EXERCISE SHEET #3

Prove that sub-exponential paths are quasi-geodesic paths in hyperbolic spaces.

Exercise 0.1. *Let p be a path in a δ-hyperbolic space. Given a non-decreasing function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, let p be a path such that $Len(q) \le f(d(q_-, q_+))$ for any subpath q of p. Assume that f is sub-exponential (i.e.: $\lim_{n\to\infty} \log f(n)/n = 0$). Then p is a quasi-geodesic path.*

(Tips: Prove first that p is contained in a uniform neighborhood of the corresponding geodesic. Then subdivide the geodesic into subsegments to control the length of path p.)

Prove that there are only finitely many conjugacy classes of finite subgroups in a hyperbolic group. You may proceed by the following steps:

Exercise 0.2. *Assume that a group G acts geometrically on a proper hyperbolic space (X, d).*

(1) *Define a notion of the center for any bounded set B in a metric X. Define first the radius of B:*

$$r_B := \inf\{r : B \subset B(x, r), r \ge 0, x \in X\}.$$

where $B(x, r)$ is the closed ball of radius r at x. The center of B is then defined to be set of points $o \in X$ such that

$$B \subset B(o, r_B + 1).$$

(2) *Prove that if X is δ-hyperbolic space, the center of any bounded set is bounded by a constant depending only on δ.*
(3) *Apply the assertion (2) to the orbit $B = F \cdot x$ of a finite group F of G, and conclude the proof that there are finitely many conjugacy classes of finite subgroups F.*

Exercise 0.3. *Let (X, d) be a geodesic metric space. If there exists $\delta > 0$ such that the following inequality holds*

$$\langle x, y\rangle_w \ge \min\{\langle x, z\rangle_w, \langle y, z\rangle_w\} - \delta \tag{1}$$

for any $x, y, z, o \in X$, then (X, d) is Gromov-hyperbolic.

You may proceed by the following steps:

(1) *Prove first that there exists a point $w \in [x, y]$ such that $\langle x, z\rangle_w, \langle y, z\rangle_w \le \delta$.*
(2) *Then prove that if $\langle x, z\rangle_w \le \delta$, then $d(w, [x, z])$ is bounded by a constant depending on δ.*

   (apply (1) to find a point $u \in [x, z]$ so that $\langle x, w\rangle_u, \langle z, z\rangle_u \le \delta$, and then prove $d(u, w)$ is bounded)