t \in [0, 1] \). Prove that for any \( v \in T_{\gamma(0)}U \), there exists a unique parallel vector field \( X \) along \( \gamma \) with \( X(0) = v \).

Exercise 2.2. Let \( \nabla \) be the Levi-Civita connection on \((U, g)\). Prove the following Koszul formula

\[
2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \]
\[
+ g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X). \tag{2.1}
\]

Exercise 2.3. Let \( M^n \) be a differentiable \( n \)-manifold. Then \( \dim(T_p M^n) = n \) for any \( p \in M^n \).

Exercise 2.4. Let \( (\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, g_{\text{flat}}) \) be a flat torus. Write a geodesic \( \gamma \subset \mathbb{T}^2 \) which is dense \( \mathbb{T}^2 \).

Exercise 2.5. Let \( \varphi : (U, g_0) \to (\varphi(U), g_0) \) be an isometry. Show that there exists an isometry \( \bar{\varphi} \) on \( \mathbb{R}^n \) such that \( \varphi = \bar{\varphi}|_U \).

Exercise 2.6. Show that \( \Delta(fg) = f\Delta g + g\Delta f + 2g(\nabla f, \nabla g) \).
Exercise 2.7. Let \((M^n, g)\) and \((N^k, h)\) be Riemannian manifolds. Then the Levi-Civita connection \(\nabla\) of \((M^n \times N^k, g \oplus h)\) satisfies
\[
\nabla_{Y_1+Y_2}(X_1 + X_2) = \nabla_{Y_1}^h X_1 + \nabla_{Y_2}^h X_2, \quad X_1 + X_2 \in TM^n \oplus TN^k, \quad i = 1, 2. \tag{2.2}
\]
In particular, \(\gamma(t) = (\gamma_1(t), \gamma_2(t))\) is a geodesic if \(\gamma_1\) and \(\gamma_2\) are geodesics on \(M^n\) and \(N^k\), respectively.

Exercise 2.8. Let \(p = (x_1, y_1)\) and \(q = (x_2, y_2)\) be any points on the product manifold \((M^n \times N^k, g \oplus h)\). Then
\[
d^2_{g \oplus h}(p, q) = d^2_g(x_1, x_2) + d^2_h(y_1, y_2). \tag{2.3}
\]

Exercise 2.9. Let \((M^n, g)\) be a Riemannian manifold. Show that for any \(r > 0\), \(Sc_g = r^{-2}Sc_{g_r}\). Then establish the similar rescaling relation for sectional and Ricci curvatures.

Exercise 2.10. Consider the Euclidean metric \(g_0\) on \(\mathbb{R}^{n+1}\) in polar coordinates \(g_0 = dr^2 + r^2g^n\). Prove that \(\sec_{g^n} \equiv +1\).

Exercise 2.11. Let \(\eta : TM \times \ldots \times TM \to C^\infty(M^n)\) be a tensor multilinear map on \(M^n\), called a \((0, r)\)-tensor field. Let \(X \in \mathfrak{X}(M^n)\). We define
\[
\mathfrak{L}_X \eta(Y_1, \ldots, Y_r) \equiv X(\eta(Y_1, \ldots, Y_r)) - \sum_{i=1}^r \eta(X_1, \ldots, [X, Y_i], \ldots, X_r). \tag{2.4}
\]

Exercise 2.12. Let \(\eta\) and \(\zeta\) be \((0, r)\) and \((0, s)\) tensor fields on \(M^n\), respectively. For any \(X \in \mathfrak{X}(M^n)\), show that
\[
\mathfrak{L}_X(T \otimes S) = (\mathfrak{L}_X T) \otimes S + T \otimes (\mathfrak{L}_X S). \tag{2.5}
\]
As a special case, \(\mathfrak{L}_X(fT) = X(f) + f\mathfrak{L}_X T\) for any \(f \in C^\infty(M^n)\).

Exercise 2.13. Prove that in the above warped product metric, \(2\text{Hess}(r) = \mathfrak{L}_{\partial_r} g\) and \(\text{Hess}(r) = (f \cdot \partial_r f) h\).

Exercise 2.14. Let \((r, x_1, \ldots, x_{n-1})\) be local coordinates on \((a, b) \times \mathbb{R}^{n-1}\). Prove the following curvature identities for the warped product metric \(dr^2 + f^2(r)h\):

1. \(R^g_{jk \ell} = f^2(r)R^h_{jk \ell} + f^2(r)(f'(r)^2)(h_{ik}h_{j \ell} - h_{i k}h_{j \ell})\).
2. \(R^g_{ij \ell} = 0\) and \(R^g_{ij \ell r} = -f(r) \cdot f''(r) g_{ij}\).
3. \(\sec_g(\partial_i, \partial_j) = f^{-2}(r)(\sec_h(\partial_i, \partial_j) - (f'(r))^2)\) and \(\sec_g(\partial_r, \partial_i) = -f^{-1}(r)f''(r)\).
4. \(\text{Ric}_g^{ij} = \text{Ric}_h^{ij} - \left((n-2)(f'(r))^2 + f(r)f''(r)\right) g_{ij}\).
5. \(\text{Ric}_g^{ir} = 0\) and \(\text{Ric}_g^{rr} = -(n-1)f^{-1}(r)f''(r)\).

3. More examples of Riemannian structures

Exercise 3.1. Let \(g_C = dr^2 + r^2 \cdot h\) be the cone metric of \(C(\Sigma)\). Show that \(\mathfrak{L}_{\partial_r} g = \frac{2}{r} g\).

Exercise 3.2. Prove that a metric cone \(C(\Sigma)\) is smooth everywhere if and only if \(C(\Sigma)\) is flat which is equivalent to say the cross-section \(\Sigma\) is isometric to the round sphere of curvature +1.

Exercise 3.3. Let \((C(\Sigma), g_C, z_*)\) be a metric cone over a compact manifold \((\Sigma, h)\), where \(z_*\) is the cone tip. Prove that away from the cone tip \(z_*\), \(\text{Ric}_{g_C} \equiv 0\) iff \(\text{Ric}_h \equiv (n-2)h\), and \(g_C\) is flat iff \(\sec_h \equiv +1\).
Exercise 3.4. Let $Z \equiv \text{Susp}_k(\Sigma)$ with $k \in \{-1, 1\}$. Show that $\sec_\Sigma \equiv 1$ if and only if $\sec_Z \equiv k$.

Exercise 3.5. Let $M^n$ be a differentiable manifold. Show that for any $X, Y, Z \in \mathfrak{X}(M^n)$, we have the following identities:

2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

As a result, $\mathfrak{X}(M^n)$ is a Lie algebra (of infinite dimension).

Exercise 3.6 (An alternative definition of Lie derivative). Let $R$ be a $(0, r)$ tensor field on $M^n$. Let $X$ be a vector field with a flow $\phi_t$. Then we define

$$ (\mathcal{L}_X R)_p(v_1, \ldots, v_r) = \left. \frac{d}{dt} \right|_{t=0} (\phi^*_t R_p(v_1, \ldots, v_r)), $$

(3.1)

where $\phi^*_t R(v_1, \ldots, v_r) \equiv R(D\phi_t(v_1), \ldots, D\phi_t(v_r))$ and $v_1, \ldots, v_r \in T_p M^n$. Show that this definition coincides with the previous one involving the Lie bracket.

Exercise 3.7. Let $X \in \mathfrak{X}(M)$. Show that $\mathcal{L}_X g = 0$ if and only if the flow of $X$ is an isometric action.

4. The space of metric structures

Exercise 4.1. Show that the tangent cone at any point in Riemannian $n$-manifold is isometric to $\mathbb{R}^n$.

Exercise 4.2. Let $(\Sigma, h)$ be any closed Riemannian manifold with $\text{diam}_h(\Sigma) \leq \pi$. Let $(C(\Sigma), d_C)$ be the metric cone over $\Sigma$ with a vertex $z_\ast$. Show that the tangent cone of $C(\Sigma)$ at $p \neq z_\ast$ is isometric to $\mathbb{R}^n$, and the tangent cone of $C(\Sigma)$ at $z_\ast$ is isometric to itself.

Exercise 4.3. Let $(\Sigma, h)$ be any closed Riemannian manifold with $\text{diam}_h(\Sigma) \leq \pi$. Let $(\text{Susp}_{+1}(\Sigma), d_C)$ be the spherical suspension over $\Sigma$ with vertices $z_\ast$ and $w_\ast$. Show that the tangent cone of $\text{Susp}_{+1}(\Sigma)$ at any vertex is a metric cone.

Exercise 4.4. Show that the asymptotic cone of a complete non-compact metric space $(X, d)$ is independent of the choice of the reference point $p$.

Exercise 4.5. Let $(X, d)$ be a compact metric space. Show that for any $\epsilon > 0$, there is a finite $\epsilon$-net $X(\epsilon) \subset X$.

Exercise 4.6. Read Chapter 5 of Petersen’s textbook. Understand the concepts: conjugate point, injectivity radius, and segment domain.