INTRODUCTION TO METRIC RIEMANIAN GEOMETRY

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Abstract. This minicourse is concerned with some basic concepts and tools in metric Riemannian geometry with a specific focus on the degenerations of Riemannian manifolds with certain curvature bounds (sectional and Ricci). Applications include the structure theory and moduli of Einstein manifolds. We will also give some quick introduction to the current studies on the special holonomy spaces such as hyperkähler manifolds, Calabi-Yau manifolds, and G_2-Einstein manifolds.

The concrete plan of the course is as follows. We will start with basic notions in metric geometry together with a large variety of intuitive examples at hand. Also basic comparison theorems and examples of smooth Riemannian manifolds with bounded curvature will be discussed in detail. For the sake of connecting more advanced topics, we will try to summarize preliminary tools in understanding the structure theory of Ricci curvature. With all the above tools, we are ready to present the fundamental theorems in the metric geometry of sectional and Ricci curvature which are the milestones of the whole field. Depending upon the interests, the course could be extended for more advanced discussions.

Contents

1. Basic Riemannian geometry

1.1. **Riemannian structure and geodesics.** Pending the modern language, we start with rather explicit computations for Euclidean domains. The Euclidean distance between two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is defined by

$$d_0(x, y) \equiv \|x - y\| = \Big(\sum_{j=1}^{n}(x_j - y_j)^2\Big)^{\frac{1}{2}}. \tag{1.1}$$

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuous curve. Then its Euclidean length is defined by

$$L_0(\gamma) \equiv \sup_P \sum_{j=1}^{k} d_0(c(t_{j-1}), c(t_j)), \tag{1.2}$$

where $P = \{a = t_0 < t_1 < \ldots < t_k = b\}$ is a partition of $[a, b]$. We say $\gamma : [a, b] \to \mathbb{R}^n$ is *rectifiable* if $L(\gamma) < \infty$. In particular, any C^1 curve γ is rectifiable with

$$L_0 = \int_0^1 \|c'(t)\| dt. \tag{1.3}$$

Exercise 1.1. *Let $\gamma(t) = (t, y(t))$, $t \in [0, 1]$ be the curve defined by*

$$y(t) = \begin{cases} t\cos(\frac{\pi}{2t}), & t \neq 0, \\ 0, & t = 0. \end{cases} \tag{1.4}$$

Show that γ is non-rectifiable.

Let $U \subset \mathbb{R}^n$ be a domain and let $x, y \in U$. We define the Euclidean distance between x and y in U as follows:

$$d_0(x, y) \equiv \inf\Big\{L_0(\gamma)\Big|\gamma \text{ is a piecewise smooth curve connecting } x \text{ and } y\Big\}. \tag{1.5}$$

Definition 1.2 (Riemannian structure)**.** A Riemannian C^k-structure $g : U \to (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$ is a C^k map on a domain $U \subset \mathbb{R}^n$ such that for any $x \in U$, there exists a nonnegative symmetric $n \times n$ matrix G_x such that

$$g_x(u, v) = \langle u, G_x v\rangle, \quad \forall u, v \in \mathbb{R}^n. \tag{1.6}$$

For any piecewise C^1-curve $\gamma : [a, b] \to U$, its Riemannian length is $L_g(\gamma)$ defined by

$$L_g(\gamma) \equiv \int_a^b (g_{\gamma(t)}(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt. \tag{1.7}$$

Let $U \subset \mathbb{R}^n$ be a domain equipped with a C^k-Riemannian structure g. Then the Riemannian distance between $x, y \in U$ is defined by

$$d_g(x, y) \equiv \inf\Big\{L_g(\gamma)\Big|\gamma \text{ is a piecewise } C^1\text{-curve connecting } x \text{ and } y\Big\}. \tag{1.8}$$

Definition 1.3 (Metric space)**.** Let X be a set. A pair (X, d) is said to be a metric space if $d : X \to X[0, \infty)$ satisfies

(1) $d(x, y) \geq 0$ for any $x, y \in X$. Moreover, "=" holds iff $x = y$,
(2) $d(x, y) = d(y, x)$ for any $x, y \in X$,
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.

Definition 1.4 (Intrinsic distance and length space)**.** Let (X, d) be a metric space. The intrinsic distance $d^{\#}$ between $x, y \in X$ are defined by

$$d^{\#}(x,y) = \inf \left\{ L(\gamma) \middle| \gamma : [0,1] \to X \text{ is a curve connecting } x \text{ and } y \right\}, \tag{1.9}$$

where $L(\gamma) \equiv \sup_{P} \sum_{j=1}^{n} d(\gamma(t_{k-1}), \gamma(t_k))$, and P is a partition of $[0,1]$. (X, d) is called a length space if $d = d^{\#}$.

Definition 1.5 (Geodesic)**.** Let (X, d) be a metric space and let $U \subset X$ be a domain. A curve $\gamma : [a,b] \to U$ is called a geodesic if $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for any $t_1, t_2 \subset [a,b]$. A geodesic $\gamma : [a,b] \to U$ is minimal if $d(\gamma(a), \gamma(b)) = L(\gamma)$.

Example 1.6. *Let U be a domain in $\mathbb{R}^n$. Then (U, d_0) is a length space.*

Theorem 1.7. *Let $U \subset \mathbb{R}^n$ be a domain and let (U, g) be a complete metric space. For any points $p, q \in U$, there exists a minimal geodesic connecting p and q.*

Exercise 1.8. *Show that $d_g : U \times U \to [0, \infty)$ gives a metric structure on U such that (U, g) becomes a metric space.*

1.2. **Exponential map, connection, variation of arc length.** Let $U \subset \mathbb{R}^n$ be a domain and let $\gamma : [a,b] \to U$ be a C^1-curve. A one-parameter variation of γ is a map

$$V : [a,b] \times (-1,1) \to U \tag{1.10}$$

such that $V(t,0) = \gamma(t)$ for all $t \in [a,b]$. From now on, we assume that V is piecewise C^k with $k \geq 1$. Note that there are two families of curves $\gamma_s(\cdot) \equiv V(\cdot, s)$ and $\sigma_t(\cdot) \equiv V(t, \cdot)$. The variation vector field $X(t)$ along γ is defined by

$$X(t) \equiv \frac{\partial}{\partial s}\Big|_{s=0} V(t,s) = \sigma_t'(0). \tag{1.11}$$

For the above family of curves γ_s, let us denote $L_s \equiv L_g(\gamma_s)$ for any $s \in (-1,1)$. Then we have the following first variation formula of arc length.

Lemma 1.9 (First variation of arc length)**.** *Let γ be parametrized by the arc length with $\ell = L_g(\gamma)$. Then the following holds,*

$$\frac{d}{ds}\Big|_{s=0} L_s = \int_0^{\ell} g_{\gamma(t)}(\nabla_\gamma X(t), \gamma'(t)), \tag{1.12}$$

where

$$\nabla_\gamma X(t) \equiv \frac{d}{dt} X(t) + \Gamma_{\gamma(t)}(X(t), \gamma'(t)), \quad \Gamma_x(u,v) \equiv \frac{1}{2} G_x^{-1} D G_x(u) v. \tag{1.13}$$

Definition 1.10 (Covariant derivative)**.** For any smooth map $\Gamma : U \to (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$ that maps every $x \in U$ to a bilinear map $\Gamma_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the corresponding covariant derivative X along γ is defined by

$$\nabla_\gamma X = \frac{d}{dt} X(t) + \Gamma_{\gamma(t)}(\gamma'(t), X). \tag{1.14}$$

Exercise 1.11. *In* (1.14)*, assume that Γ is chosen such that $\Gamma_x(u,v) \equiv \frac{1}{2} G_x^{-1} D G_x(u) v$ for any $x \in U$. Let X, Y be C^1-vector fields along γ and let $f : U \to \mathbb{R}$ be any C^1-function. Show that*

(1) $\nabla_\gamma(\alpha X + \beta Y) = \alpha\nabla_\gamma X + \beta\nabla_\gamma Y$,
(2) $\nabla_\gamma(fX) = f' \cdot X + f\nabla_\gamma X$,
(3) $\frac{d}{dt}g_\gamma(X, Y) = g_\gamma(\nabla_\gamma X, Y) + (X, \nabla_\gamma Y)$.

If the bilinear map Γ is symmetric, then the first variation formula can be written as

$$\frac{d}{ds}\Big|_{s=0} L_s = g_{\gamma(\ell)}(X(\ell), \gamma'(\ell)) - g_{\gamma(0)}(X(0), \gamma'(0)) - \int_0^\ell g_{\gamma(t)}(X, \nabla_\gamma\gamma'(t)). \tag{1.15}$$

If γ is a geodesic, then γ satisfies

$$\nabla_\gamma\gamma' \equiv 0. \tag{1.16}$$

Let X be a vector field and let f be a function. Then we define $X(f) \equiv Df(X)$.

Definition 1.12 (Linear connection)**.** A map $\nabla : \mathfrak{X}(U)\times\mathfrak{X}(U) \to \mathfrak{X}(U)$ is called a connection if for any $X, Y, Z \in \mathfrak{X}(U)$ and any C^1-function f on U, we have

(1) $\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z$,
(2) $\nabla_{fX}Y = f\nabla_X Y$,
(3) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
(4) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

Definition 1.13 (Levi-Civita connection)**.** A linear connection $\nabla : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ is called a Levi-Civita connection if for any $X, Y, Z \in \mathfrak{X}(U)$

(1) (Symmetric or torsion-free) $\nabla_X Y - \nabla_Y X - [X, Y] \equiv 0$,
(2) (g-parallel) $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

Recall the covariant derivative given in Lemma 1.9. One can prove that ∇ gives a Levi-Civita connection if for any $x \in U$, the bilinear map Γ_x is symmetric.

Lemma 1.14. *Let g be a Riemannian metric on a domain $U \subset \mathbb{R}^n$. Then there exists a unique Levi-Civita connection ∇.*

Proof. It suffices to prove the following formula (called the Koszul formula) for a Levi-Citiva connection

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ & + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y). \end{aligned} \tag{1.17}$$

Then both existence and uniqueness immediately follow. □

From now on, we will always focus on the Levi-Civita connection of a Riemannian metric.

Definition 1.15 (Exponential map)**.** Let $U \subset \mathbb{R}^n$ be a domain equipped with a Riemannian structure g. Then for any $p \in U$, the exponential map Exp at p is defined by

$$\mathrm{Exp}_p(v) = \gamma(1), \tag{1.18}$$

where $\gamma : [0, 1] \to U$ is the geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$.

Example 1.16. *Let $p \in U \subset \mathbb{R}^n$. Assume that $\mathrm{Exp}_p(tv)$ is welled defined for any $t \in [0, t_0]$. Then $\gamma(t) \equiv \mathrm{Exp}_p(tv)$ is the geodesic starting from $\gamma(0) = p$ with the initial tangent vector $\gamma'(0) = v$.*

Lemma 1.17. *Let $\mathfrak{E}(x,v) \equiv (x, \mathrm{Exp}_x(v))$. Then*

$$D(\mathfrak{E}) = \begin{pmatrix} \mathrm{Id} & 0 \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix}. \tag{1.19}$$

In particular, for any $p \in U$, $\mathrm{Exp}_p : \mathcal{O}_p \to U$ is a local diffeomorphism, where $\mathcal{O}_p \subset T_pU$ is some open subset.

Lemma 1.18 (Gauß Lemma)**.** *The exponential map is a radial isometry, i.e.,*

$$g_{\mathrm{Exp}_p(v)}(D\,\mathrm{Exp}_p(v), D\,\mathrm{Exp}_p(u)) = g_p(v,u). \tag{1.20}$$

Definition 1.19 (Geodesic normal coordinates)**.** Let $p \in U \subset \mathbb{R}^n$. Denote by $\{e_1, \ldots, e_n\}$ an g-orthornormal basis of T_pU. Assume that Exp_p is a diffeomorphism from $B_\delta(0^n) \subset T_pU$ to $B_\delta(p)$ for some sufficiently small $\delta > 0$ (Such a δ always exists by Lemma 1.17!). Then any $v \in B_\delta(0^n)$ has a unique coordinate representation,

$$v = \sum_{i=1}^n x_i e_i. \tag{1.21}$$

Then $\{x_1, \ldots, x_n\}$ can be viewed as a coordinate system on $B_\delta(p)$ called the *geodesic normal coordinates* via the exponential map. We also denote by ∂_i the coordinate frame corresponding to x_i.

Exercise 1.20. *In geodesic normal coordinates, for any $(x_1, \ldots, x_n) \in B_\delta(0^n)$, $\gamma(t) \equiv (tx_1, \ldots, tx_n)$ is a geodesic. Moreover,*

$$g_{ij}(p) = \delta_{ij}, \quad \Gamma_{ij}^k(p) = 0. \tag{1.22}$$

1.3. **Curvature and Jacobi field.** Let $U \subset \mathbb{R}^n$ be a domain and let $\gamma : [a,b] \to U$ be a geodesic. A variation $V : [a,b] \times (-1,1) \to U$ of γ is called a geodesic variation if $\gamma_s(\cdot) \equiv V(\cdot, s)$ is a geodesic for any $s \in (-1,1)$. The variation vector field $J(t) \equiv \frac{\partial}{\partial s}|_{s=0} V(t,s)$ is called a Jacofi field along γ. Let $p \in U \subset \mathbb{R}^n$ and we consider a specific geodesic variation

$$V(t,s) = \mathrm{Exp}_p(t(u+sv)). \tag{1.23}$$

Then its variation vector field $J(t)$ is expressed as $J(t) = t \cdot (D\,\mathrm{Exp})_{tu}(v)$. Obviously, $J(0) = 0$. Next, we will show that $J(t)$ satisfies a linear ODE involving curvature tensor. Since $V(t,s)$ is a geodesic variation, $\nabla_{\gamma_s}\gamma_s'(t) \equiv 0$. Then

$$0 = \nabla_{\sigma_t}\nabla_{\gamma_s}\gamma_s'(t) = \nabla_{\sigma_t}\nabla_{\gamma_s}\gamma_s'(t) - \nabla_{\gamma_s}\nabla_{\sigma_t}\gamma_s'(t) + \nabla_{\gamma_s}\nabla_{\sigma_t}\gamma_s'(t). \tag{1.24}$$

Notice that

$$\nabla_{\sigma_t}\gamma_s'(t) - \nabla_{\gamma_s}\sigma_t'(s) = \nabla_{DV(\partial_s)}DV(\partial_t) - \nabla_{DV(\partial_t)}DV(\partial_s) = DV([\partial_s, \partial_t]) = 0. \tag{1.25}$$

Therefore, evaluating (1.24) at $s = 0$,

$$\nabla_{\gamma_s}\nabla_{\gamma_s}X + \nabla_{\sigma_t}\nabla_{\gamma_s}\gamma_s'(t) - \nabla_{\gamma_s}\nabla_{\sigma_t}\gamma_s'(t) = 0. \tag{1.26}$$

By computations, we have

$$\begin{aligned} &\nabla_{\sigma_t}\nabla_{\gamma_s}Z - \nabla_{\gamma_s}\nabla_{\sigma_t}Z \\ &= D\Gamma_{V(t,s)}(\sigma_t')(\gamma_s', Z) - D\Gamma_{V(t,s)}(\gamma_s')(\sigma_t', Z) \\ &\quad + \Gamma_{V(t,s)}(\sigma_t', \Gamma(\gamma_s', Z)) - \Gamma_{V(t,s)}(\gamma_s', \Gamma(\sigma_t', Z)). \end{aligned} \tag{1.27}$$

Definition 1.21 (Riemann curvature)**.** The Riemann curvature $R_p : T_pU \times T_pU \times T_pU \to T_pU$ is defined as

$$R(u,v)w = D\Gamma(u)(v,w) - D\Gamma(v)(u,w) + \Gamma(u,\Gamma(v,w)) - \Gamma(v,\Gamma(u,w)), \tag{1.28}$$

where $u, v, w \in T_pU$.

By the above definition, the Jacofi field $J(t)$ satisfies

$$J''(t) + R(J(t),\gamma'(t))\gamma'(t) = 0. \tag{1.29}$$

Exercise 1.22. *Let* $\mathscr{R}(X,Y)Z \equiv \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ *for* $X, Y, Z \in \mathfrak{X}(U)$*. Show that* $\mathscr{R}$ *is tensorial and* $R = \mathscr{R}$*.*

We have more notions of curvatures.

Definition 1.23. Let $u, v, w, z \in T_pU$. Then $R(u,v,w,z) \equiv g(R(u,v)w, z)$.

Definition 1.24 (Sectional curvature)**.** Let $u, v \in T_pU$. Then

$$\sec(u,v) \equiv \frac{R(u,v,v,u)}{\|u\|_g^2 \cdot \|v\|_g^2 - g(u,v)^2}. \tag{1.30}$$

Definition 1.25 (Ricci curvature)**.** Let $\{e_i\}_{i=1}^n$ be an orthonormal basis. Then

$$\mathrm{Ric}(u,v) \equiv \sum_{i=1}^n R(e_i,u,v,e_i). \tag{1.31}$$

We define $\mathrm{Ric}(v) = \frac{\mathrm{Ric}(v,v)}{g(v,v)}$.

Definition 1.26 (Scalar curvature)**.** $\mathrm{Sc} \equiv \mathrm{Tr}(\mathrm{Ric}) = \sum\limits_{j=1}^n \mathrm{Ric}(e_j, e_j)$.

The following are some elementary properties of curvature tensors (exercise!).

Theorem 1.27. *The curvature tensors* R *and* Ric *satisfies the identities*

(1) $R(u,v,w,z) = -R(v,u,w,z)$
(2) $R(u,v,w,z) = R(w,z,u,z)$
(3) *(1st Bianchi)* $R(u,v)w + R(v,w)u + R(w,u)v = 0$
(4) $\mathrm{Ric}(u,v) = \mathrm{Ric}(v,u)$.

Theorem 1.28. *Both* R *and* Ric *are tensorial. For example,* $R(fX,\cdot,\cdot,\cdot) = fR(X,\cdot,\cdot,\cdot)$ *for any* $f \in C^\infty(U)$ *and* $X \in \mathfrak{X}(U)$*.*

2. Geometry of Riemannian manifolds

2.1. **Some applications of Jacobi fields.**

Lemma 2.1. *Let $\gamma : [0,1] \to U$ be a geodesic in (U,g). Then for any $\zeta \in T_{\gamma(0)}U$, there exists a unique Jacobi field $J(t)$ along γ with $J(0) = 0$ and $J'(0) = \zeta$.*

Let $\gamma : [0,1] \to U$ be the geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Let $U(t)$ and $W(t)$ be the Jacobi fields along a geodesic γ that determined by the initial values $J(0) = W(0) = 0$, $U'(0) = u$, and $W'(0) = w$. Let us compute the Taylor expansion of the 1-variable function $g(U(t), W(t))$ at $t = 0$. To do this, the coefficients at 0 are given by

$$g(U(0), W(0)) = 0, \quad (g(U,W))'(0) = 0, \quad (g(U,W))''(0) = 2g(u,w), \tag{2.1}$$

$$(g(U,W))'''(0) = 0, \quad (g(U,W))^{(4)}(0) = -8R(v,u,w,v). \tag{2.2}$$

Let U be the Jacobi field along γ with $U(0) = 0$ and $U'(0) = u$. Then the Taylor expansion of $|U(t)|^2$ along γ at $t = 0$ is given by

$$|J(t)|^2 = t^2 - \frac{1}{3}\sec(v,u)t^4 + O(|t|^5). \tag{2.3}$$

Using the above calculation, we are able to compute the Taylor expansion of (g_{ij}) along a geodesic γ. To do this, now let us consider the geodesic variation in geodesic normal coordinates $\{x_i\}_{i=1}^n$,

$$V_i(t,s) \equiv (tx_1, \ldots, t(x_i + s), tx_n). \tag{2.4}$$

Then the Jacobi field $J_i(t)$ can be expressed as $J_i(t) = t\partial_i$ which implies that

$$g_{ij} = g(\partial_i, \partial_j) = t^{-2} g(J_i(t), J_j(t)). \tag{2.5}$$

Therefore,

$$g_{ij} = \delta_{ij} - \frac{R(x, e_i, e_j, x)}{3} t^2 + O(|x|^3) = \delta_{ij} - \frac{1}{3} R_{kij\ell} x^k x^\ell + O(|x|^3), \tag{2.6}$$

where $x^k \equiv t \cdot x_k$. Moreover, we can compute the Taylor expansion of the volume form $\sqrt{G} \equiv (\det(g_{ij}))^{\frac{1}{2}}$, $\mathrm{Vol}(B_r(p))$, and $\mathrm{Area}(S_r(p))$.

Example 2.2. *Consider the unit 2-sphere $x^2 + y^2 + z^2 = 1$. For any point $p \in \mathbb{S}^2$, one can write the metric in terms of the warped product*

$$g = dr^2 + \sin^2(r) d\theta^2, \quad r \in [0,\pi], \quad \theta \in [0, 2\pi]. \tag{2.7}$$

Using the volume expansion, it is easy to check that the Gauß(sectional) curvature of $\mathbb{S}^2$ is a constant $+1$. Then any vector field $(0, 0, c_1 \cos t + c_2 \sin t)$ is a Jacobi field along the geodesic $(\cos t, \sin t, 0)$.

2.2. **A quick introduction to Riemannian manifolds.**

Definition 2.3 (Differentiable manifold)**.** A differentiable n-manifold is a Hausdorff and second countable topological space M^n, together with a C^∞-atlas $\{(U_\alpha, \varphi_\alpha)\}$ on M such that the following properties hold:

(1) $M^n \equiv \bigcup U_\alpha$,
(2) $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism,
(3) the transition map $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is C^∞.

Definition 2.4 (Tangent vector and tangent space)**.** Let M^n be a differentiable manifold. A tangent vector at p is a linear operator $v : C^\infty(M^n) \to \mathbb{R}$ called a derivation at p which satisfies

$$v(fg) = f(p)v(g) + g(p)v(f), \quad \forall f, g \in C^\infty(M^n). \tag{2.8}$$

The linear space of all the above tangent vectors at p is called the tangent space at p and denoted by T_pM^n.

Exercise 2.5. *Let M^n be a differentiable n-manifold. Then $\dim(T_pM^n) = n$ for any $p \in M^n$.*

Definition 2.6 (Vector field)**.** A vector field X on M^n is a C^∞-map that assigns every point $p \in M^n$ to a tangent vector $X(p) \in T_pM^n$. The space of vector fields on M^n is denoted by $\mathfrak{X}(M^n)$.

Definition 2.7 (Riemannian manifold)**.** A Riemannian manifold (M^n, g) is a differentiable n-manifold equipped with a Riemannian metric g which is a symmetric nonnegative bilinear tensor field $g : TM^n \times TM^n \to [0, \infty)$. In other words, every tangent space T_pM^n is equipped with an inner product $g_p(\cdot, \cdot)$.

Definition 2.8 (Completeness)**.** A Riemannian manifold (M^n, g) is said to be geodesically complete if for any $p \in M^n$, the exponential map Exp_p is welled defined on the entire T_pM^n.

Lemma 2.9. *Every Riemannian manifold is locally compact.*

Theorem 2.10 (Hopf-Rinow)**.** *The following statements are equivalent:*

(1) *(M^n, g) is geodesically complete.*
(2) *(M^n, g) is a complete metric space.*
(3) *Every bounded closed subset in M^n is compact.*
(4) *Every geodesic $\gamma : [0, a) \to M^n$ can be extended to a continuous path $\bar{\gamma} : [0, a] \to M^n$.*
(5) *For any $p \in M^n$, the exponential map Exp_p is well defined on the entire T_pM^n.*

If one of the above holds, then for any $p, q \in M^n$, there exists a minimal geodesic (not necessarily unique!) connecting p and q.

Example 2.11. *Sphere, torus and flat manifold, closed geodesic in $\mathbb{T}^2$*

Exercise 2.12. *Write a geodesic $\gamma \subset \mathbb{T}^2$ which is dense $\mathbb{T}^2$.*

Example 2.13. *$(\mathring{\mathbb{D}}, g_0)$ and $(\mathbb{R}^2 \setminus \{0\}, g_0)$ are incomplete manifolds*

Definition 2.14 (Isometry)**.** A map $\varphi : (M^n, g) \to (N^k, h)$ is called an isometric embedding if

$$d_h(\varphi(p), \varphi(q)) = d_g(p, q), \quad \forall p, q \in M^n. \tag{2.9}$$

An isometric embedding φ is called an isometry if φ is surjective.

Lemma 2.15. *If $\varphi : (M^n, g) \to (N^k, h)$ is an isometry. Then the following holds:*

(1) *M^n is diffeomorphic to N^k. In particular, $n = k$.*
(2) *φ preserves the Riemannian structure, i.e.,*

$$h_{\varphi(p)}(D\varphi(u)D\varphi(v)) = g_p(u, v), \quad p \in M^n, \quad u, v \in T_pM^n. \tag{2.10}$$

(3) *For any $p \in M^n$, $\varphi \circ \mathrm{Exp}_p = \mathrm{Exp}_{\varphi(p)} \circ D\varphi_p$.*

Example 2.16. *Let φ be an isometry on $(\mathbb{R}^n, g_0)$. Then there exists some matrix $A \in O(n)$ such that $\varphi(x) = A \cdot x + \varphi(0)$ for all $x \in \mathbb{R}^n$. Indeed, up to a translation, we can assume $\varphi(0) = 0$. Now we work with the isometry $\psi \equiv \varphi \circ (D\varphi_0)^{-1}$. Obviously, $D\psi_0 = \mathrm{Id}$. For any $v \in \mathbb{R}^n$, the curve $\psi(tv)$ must be a geodesic. Since $D\psi_0 = \mathrm{Id}$ and $\psi(0) = 0$, by the uniqueness property of geodesics, $\psi(tv) = tv$. In particular, $\psi(v) = v$. Therefore, $\psi = \mathrm{Id}$.*

Exercise 2.17. *Let $\varphi : (U, g_0) \to (\varphi(U), g_0)$ be an isometry. Show that there exists an isometry $\bar{\varphi}$ on $\mathbb{R}^n$ such that $\varphi = \bar{\varphi}|_U$.*

2.3. **Operators on a Riemannian manifold.**

Definition 2.18 (Gradient)**.** Let (M^n, g) be a Riemannian manifold. Given any C^1-function $f : M^n \to \mathbb{R}$, then ∇f is the vector field which is uniquely determined by the relation

$$\langle \nabla f, X\rangle = X(f), \quad X \in \mathfrak{X}(M^n). \tag{2.11}$$

In coordinates, we can write $\nabla f = g^{ij} \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\partial}{\partial x_j}$ and $|\nabla f|^2 = g^{ij} \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}$.

Lemma 2.19. *Let (M^n, g) be a Riemannian manifold. Given any $p \in M^n$, define $r(x) \equiv d_g(x, p)$. Then $|\nabla r| \equiv 1$ when r is smooth.*

Proof. We can use the geodesic *polar* coordinates (r, Θ) centered at p. Denote by ∂_r the unit radial vector. By Gauß lemma, $g_{rr} = 1$ and $g_{ri} = 0$. The conclusion immediately follows. □

Definition 2.20 (Hessian and Laplacian)**.** Let (M^n, g) be a Riemannian manifold. Given any C^1-function $f : M^n \to \mathbb{R}$, then $\nabla^2 f$ (sometimes we also denote $\nabla^2 f = \mathrm{Hess}\, f$) is the symmetric tensor field defined by

$$\nabla^2 f(X, Y) = \langle \nabla_X \nabla f, Y\rangle, \quad X \in \mathfrak{X}(M^n). \tag{2.12}$$

We define $\Delta f \equiv \mathrm{Tr}(\nabla^2 f)$.

Since ∇ is the Levi-Civita connection, we have

$$\nabla^2 f(X, Y) = XYf - (\nabla_X Y)f. \tag{2.13}$$

Let $\{E_i\}_{i=1}^n$ be an orthornormal basis. Then

$$\Delta f = \sum_{i=1}^{n} (E_i E_i(f) - \nabla_{E_i} E_i f). \tag{2.14}$$

In local coordinate frames $\{\partial_i\}_{i=1}^n$,

$$\Delta f = \frac{1}{\sqrt{G}} \partial_i (\sqrt{G} g^{ij} \partial_j f), \tag{2.15}$$

where $G = \det(g_{ij})$.

Example 2.21. *Let $\gamma : [a, b] \to M^n$ be a geodesic. Then $\nabla^2 f(\gamma'(t), \gamma'(t)) = \frac{d^2 f}{dt^2}$.*

Example 2.22. *Let $\{x_i\}_{i=1}^n$ be the geodesic normal coordinates at $p \in M^n$. Then $\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial}{\partial x_i}$ and $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$ at point p.*

Lemma 2.23. ∇^2 *is tensorial and symmetric.*

Exercise 2.24. *Show that $\Delta(fg) = f\Delta g + g\Delta f + 2g(\nabla f, \nabla g)$.*

2.4. **Constructing Riemannian metrics.** In this subsection, we exhibit some elementary ways to construct new Riemannian metrics.

Example 2.25 (Product metric). *Let (M^n, g) and (N^k, h) be Riemannian manifolds. Then $(M^n \times N^k, g \oplus h)$ is called a Riemannian product, where the product metric $g \oplus h : (TM^n \oplus TN^k) \otimes (TM^n \oplus TN^k) \to [0, \infty)$ is defined by*

$$(g \oplus h)(X_1 + Y_1, X_2 + Y_2) \equiv g(X_1, X_2) + h(Y_1, Y_2), \tag{2.16}$$

and where $X_i + Y_i \in TM^n \oplus TN^k$, $i = 1, 2$.

Exercise 2.26. *Let (M^n, g) and (N^k, h) be Riemannian manifolds. Then the Levi-Civita connection ∇ of $(M^n \times N^k, g \oplus h)$ satisfies*

$$\nabla_{Y_1+Y_2}(X_1 + X_2) = \nabla^g_{Y_1} X_1 + \nabla^h_{Y_2} X_2, \quad X_i + Y_i \in TM^n \oplus TN^k,\ i = 1, 2. \tag{2.17}$$

In particular, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a geodesic if γ_1 and γ_2 are geodesics on M^n and N^k, respectively.

Theorem 2.27. *Let (M^n, g) be a complete Riemannian manifold. Then there exists a full measure subset $\mathscr{R}$ in $M^n \times M^n$ with respect to the product measure such that for any $(p, q) \in \mathscr{R}$, there exists a unique minimal geodesic connecting p and q.*

Exercise 2.28. *Let $p = (x_1, y_1)$ and $q = (x_2, y_2)$ be any points on the product manifold $(M^n \times N^k, g \oplus h)$. Then*

$$d^2_{g\oplus h}(p, q) = d^2_g(x_1, x_2) + d^2_h(y_1, y_2). \tag{2.18}$$

Example 2.29. *Now we consider a special case of Riemannian product. Let (X^n, g) be a Riemannian manifold. We define the Riemannian product $Y^{n+1} \equiv \mathbb{R} \times X^n$ with a natural coordinate system (t, x). We define a function $h(t, x) \equiv t$. Then one can check that*

$$|\nabla h| \equiv 1, \quad |\nabla^2 h| \equiv 0. \tag{2.19}$$

We take three points $z, x, w \in Y^{n+1}$ such that $h(x) = 0$ and w is the orthogonal projection of z onto the fiber $X^n \times \{0\}$. In particular, $h(w) = 0$ as well. Let $d_0 \equiv d(z, w)$. We will prove that $d^2(z, x) = d^2(z, w) + d^2(x, w)$. NOTE: we will always work with the "regular points" as described in Theorem 2.27.

Let $\sigma : [0, d_0] \to Y^{n+1}$ be the unique minimal geodesic connecting w and z. For any $s \in [0, d_0]$, there exists a geodesic $\tau_s : [0, \ell(s)] \to Y^{n+1}$ with $\tau_s(0) = x$ and $\tau_s(\ell(s)) = \sigma(s)$. Then we have that

$$\begin{aligned} d^2(z, w) = d_0^2 &= \frac{1}{2}\int_0^{d_0} s ds \\ &= \frac{1}{2}\int_I (h(\sigma(s)) - h(\sigma(0))) ds \\ &= \frac{1}{2}\int_I (h(\tau_s(\ell_s)) - h(\tau_s(0))) ds \\ &= \frac{1}{2}\int_I \int_0^{\ell_s} \langle \nabla h(\tau_s(t)), \tau_s'(t)\rangle dt ds. \end{aligned} \tag{2.20}$$

Next, we show that $\langle \nabla h, \tau_s' \rangle$ is constant along the geodesic τ_s. In fact, for any $t \in [0, s]$

$$\Big|\langle \nabla h(\tau_s(t)), \tau_s'(t)\rangle - \langle \nabla h, \tau_s'(t)\rangle(\tau_s(\ell_s))\Big| = \Big|\int_t^s \langle \nabla_{\tau_s'(u)} \nabla h, \tau_s'(u)\rangle du\Big| \equiv 0. \tag{2.21}$$

Therefore,

$$d^2(z,w) = \frac{1}{2}\int_I \int_0^{\ell_s} \langle \nabla h, \tau_s'\rangle(\sigma_s) dt ds = \frac{1}{2}\int_I \ell_s \langle \sigma', \tau_s'\rangle(\sigma(s)) ds. \tag{2.22}$$

Applying the first variation formula, $\frac{d}{ds}\ell_s = \langle \sigma', \tau_s'\rangle$, which implies that

$$d^2(z,w) = \frac{1}{2}\int_I \ell_s \ell_s' ds = \int_0^{d_0} \ell_s \ell_s' ds = \ell^2(d_0) - \ell^2(0) = d^2(x,z) - d^2(x,w). \tag{2.23}$$

Example 2.30 (Rescaling). *Let (M^n, g) be a Riemannian manifold. For any $r > 0$, we define the rescaled metrics $\bar{g}_r$ and $\underline{g}_r$ as follows:*

$$\bar{g}_r(X,Y) = r^2 g(X,Y), \quad \underline{g}_r(X,Y) = r^{-2} g(X,Y). \tag{2.24}$$

Exercise 2.31. *Let (M^n, g) be a Riemannian manifold. Show that for any $r > 0$, $\operatorname{Sc}_{\bar{g}_r} = r^{-2}\operatorname{Sc}_{g_r}$. Then establish the similar rescaling relation for sectional and Ricci curvatures.*

Example 2.32 (Warped product). *Let (N^{n-1}, h) be a Riemannian manifold and let $f(r)$ be a smooth function on (a,b). We construct $g = dr^2 + f^2(r)h$ the warped product metricon $M^n \equiv (a,b) \times N^{n-1}$. The second fundamental form II of N^{n-1} in M^n is defined by*

$$\mathrm{II}(X,Y) \equiv g(\nabla_X \partial_r, Y), \quad X, Y \in T_p N^{n-1}. \tag{2.25}$$

Since $\partial_r = \nabla r$, we have $\mathrm{II} = \nabla^2 r$. Then we have the following fundamental curvature equations

(1) *Radial curvature equation (Riccati)*

$$R_g(\partial_r, X, Y, \partial_r) = -(\nabla_{\partial_r} \operatorname{Hess} r)(X,Y) - \operatorname{Hess}^2 r(X,Y) \tag{2.26}$$

(2) *Tangential curvature equation (Gauß-Codazzi)*

$$R_g(X,Y,Z,W) = R_h(X,Y,Z,W) + \mathrm{II}(X,Z)\,\mathrm{II}(Y,W) - \mathrm{II}(X,W)\,\mathrm{II}(Y,Z) \tag{2.27}$$

(3) *Mixed curvature equation*

$$R(X,Y,Z,\partial_r) = -(\nabla_X \mathrm{II})(Y,Z) + (\nabla_Y \mathrm{II})(X,Z). \tag{2.28}$$

Here X, Y, Z, W are tangential to the fiber N^{n-1}.

Exercise 2.33. *Consider the Euclidean metric g_0 on $\mathbb{R}^{n+1}$ in polar coordinates $g_0 = dr^2 + r^2 g_{\mathbb{S}^n}$. Prove that $\sec_{\mathbb{S}^n} \equiv +1$.*

Exercise 2.34. *Let $\eta : \underbrace{TM \times \ldots \times TM}_{r} \to C^\infty(M^n)$ be a tensor multilinear map on M^n, called a $(0,r)$-tensor field. Let $X \in \mathfrak{X}(M^n)$. We define*

$$\mathfrak{L}_X \eta(Y_1, \ldots, Y_r) \equiv X(\eta(Y_1, \ldots, Y_r)) - \sum_{i=1}^r \eta(X_1, \ldots, [X, Y_i], \ldots, X_r). \tag{2.29}$$

Exercise 2.35. *Let η and ζ be $(0,r)$ and $(0,s)$ tensor fields on M^n, repsectively. For any $X \in \mathfrak{X}(M^n)$, show that*

$$\mathfrak{L}_X(T \otimes S) = (\mathfrak{L}_X T) \otimes S + T \otimes (\mathfrak{L}_X S). \tag{2.30}$$

As a special case, $\mathfrak{L}_X(fT) = X(f) + f\mathfrak{L}_X T$ for any $f \in C^\infty(M^n)$.

Exercise 2.36. *Prove that in the above warped product metric, $2\,\mathrm{Hess}(r) = \mathfrak{L}_{\partial_r} g$ and $\mathrm{Hess}(r) = (f \cdot \partial_r f)h$.*

Exercise 2.37. *Let $(r, x_1, \ldots, x_{n-1})$ be local coordinates on $(a,b) \times N^{n-1}$. Prove the following curvature identities for the warped product metric $dr^2 + f^2(r)h$:*

1. $R^g_{ijk\ell} = f^2(r) R^h_{ijk\ell} + f^2(r)(f'(r))^2(h_{ik}h_{j\ell} - h_{i\ell}h_{jk})$.
2. $R^g_{ij\ell r} = 0$ *and* $R^g_{rijr} = -f(r) \cdot f''(r) g_{ij}$.
3. $\sec_g(\partial_i, \partial_j) = f^{-2}(r)(\sec_h(\partial_i, \partial_j) - (f'(r))^2)$ *and* $\sec_g(\partial_r, \partial_i) = -f^{-1}(r) f''(r)$.
4. $\mathrm{Ric}^g_{ij} = \mathrm{Ric}^h_{ij} - \Big((n-2)(f'(r))^2 + f(r)f''(r)\Big) g_{ij}$.
5. $\mathrm{Ric}^g_{ir} = 0$ *and* $\mathrm{Ric}^g_{rr} = -(n-1) f^{-1}(r) f''(r)$.

Reference for the above curvature formulae: Chapter 3-4 in Petersen's book.

3. More examples of metric structures

3.1. Geometry of warped products.

Definition 3.1 (Metric cone (Euclidean cone))**.** Let (Σ, d_Σ) be a compact metric space with diameter $\leq \pi$. The metric space $(Z, d_C) \equiv (C(\Sigma), d_C)$ is called the metric cone over the cross-section Σ, denoted by $C(\Sigma) \equiv C_0(\Sigma)$, if Z is homeomorphic to the topological cone $(\Sigma \times [0,\infty))/(\Sigma \times \{0\})$ and the cone metric d_C is given by

$$d^2(x,y) = d^2(z_*,x) + d^2(z_*,y) - 2d(z_*,x)d(z_*,y)\cos d_\Sigma(\bar{x},\bar{y}), \tag{3.1}$$

where $z_* \equiv \Sigma \times \{0\}$ is called the cone vertex of $C(\Sigma)$.

Lemma 3.2. *Any cone metric g_C is scaling invariant.*

Example 3.3. *Let (Σ, h) be a compact Riemannian manifold with $\mathrm{diam}_h(\Sigma) \leq \pi$. The cone metric of $(C(\Sigma), d_C, z_*)$ can be written by the warped Riemannian metric $g_C = dr^2 + r^2 \cdot h$ away from the cone tip z_*. As an exercise, one can prove the following Euclidean law of cosine on $(C(\Sigma), g_C)$.*

Exercise 3.4. *Let $g_C = dr^2 + r^2 \cdot h$ be the cone metric of $C(\Sigma)$. Show that $\mathfrak{L}_{\partial_r} g = \frac{2}{r} g$.*

Exercise 3.5. *Prove that a metric cone $C(\Sigma)$ is smooth everywhere if and only if $C(\Sigma)$ is flat which is equivalent to say the cross-section Σ is isometric to the round sphere of curvature $+1$.*

There are many examples of metric cones:

(1) $\mathbb{R}^n$ is the metric cone over the standard round sphere $\mathbb{S}^{n-1}$.
(2) $\mathbb{R}_+$ is the metric cone over a point.
(3) The half plane $\mathbb{R} \times \mathbb{R}_+$ is the metric cone over the segment $[0,\pi]$. It can be viewed as the quotient $\mathbb{R}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the reflection $(x,y) \mapsto (x,-y)$.
(4) Let $\mathbb{Z}_+$ be the group generated by the rotation $(x,y) \mapsto (-x,-y)$. Then $\mathbb{R}_+/\mathbb{Z}_2$ is the metric cone over the circle S^1 of perimeter π.
(5) The metric cone $\mathbb{R}^4/\mathbb{Z}_2 \cong C(\mathbb{R}P^3)$, where the quotient group $\mathbb{Z}_2$ is generated by the involution $\iota : (x_1,x_2,x_3,x_4) \mapsto (-x_1,-x_2,-x_3,-x_4)$.

Using the curvature equations, we can compute the curvature of the cross-section of $C(\Sigma)$.

Exercise 3.6. *Let $(C(\Sigma), g_C, z_*)$ be a metric cone over a compact manifold (Σ, h), where z_* is the cone tip. Prove that away from the cone tip z_*, $\mathrm{Ric}_{g_C} \equiv 0$ iff $\mathrm{Ric}_h \equiv (n-2)h$, and g_C is flat iff $\sec_h \equiv +1$.*

Definition 3.7 (Spherical suspension)**.** Let (Σ, d_Σ) be a metric space of diameter $\leq \pi$. A metric space (Z, d) is called the spherical suspension (or spherical cone) over Σ, denoted by $\mathrm{Susp}_{+1}(\Sigma) \equiv C_{+1}(\Sigma)$, if Z is homeomorphic to $(\Sigma \times [0,\pi])/(\Sigma \times \{0,\pi\})$ and the spherical metric d is given by

$$\cos d(x,y) = \cos d(z_*,x)\cos d(z_*,y) + \sin d(z_*,x)\sin d(z_*,y)\cos d_\Sigma(\bar{x},\bar{y}). \tag{3.2}$$

Notice that, there are two vertices $z_* \equiv \Sigma \times \{0\}$ and $w_* \equiv \Sigma \times \{\pi\}$ on the spherical suspension $C_{+1}(\Sigma)$.

Theorem 3.8. *Let $C(Z)$ be a metric cone over a compact Riemannian manifold (Z,h). Assume that $C(Z)$ is isometric to $C(W) \times \mathbb{R}$. Then both Z and W are round spheres of curvature $+1$. Moreover, Z is the spherical suspension over W.*

Definition 3.9 (Hyperbolic suspension)**.** Let (Σ, d_Σ) be a metric space of diameter $\leq \pi$. A metric space (Z, d) is a called the hyperbolic suspension (or hyperbolic cone) over Σ, denoted by $\mathrm{Susp}_{-1}(\Sigma) \equiv C_{-1}(\Sigma)$ if Z is homeomorphic to the topological cone $(\Sigma \times [0, \infty))/(\Sigma \times \{0\})$ and the is given by

$$\cosh d(x, y) = \cosh d(z_*, x) \cosh d(z_*, y) - \sinh d(z_*, x) \sinh d(z_*, y) \cos d_\Sigma(\bar{x}, \bar{y}), \tag{3.3}$$

where $z_* \equiv \Sigma \times \{0\}$ is called the cone vertex of $C_{-1}(\Sigma)$.

Exercise 3.10. *Let $Z \equiv \mathrm{Susp}_k(\Sigma)$ with $k \in \{-1, 1\}$. Show that $\sec_\Sigma \equiv 1$ if and only if $\sec_Z \equiv k$.*

3.2. **Riemannian geometry of Lie groups: a crash course.**

Definition 3.11 (Lie group)**.** A group G is called a Lie group if the following properties hold:

1. G is a differentiable manifold.
2. Both the multiplication $(g, h) \mapsto gh$ and the inversion $g \mapsto g^{-1}$ are C^∞ for any $g, h \in G$.

On a Lie group G, one can define the left and right translations as follows,

$$L_g(h) \equiv g \cdot h, \quad \forall h \in G, \tag{3.4}$$

$$R_h(h) \equiv h \cdot g, \quad \forall h \in G. \tag{3.5}$$

Definition 3.12 (Left invariant vector field)**.** Let G be a Lie group. A vector field X is called a left invariant vector field if for any $g, h \in G$,

$$DL_g(X_h) = X_{L_g(h)} = X_{gh}. \tag{3.6}$$

Similarly, one can define the notion of a right invariant vector field. By definition, for a left invariant vector field X, we have $X_g = dL_g(X_e)$. Similarly, $X_g = dR_g(X_e)$ if X is a right invariant vetor field.

Definition 3.13 (Lie algebra)**.** A linear space V is called a Lie algebra if V is equipped with a bilinear operation $[\cdot, \cdot] : V \times V \to V$ such that for all $u, v, w \in V$

1. (Skew symmetry) $[v, w] = -[w, v]$,
2. (Jacobi identity) $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0$.

Exercise 3.14. *Let M^n be a differentiable manifold. Show that for any $X, Y, Z \in \mathfrak{X}(M^n)$, we have the following identities:*

1. $[Y, X] = -[X, Y]$.
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

As a result, $\mathfrak{X}(M^n)$ is a Lie algebra (of infinite dimension).

Let G be a Lie group, one can prove that $[X, Y]$ is a left invariant vector field if both X and Y are left invariant vector fields. Then we denote by $\mathfrak{g}$ the linear space of all left invariant vector fields on G. In the above notations, it follows that $\mathfrak{g}$ is a Lie algebra with $\dim(\mathfrak{g}) = \dim(T_eG) = \dim(G)$, and $\mathfrak{g}$ is called the Lie algebra of G.

Example 3.15. *Let $\mathrm{SO}(2)$ be the space of rotation matrices*

$$\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad t \in [0, 2\pi]. \tag{3.7}$$

Then SO(2) *is homeomorphic to* S^1. *Its Lie algebra is*

$$\mathfrak{so}(2) \equiv \left\{ \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \Big| x \in \mathbb{R} \right\}. \tag{3.8}$$

Definition 3.16. Let M^n be a differentiable manifold and let $X \in \mathfrak{X}(M^n)$. A curve $\gamma : [0,1] \to X$ is called the integral curve of X if for any $t \in [0,1]$,

$$\frac{d}{dt}\gamma(t) = X(t). \tag{3.9}$$

Given $X \in \mathfrak{X}(M^n)$, the map $\phi_t : M^n \to M^n$ defined by $\phi_t(p) \equiv \gamma(t)$ is called the flow of X, where $\gamma : [0,1]$ is the integral curve of X with $\gamma(0) = p$.

Definition 3.17. Let G be a Lie group. A 1-parameter Lie group subgroup of G is a Lie group homomorphism $\gamma : \mathbb{R} \to G$ from the additive group of $\mathbb{R}$ into G.

Theorem 3.18. *Let G be a Lie group. For any $v \in T_eG$, there exists a unique* 1*-parameter subgroup of G such that $\gamma'(0) = v$.*

Now we are ready to define the exponential map. Given any $X \in \mathfrak{g}$, the integral curve through of X through e is the unique 1-parameter subgroup γ of G with $\gamma(0) = e$ and $\gamma'(0) = X_e$. This integral curve will be denoted by $\exp_X$.

Definition 3.19 (Exponential map)**.** The exponential map $\exp : \mathfrak{g} \to G$ is the map defined by $\exp(X) \equiv \exp_X(1)$.

Proposition 3.20. *Let G be a Lie group, $X \in \mathfrak{g}$, $s, t \in \mathbb{R}$ and $r \in G$.*

(1) $\exp(tX) = \exp_X(t)$ *for all* $t \in \mathbb{R}$.
(2) $\exp((t+s)X) = \exp(sX) \cdot \exp(tX)$.
(3) $L_g \circ \exp_X$ *is the unique integral curve of X through g.*
(4) *Let ρ_t be the flow of X. Then $\rho_t = R_{\exp_X(t)}$.*

Proof. The proof of items (1)-(3) demands the standard ODE uniqueness. We only prove item (4). For any $g \in G$, ρ_t maps g to $\gamma_g(t)$, where γ_g is the integral curve of X through g. It follows from item (3) that $\gamma_g(t) = L_g \circ \exp_X(t)$. Therefore,

$$\rho_t(g) = L_g \circ \exp_X(t) = g \cdot \exp_X(t) = R_{\exp_X(t)}(g), \quad \forall g \in G. \tag{3.10}$$

The proof is done. ∎

Example 3.21 (Group of matrices)**.** *Compute Lie algebras of* GL(n) *and* SL(n).

Example 3.22. *Compute the Lie algebras of* SU(n) *and* U(n).

Definition 3.23 (Left invariant metric)**.** Let G be a Lie group. A Riemannian metric $\langle\cdot,\cdot\rangle$ on G is said to be left invariant if every left translation is an isometry, i.e., for any $g, h \in G$ and for any $u, v \in T_gG$,

$$\langle u, v\rangle_g = \langle (DL_h)_g u, (DL_h)_g v\rangle_{gh}. \tag{3.11}$$

Right invariant metrics are defined correspondingly.

Any inner product $\langle\cdot,\cdot\rangle_e$ on T_eG can be extended to be a left invariant metric on G by defining,

$$\langle u, v\rangle_g \equiv \langle (DL_{g^{-1}})_g(u), (DL_{g^{-1}})_g(v)\rangle_e. \tag{3.12}$$

Definition 3.24 (Bi-invariant metric)**.** A metric on a Lie group G is said to be bi-invariant if it is both left invariant and right invariant.

Theorem 3.25 (Milnor 1976)**.** *Let G be a Lie group. Then the following holds.*

(1) *If G is compact, then G admits a bi-invariant metric.*
(2) *A Lie group admits a bi-invariant metric if and only if it is isomorphic to $G \times H$ with G compact and H abelian.*

We list some properties of bi-invariant metrics.

Proposition 3.26. *Let G be a Lie group. If $\langle \cdot, \cdot \rangle$ is bi-invariant, then the following holds for every $X, Y, Z \in \mathfrak{g}$:*

(1) $\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = 0$.
(2) *If left invariant metric on G satisfies (1), then it is a bi-invariant metric.*
(3) $\nabla_X X = 0$. *That is, any 1-parameter subgroup of G is a geodesic. Therefore, the two notions of exponential maps* exp *(Lie theoretic) and* Exp *(Riemannian) coincide.*
(4) $\nabla_X Y = \frac{1}{2}[X,Y]$.
(5) $R(X,Y,Z,W) = \frac{1}{4}\langle [X,Y],[W,Z]\rangle$. *Consequently,* $\sec_G \geq 0$.

Exercise 3.27 (An alternative definition of Lie derivative)**.** *Let R be a $(0,r)$ tensor field on M^n. Let X be a vector field with a flow ϕ_t. Then we define*

$$(\mathfrak{L}_X \mathcal{R})_p(v_1,\ldots,v_r) = \frac{d}{dt}\Big|_{t=0}(\phi_t^* R_p(v_1,\ldots,v_r)), \tag{3.13}$$

where $\phi_t^*\mathcal{R}(v_1,\ldots,v_r) \equiv \mathcal{R}(D\phi_t(v_1),\ldots,D\phi_t(v_r))$ and $v_1,\ldots,v_r \in T_pM^n$. Show that this definition coincides with the previous one involving the Lie bracket.

Exercise 3.28. *Let $X \in \mathfrak{X}(M)$. Show that $\mathfrak{L}_X g = 0$ if and only if the flow of X is an isometric action.*

Example 3.29 (Hopf sphere)**.** *Consider the Lie group*

$$\mathrm{SU}(2) \equiv \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} : z, w \in \mathbb{C},\ |z|^2 + |w|^2 = 1 \right\}. \tag{3.14}$$

Obviously, SU(2) *is diffeomorphic to $\mathbb{S}^3$. We define a map $\pi : (z,w) \mapsto z/w$. Then π gives a fiber bundle map $S^1 \to \mathrm{SU}(2) \xrightarrow{\pi} S^2$ (called Hopf fibration). One can also represents π as*

$$(z,w) \mapsto \Big(Re(zw), Im(zw), \frac{|z|^2 - |w|^2}{2}\Big) \in \mathbb{R}^3. \tag{3.15}$$

There is a natural S^1-acting on SU(2)*: $t \cdot (z,w) \equiv (e^{\sqrt{-1}t}z, e^{-\sqrt{-1}t}w)$.*

Next we construct a family of collapsing metrics on SU(2)*. Let X, Y, Z be the left invariant metrics on* SU(2) *such that $[X,Y] = Z$, $[Y,Z] = X$, and $[Z,X] = Y$. Indeed, one can just use the Pauli matrices*

$$X = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z = \frac{1}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}. \tag{3.16}$$

Denote by η_1, η_2 and η_3 the dual frames of X, Y, Z, respectively. We define a family of Riemannian metrics

$$g_t \equiv t^2 \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3. \tag{3.17}$$

Then we can compute the connection terms,

$$\nabla_X X = 0, \quad \nabla_X Y = (2 - t^2)Z, \quad \nabla_X Z = (-2 + t^2)Y, \tag{3.18}$$

$$\nabla_Y X = -t^2 Z, \quad \nabla_Y Y = 0, \quad \nabla_Y Z = X, \tag{3.19}$$

$$\nabla_Z X = t^2 Y, \quad \nabla_Z Y = -X, \quad \nabla_Z Z = 0. \tag{3.20}$$

Furthermore, $\sec(X, Y) = t^2$, $\sec(X, Z) = t^2$, *and* $\sec(Z, Y) = 4 - 3t^2$.

Note that, as $t \to 0$*, the direction* X *is collapsing since its integral curve has length* $2t\pi$*. The "limiting metric" is the round metric of curvature* $+4$ *on the* 2*-sphere of radius* $\frac{1}{2}$*. We will rigorously formulate the above convergence in terms of the Gromov-Hausdorff distance.*

4. The space of metric structures

4.1. **Modern metric Riemannian geometry: natural questions.** In the past half century, metric Riemannian geometry focuses on the following typical problems.

(A) Geometric rigidity

Let $\mathfrak{M}$ be a class of Riemannian manifolds such that there is a geometric quantity (functional)

$$\mathcal{Q} : \mathfrak{M} \to \mathbb{R} \tag{4.1}$$

has an optimal upper (lower) bound Q_0. A geometric rigidity result usually states the isometric classification or characterization of the *optimal solution* $\mathcal{Q}(M^n, g) = Q_0$ for some $(M^n, g) \in \mathfrak{M}$. We list several typical examples.

Theorem 4.1 (Bonnet-Myers). *Let (M^n, g) satisfy $\mathrm{Ric}_g \geq n-1$. Then $\mathrm{diam}_g(M^n) \leq \pi$.*

Theorem 4.2 (Cheng's maximal diameter theorem). *If (M^n, g) satisfies $\mathrm{Ric}_g \geq n-1$ and $\mathrm{diam}_g(M^n) = \pi$. Then (M^n, g) is isometric to the round sphere of curvature $+1$.*

Theorem 4.3 (Bishop-Gromov). *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_g \geq (n-1)k$ with $k \in \{-1, 0, 1\}$. Then for any $r > 0$, $\mathrm{Vol}_g(B_r(p)) \leq V_k^n(r)$. Furthermore, if $Vol_g(B_r(p)) \leq V_k^n(r)$, then $B_r(p)$ is isometric to $B_r(0^n)$, where $B_r(0^n)$ is a geodesic ball of radius r in S_k^n, $V_k^n(r) = \mathrm{Vol}_{g_k}(B_r(0^n))$, and*

$$S_k^n \equiv \begin{cases} \mathbb{S}^n, & k = 1, \\ \mathbb{R}^n, & k = 0, \\ \mathbb{H}^n, & k = -1. \end{cases} \tag{4.2}$$

Theorem 4.4 (Bochner). *Let (M^n, g) be compact and satisfy $\mathrm{Ric}_g \geq 0$. Then $b_1(M^n) \leq \dim(M^n) = n$. Moreover, equality holds iff M^n is isometric to the flat torus $\mathbb{T}^n$.*

B. Stability

Let $\mathfrak{M}$ be a class of Riemannian manifolds. It is a common principle that the topologies of the underlying manifolds M^n change in a discrete manner when the metrics vary in $\mathfrak{M}$. This indicates that certain geometry quantity may have a definite gap when the topology jumps. The topological stability usually refers to phenomenon that the topology remains the same as slightly perturbing geometric quantity.

Theorem 4.5 (Grove-Shiohama). *Let (M^n, g) satisfy $\sec_g \geq 1$ and $\mathrm{diam}_g(M^n) > \frac{\pi}{2}$. Then M^n is homeomorphic to $\mathbb{S}^n$.*

Theorem 4.6 (Perel'man). *There exists a dimensional constant $\delta > 0$ such that if (M^n, g) satisfies $\mathrm{Ric}_g \geq n-1$ and $\mathrm{Vol}(M^n) \geq \mathrm{Vol}(\mathbb{S}^n) - \delta$, then M^n is diffeomorphic to S^n.*

Theorem 4.7 (Colding). *There exists $\delta(n) > 0$ such that if M^n is closed manifold with $\mathrm{diam}^2(M^n) \cdot \mathrm{Ric}_g \geq -\delta$ and $b_1(M^n) = n$. Then M^n is diffeomorphic to a torus.*

C. Almost rigidity (quantitative rigitity)

Given a class of Riemannian manifolds $\mathfrak{M}$ with a geometric quantity $\mathcal{Q}$. We are also interested in the problem that when $\mathcal{Q}((M^n, g))$ is sufficiently close to the optimal bound Q_0, in what sense is (M^n, g) close to the optimal solution (M_0^n, g_0)? The rigorous formulation of this question demands the geometric distance between two Riemannian manifolds such as Gromov-Hausdorff distance and other distances with higher regularity.

D. Moduli space problem

Let M^n be a differentiable manifold. Given certain curvature condition $\mathscr{C}$ (e.g. Einstein condition $\mathrm{Ric}_g \equiv \lambda$, $\mathrm{Sc} \geq 0$, etc), we are interested in the space of Riemannian metrics

$$\mathscr{M} \equiv \{g|\ g \text{ is a Riemannian metric on } M^n \text{ that satisfies Condition } \mathscr{C}\}/\operatorname{Diff}(M^n) \tag{4.3}$$

is called the moduli space under Condition $\mathscr{C}$. A challenging problem is to investigate the behaviors of g as approaching to the "boundary" $\partial\mathscr{M}$. Both the definition of $\partial\mathscr{M}$ and the delicate geometric behaviors near $\partial\mathscr{M}$ requires very deep understanding of the geometric convergence of metric spaces. Usually the Gromov-Hausdorff convergence gives a convenient formulation in this context.

E. Geometric evolution along geometric flows

Let M^n be a differentiable manifold. A very important research area is to study the geometric evolution of a family of Riemannian metrics $g(t)$ on M^n that satisfies certain equation in t. For example, the geometry of the Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}_{g(t)} \tag{4.4}$$

is a central area in geometric analysis and has an exciting history of development.

F. Geometric Group Theory

Definition 4.8 (Word metric)**.**

Theorem 4.9 (Gromov)**.** *virtually nilpotent*

4.2. **Gromov-Hausdorff distance.** Let X be a metric space and $A \subset X$. Then we denote $T_r(A) \equiv \{x \in X | d(x, A) \leq r\}$.

Definition 4.10 (Hausdorff distance)**.** Let (Z, d) be a metric space. Let $A, B \subset Z$ be compact. Then

$$d_H(A, B) \equiv \inf\{r > 0 | B \subset T_r(A) \text{ and } A \subset T_r(B)\}. \tag{4.5}$$

Theorem 4.11. *Denote by $\mathfrak{M}(Z)$ the collection of all subsets in the metric space (Z, d). Then $(\mathfrak{M}(Z), d_H)$ is a metric space.*

Theorem 4.12. *If (Z, d) is compact, then $(\mathfrak{M}(Z), d_H)$ is also compact.*

Definition 4.13 (Gromov-Hausdorff distance)**.** Let (X, d_X) and (Y, d_Y) be compact metric spaces. Then we define

$$d_{GH}(X, Y) \equiv \inf_{Z,\phi,\psi} \left\{ d_H^Z(\phi(X), \psi(Y)) : \phi : X \to Z,\ \psi : Y \to Z \text{ are isometric embeddings} \right\}.$$

Let A and B be two sets. Then we define

$$A \sqcup B \equiv \{(x, 0) : x \in A\} \cup \{(y, 1) : y \in B\}. \tag{4.6}$$

Let (X, d_X) and (Y, d_Y) be metric spaces. We denote by $\bar{d}$ the *admissible metric* on $X \sqcup Y$ which isometrically extends d_X and d_Y into $X \sqcup Y$.

Lemma 4.14. *Let (X, d_X) and (Y, d_Y) be compact metric spaces. We define*

$$\hat{d}_{GH}(X, Y) \equiv \inf_{\bar{d}} \left\{ d_H(X, Y) : \bar{d} \text{ is an admissible metric on } X \sqcup Y \right\}. \tag{4.7}$$

Then $d_{GH} = \hat{d}_{GH}$.

Proof. By definition, $d_{GH} \leq \hat{d}_{GH}$. We will prove $\hat{d}_{GH} \leq d_{GH}$. For any $\epsilon > 0$, there exist a metric space (Z, d) and isometric embeddings

$$\phi : X \hookrightarrow Z, \quad \psi : Y \hookrightarrow Z \tag{4.8}$$

such that $d_{GH}(X, Y) \leq d_H(\phi(X), \psi(Y)) + \epsilon$. Let us consider the product metric d^ϵ on $Z \times [0, \epsilon]$ and isometric embeddings

$$\phi_0 \equiv (\phi, 0) : X \times \{0\} \hookrightarrow Z \times \{0\}, \quad \psi_\epsilon \equiv (\psi, \epsilon) : Y \times \{\epsilon\} \hookrightarrow Z \times \{\epsilon\}. \tag{4.9}$$

The restriction of d_ϵ onto $(\phi(X) \times \{0\}) \cup (\psi(Y) \times \{\epsilon\}) \subset Z \times [0, \epsilon]$ gives an admissible metric $\bar{d}^\epsilon$ on $X \sqcup Y$ (realized by $\phi_0(X) \sqcup \psi_\epsilon(Y)$). Then

$$\begin{aligned} \hat{d}_{GH}(X, Y) &\leq \bar{d}^\epsilon_H(X, Y) \\ &= \bar{d}^\epsilon_H(\phi_0(X), \psi_\epsilon(Y)) \\ &\leq \bar{d}^\epsilon_H(\phi_0(X), X \times \{\epsilon\}) + \bar{d}^\epsilon_H(\phi(X) \times \{\epsilon\}, \psi(Y) \times \{\epsilon\}) \\ &\leq 2\epsilon + d_{GH}(X, Y), \end{aligned} \tag{4.10}$$

which completes the proof. $\square$

Lemma 4.15. *Denote by $\mathcal{M}et$ the collection of all compact metric spaces. Then d_{GH} is a pseudo metric on $\mathcal{M}et$. Furthermore, $d_{GH}(X, Y) = 0$ if and only if X is isometric to Y.*

Proof. First, let us prove the triangle inequality. Given any compact metric spaces (X, d_X), (Y, d_Y) and (Z, d_Z), we will show that for any $\epsilon > 0$,

$$d_{GH}(X, Z) \leq d_{GH}(X, Y) + d_{GH}(Y, Z) + \epsilon. \tag{4.11}$$

Taking admissible metrics d_{XY} and d_{YZ} on $X \sqcup Y$ and $Y \sqcup Z$, respectively, such that

$$d_{XY,H}(X, Y) \leq d_{GH}(X, Y) + \frac{\epsilon}{2}, \quad d_{YZ,H}(Y, Z) \leq d_{GH}(Y, Z) + \frac{\epsilon}{2}. \tag{4.12}$$

One can check that

$$d_{XZ}(x, z) \equiv \inf \{d_{XY}(x, y) + d_{YZ}(y, z) : y \in Y\} \tag{4.13}$$

is an admissible metric on $X \sqcup Z$. Then the collection $\{d_{XY}, d_{YZ}, d_{XZ}\}$ gives an admissible metric on $X \sqcup Y \sqcup Z$. Therefore, using the above admissible metrics,

$$d_{GH}(X, Z) \leq d_H(X, Z) \leq d_H(X, Y) + d_H(Y, Z) \leq d_{GH}(X, Y) + d_{GH}(Y, Z) + \epsilon, \tag{4.14}$$

which completes the proof of the triangle inequality.

Now we are in a position to show that X is isometric to Y if $d_{GH}(X, Y) = 0$. The goal is to construct an isometry from X to Y. By $d_{GH}(X, Y) = 0$, there exist d^i on $X \sqcup Y$ such that $d^i_H(X, Y) \leq 2^{-i} \to 0$. Let $A \equiv \{x_k\}_{k=1}^\infty \subset X$ be countable and dense (Why does A exist?). For $x_1 \in A$, let $\{y_{1,i}\} \subset Y$ be a sequence such that $d^i(x_1, y_{1,i}) < 2^{-i}$. Since (Y, d_Y) is compact, $\{y_{1,i}\}$ has a subsequence $\{y_{1,i_1}\}$ which converges to $y_1 \in Y$. Then

$$d_{i_1}(x_1, y_1) \leq d_{i_1}(x_1, y_{1,i_1}) + d_{i_1}(y_{1,i_1}, y_1) \to 0. \tag{4.15}$$

Similarly, for x_2 and the sequence $\{d_{i_1}\}$, we choose a subsequence $\{d_{i_2}\} \subset \{d_{i_1}\}$ and a point $y_2 \in Y$ such that $d_{i_2}(x_2, y_2) \to 0$. Iterating the above process and applying the diagonal argument, one can select a subsequence of admissible metrics $\{d_\ell\} \subset \{d_i\}$ and a sequence of points $y_k \in Y$ such that for any $k \in \mathbb{Z}_+$,

$$d_\ell(x_k, y_k) \to 0, \quad \text{as } \ell \to 0. \tag{4.16}$$

Now we define $f : A \to Y$ by $f(x_k) \equiv y_k$. Then

$$d_Y(f(x_j), f(x_k)) = d_\ell(f(x_j), f(x_k)) = d_\ell(y_j, y_k) \le d_\ell(y_j, x_j) + d_\ell(x_j, x_k) + d_\ell(x_k, y_k).$$

Taking the limit for $\ell \to \infty$,

$$d_Y(f(x_j), f(x_k)) \le \lim_{\ell\to\infty} d_\ell(x_j, x_k) = \lim_{\ell\to\infty} d_X(x_j, x_k) = d_X(x_j, x_k). \tag{4.17}$$

Using the triangle inequality

$$d_\ell(y_j, y_k) \ge -d_\ell(y_j, x_j) + d_\ell(x_j, x_k) - d_\ell(x_k, y_k), \tag{4.18}$$

one can prove $d_Y(f(x_j), f(x_k)) \ge d(x_j, x_k)$. The above shows that $f : A \to X$ is an isometric embedding. Since A is dense in X, f extends uniquely to an isometric embedding $X \to Y$. One can use the same argument to construct an isometric embedding $h : Y \to X$. ☐

Theorem 4.16. *We denote by $\mathcal{M}et$ the collection of all isometry classes of compact metric spaces. Then $(\mathcal{M}et, d_{GH})$ is a complete metric space.*

Definition 4.17 (Gromov-Hausdorff convergence)**.** We say a sequence of compact metric spaces (X_j, d_j) GH-converge to (X, d) if $d_{GH}(X_j, X) \to 0$.

Definition 4.18 (Pointed Gromov-Hausdorff convergence)**.** We say a sequence of metric spaces (X_j, d_j, p_j) pointed Gromov-Hausdorff converges to (X, d, p) if $(X_j, d_j, p_j) \xrightarrow{GH} (X, d, p)$ for any $R > 0$.

Theorem 4.19. *Let $(X_j, d_j), (X, d) \in \mathcal{M}et$. Then the following statements are equivalent:*

(1) $(X_j, d_j) \xrightarrow{GH} (X, d)$.
(2) *There exists a sequence of ϵ_j-GHAs $f_j : (X_j, d_j) \to (X, d)$ such that $\epsilon_j \to 0$.*
(3) *There exist ϵ_j-GHAs $f_j : X_j \to X$ and $h_j : X \to X_j$ such that*

$$d(f_j \circ h_j, \mathrm{Id}_X) < \epsilon_j, \quad d_j(h_j \circ f_j, \mathrm{Id}_{X_j}) < \epsilon_j. \tag{4.19}$$

Definition 4.20 (Tangent cone)**.** Let (X, d) be a metric space and $p \in X$. A metric space (Y, d_*) is called a tangent cone at p if there exists a sequence $\lambda_j \to \infty$ such that

$$(X, \lambda_j \cdot d, p) \xrightarrow{GH} (Y, d_*) \tag{4.20}$$

Definition 4.21 (Asymptotic cone)**.** Let (X, d) be a complete non-compact metric space. A metric space (Y, d_*) is called an asymptotic cone (or tangent cone at infinity) if there exists a sequence $\lambda_j \to 0$ such that

$$(X, \lambda_j \cdot d, p) \xrightarrow{GH} (Y, d_*). \tag{4.21}$$

Exercise 4.22. *Show that the tangent cone at any point in Riemannian n-manifold is isometric to $\mathbb{R}^n$.*

Exercise 4.23. *Let (Σ, h) be any closed Riemannian manifold with* $\mathrm{diam}_h(\Sigma) \le \pi$. *Let $(C(\Sigma), d_C)$ be the metric cone over Σ with a vertex z_*. Show that the tangent cone of $C(\Sigma)$ at $p \ne z_*$ is isometric to $\mathbb{R}^n$, and the tangent cone of $C(\Sigma)$ at z_* is isometric to itself.*

Exercise 4.24. *Let (Σ, h) be any closed Riemannian manifold with* $\mathrm{diam}_h(\Sigma) \le \pi$. *Let $(\mathrm{Susp}_{+1}(\Sigma), d_C)$ be the spherical suspension over Σ with vertices z_* and w_*. Show that the tangent cone of* $\mathrm{Susp}_{+1}(\Sigma)$ *at any vertex is a metric cone.*

Exercise 4.25. *Show that the asymptotic cone of a complete non-compact metric space (X, d) is independent of the choice of the reference point p.*

5. Volume comparison and Gromov's Compactness Theorem

5.1. Comparison theorems of the Ricci curvature.

Lemma 5.1 (Bochner formula). *Let $f \in C^\infty(M^n)$. Then*

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla \Delta f\rangle + \mathrm{Ric}(\nabla f, \nabla f). \tag{5.1}$$

Proof. Taking normal coordinates $\{x_i\}$, $(f_i^2)_{jj} = 2(f_i f_{ij})_j = 2(f_{ij})^2 + 2f_i f_{ijj}$. By the symmetry of Hess(f), $f_{ijj} = f_{jij}$. Tracing the Ricci identity $f_{jik} - f_{jki} = -f_\ell R_{jik\ell}$, we obtain

$$f_{jij} - f_{jji} = f_\ell \,\mathrm{Ric}_{i\ell}\,. \tag{5.2}$$

Therefore,

$$f_i f_{jij} - f_i f_{jji} = f_i f_\ell \,\mathrm{Ric}_{i\ell}\,. \tag{5.3}$$

So it follows that $\frac{1}{2}(f_i^2)_{jj} = (f_i f_{ij})_j = (f_{ij})^2 + f_i f_{jji} + f_i f_\ell \,\mathrm{Ric}_{i\ell}$. □

Corollary 5.2 (Riccati equation). *Let $p \in M^n$ and let $r(x) \equiv d(p,x)$ be the distance function to p. Then the following holds:*

$$\partial_r H + |\,\mathrm{II}\,|^2 + \mathrm{Ric}_{rr} = 0. \tag{5.4}$$

Theorem 5.3 (Local comparison). *Let (M^n, g) be complete with $\mathrm{Ric}_g \geq (n-1)\kappa$. Given $p \in M^n$, let $r(x) \equiv d(p,x)$ be the distance function to p. Then the following holds:*

(1) *(Laplacian) $\Delta r \leq (n-1)\frac{\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_\kappa(r)}$ when r is smooth.*
(2) *(Volume density) $\sqrt{G} \leq \mathrm{sn}_k^{n-1}(r)$. Moreover, "=" holds for all $r > 0$ iff M has constant curvature κ.*

Lemma 5.4. *If a C^1-function $\lambda(r)$ satisfies*

$$\begin{cases} -\Lambda \leq u'(r) + u^2(r) \leq -\lambda \\ u(r) = \frac{1}{r} + O(r) \text{ as } r \to 0, \end{cases} \tag{5.5}$$

or

$$\begin{cases} -K \leq u'(r) + u^2(r) \leq -k \\ u(0) = 0. \end{cases} \tag{5.6}$$

Then $u_\Lambda(r) \leq u(r) \leq u_\lambda(r)$, where $u_k(r) = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$ for any $k \in \mathbb{R}$.

Proof of Lemma 5.4. □

Proof of Theorem 5.3. Proof of item (1). First, we will prove that as $r \to 0$,

$$H(r) = (n-1)\frac{1}{r} + O(r). \tag{5.7}$$

Let us denote by g_0 the Euclidean metric on $\mathrm{Exp}_p(\overset{\circ}{\mathrm{Seg}})$. Since the Taylor expansion of the metric tensor g_{ij} (in normal coordinates) along γ is given by

$$g_{ij}(t) = g_0 + O(t^2), \tag{5.8}$$

applying Koszul's formula, we have

$$\nabla = \nabla^0 + O(r), \tag{5.9}$$

where $\nabla = \nabla^g$ and $\nabla^0 = \nabla^{g_0}$. Therefore,

$$\mathrm{II}_{\partial_r} = \nabla(\nabla r) = \nabla(\nabla^0 r_0) = (\nabla^0 + O(r))(\nabla^0(r_0)) = (\nabla^0)^2 r_0 + O(r) = \frac{1}{r}\cdot g_0 + O(r). \quad (5.10)$$

Tracing the above expansion, we have $H(r) = \frac{n-1}{r} + O(r)$. Let $u(r) = \frac{H(r)}{n-1}$. Applying Lemma 5.4 to $u(r)$, the Laplacian comparison just follows.

Now we prove item (2). Denote $g \equiv \sqrt{G}$

$$\begin{aligned}\Big(\frac{g}{g_\kappa}\Big)' &= \frac{g'(r)g_\kappa(r) - g(r)g_\kappa'(r)}{g_\kappa^2}\\ &= \frac{(H(r) - H_\kappa(r))g(r)g_\kappa(r)}{g_\kappa^2}\\ &\le 0. \end{aligned} \quad (5.11)$$

Now assuming $g(r) = g_\kappa(r)$ for all $r \ge 0$. Then $H(r)g(r) = g'(r) = g_\kappa'(r) = H_\kappa(r)g_\kappa(r)$, and hence $H(r) = m_\kappa(r) = (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$, and $g(r) = \mathrm{sn}_k^{n-1}(r)g_\kappa(r)$. Therefore,

$$-(n-1)\kappa = H' + \frac{H^2}{n-1} \le H' + \mathrm{Tr}(\mathrm{II}^2) = -\operatorname{Ric}_{rr} \le -(n-1)\kappa. \quad (5.12)$$

"=" holds in all the above inequalities, which implies $(\Delta r)^2 = (n-1)\,\mathrm{Tr}(\nabla^2 r)$. Equivalently, $\lambda_1 = \ldots = \lambda_n = \lambda$, where λ_i's are $(n-1)$ non-zero eigenvalues of $\nabla^2 r$. Consequently,

$$\nabla^2 r = \lambda \cdot \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{n-1}\end{pmatrix}, \quad (5.13)$$

and hence $G(r) = G_\kappa(r)$. It follows from the fact $\mathfrak{L}_{\partial_r} g = 2\,\mathrm{Hess}(r)g$, and the initial condition $g_{ri}(0) = g_{ij}(0) = 0$ $(i \ne j)$, $g_{rr}(0) = g_{ii}(0) = 1$ that $g_{ij} = g_{ir} = 0$ $(i \ne j)$, $g_{rr} = 1$, and $g_{ii}(r) = \mathrm{sn}_k^2(r)$. Therefore, g has constant curvature κ. □

Theorem 5.5 (Global Laplcian comparison). *Let (M^n, g) be complete with* $\operatorname{Ric} \ge (n-1)\kappa$. *Let $r(x) \equiv d(p, x)$ for $p \in M^n$. Then Δr is a signed Radon measure on (M^n, g) with the decomposition*

$$\Delta r = \mu_{ac} + \mu_{sing} \quad (5.14)$$

such that

$$\mu_{ac} \le (n-1)\frac{\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_k(r)} \quad (5.15)$$

and μ_{sing} is non-positive and supported in $\mathcal{C}_p$, where $\mathcal{C}_p$ denotes the cut locus of p. In particular, the Laplacian comparison holds in the distributional sense.

Proof. We only verify that μ_{sing} is non-positive. Let $B_R+(y) \subset B_R(y) \setminus (\{p\} \cup \mathcal{C}_p)$ be the set on which $\Delta r > 0$. Let $\partial B_R^-(y) \subset \partial B_R(y) \setminus (\{p\} \cup \mathcal{C}_p)$ denote the subset on which $\langle \nabla r, N\rangle < 0$, where N is the outward unit normal vector field. For any $\epsilon > 0$, taking some open set U_ϵ such that $\mathcal{C}_p \subset U_\epsilon \subset T_\epsilon(\mathcal{C}_p)$. Let $\widehat{N}$ be the outward unit normal vector field to $\partial(B_R(y) \setminus U_\epsilon)$ at $x \in B_R(y) \cap \partial U_\epsilon$. Then $\langle \nabla r, \widehat{N}\rangle \ge 0$.

Let $f \in C^\infty(M^n)$ with $\nabla f \equiv 0$ near $\partial B_R(y)$. Then

$$\begin{aligned}\lim_{\eta\to 0}\int_{B_R(y)\backslash B_\eta(q)} r\Delta f &= -\int_{B_R(y)}\langle\nabla r, \nabla f\rangle \\ &= \lim_{\epsilon\to 0}\int_{B_R(y)\backslash U_\epsilon} f\Delta r - \lim_{\epsilon\to 0}\int_{\partial B_R(y)\backslash U_\epsilon}\langle\nabla r, N\rangle f - \lim_{\epsilon\to 0}\int_{B_R(y)\backslash \partial U_\epsilon}\langle\nabla r, N\rangle f \\ &= \int_{B_R(y)\backslash \mathcal{C}_p} f\Delta r - \int_{\partial B_R(y)}\langle\nabla r, N\rangle f - \lim_{\epsilon\to 0}\int_{B_R(y)\cap\partial U_\epsilon}\langle\nabla r, \widehat{N}\rangle f. \quad (5.16)\end{aligned}$$

Notice that the last term in the above equality is *non-positive.* Let $f \equiv 1$ and $\mathrm{Supp}(f) \subset B_R(y)$. Then

$$\int_{\mathcal{C}_p\cap B_R(y)}\Delta r \leq \int_{B_R(y)}\Delta r - \int_{\partial B_R(p)}\langle\nabla r, N\rangle = 0, \quad (5.17)$$

which implies μ_{sing} is non-positive. □

Theorem 5.6 (General relative volume comparison)**.** *Let (M^n, g, p) be complete such that* $\mathrm{Ric} \geq (n-1)\kappa$. *Assume that $r_1 \leq r_2 \leq r_3 \leq r_4$, then*

$$\frac{\mathrm{Vol}(A_{r_3,r_4}(p))}{\mathrm{Vol}_k(A_{r_3,r_4}(0^*))} \leq \frac{\mathrm{Vol}(A_{r_1,r_2}(p))}{\mathrm{Vol}_k(A_{r_1,r_2}(0^*))}. \quad (5.18)$$

Equality holds if and only if $A_{r_1,r_4}(p)$ is isometric to $A_{r_1,r_4}(0^*) \subset S^n_\kappa$.

Proof. Let $f(r) \equiv \frac{\mathrm{Area}(\partial B_r(x))}{\mathrm{Area}_\kappa(\partial B_r(0^*))} = \frac{\mathrm{Area}(\partial B_r(x))}{\sqrt{G_\kappa(r)}\cdot\omega_{n-1}}$. Let $\{r, \Theta\}$ be the geodesic polar coordinate system. Then

$$\begin{aligned}\frac{df}{dr} &= \frac{1}{\omega_{n-1}}\int_{S^{n-1}}\frac{\sqrt{G}}{\sqrt{G_\kappa}}\cdot\Big(\frac{\partial_r\sqrt{G}}{\sqrt{G}} - \frac{\partial_r\sqrt{G_\kappa}}{\sqrt{G_\kappa}}\Big)d\Theta \\ &\leq \frac{1}{\omega_{n-1}}\int_{S^{n-1}}\frac{\sqrt{G}}{\sqrt{G_\kappa}}\cdot(H(r) - H_\kappa(r))d\Theta \leq 0, \quad (5.19)\end{aligned}$$

and $f(0) = 1$. Next,

$$\begin{aligned}\mathrm{Vol}(A_{r_3,r_4}(x))\cdot\mathrm{Vol}_\kappa(A_{r_1,r_2}(0^*)) &= \Big(\int_{r_3}^{r_4}A(t)dt\Big)\cdot\Big(\int_{r_1}^{r_2}A_\kappa(t)dt\Big) \\ &= \Big(\int_{r_3}^{r_4}f(t)A_\kappa(t)\Big)\cdot\Big(\int_{r_1}^{r_2}A_\kappa(t)dt\Big) \\ &\leq f(r_3)\Big(\int_{r_3}^{r_4}A_\kappa(t)dt\Big)\cdot\Big(\int_{r_1}^{r_2}A_\kappa(t)dt\Big) \\ &\leq \Big(\int_{r_3}^{r_4}A_\kappa(t)dt\Big)\cdot\Big(\int_{r_1}^{r_2}f(t)A_\kappa(t)dt\Big) \\ &= \mathrm{Vol}_\kappa(A_{r_3,r_4}(0^*))\cdot\mathrm{Vol}(A_{r_1,r_2}(x)). \quad (5.20)\end{aligned}$$

So the proof of the comparison is done.

Next, when equality holds, $f(r_1) = f(r_4)$. Then the proof of Laplacian comparison implies that $A_{r_1,r_4}(p)$ is isometric to $A_{r_1,r_4}(0^n)$. □

Corollary 5.7. *Let (M^n, g, p) be complete such that* $\mathrm{Ric} \geq (n-1)\kappa$. *For any $R > 0$,*

$$\frac{\mathrm{Area}(\partial B_R(p))}{\mathrm{Area}_k(\partial B_R(0^*))} \leq \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_k(B_R(0^*))}, \tag{5.21}$$

and equality holds if and only if $B_R(p)$ is isometric to $B_R(0^*) \subset S^n_k$.

Proof. For any $\epsilon > 0$ and $R > 0$, by Theorem 5.6,

$$\frac{\mathrm{Vol}(B_{R+\epsilon}(p)) - \mathrm{Vol}(B_R(p))}{\mathrm{Vol}_\kappa(B_{R+\epsilon}(0^*)) - \mathrm{Vol}_\kappa(B_R(0^*))} = \frac{\mathrm{Vol}(A_{R,R+\epsilon}(p))}{\mathrm{Vol}_\kappa(A_{R,R+\epsilon}(0^*))} \leq \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_\kappa(B_R(0^*))}, \tag{5.22}$$

which implies

$$\frac{\frac{1}{\epsilon} \cdot (\mathrm{Vol}(B_{R+\epsilon}(p)) - \mathrm{Vol}(B_R(p)))}{\frac{1}{\epsilon} \cdot (\mathrm{Vol}_\kappa(B_{R+\epsilon}(0^*)) - \mathrm{Vol}_\kappa(B_R(0^*)))} \leq \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_\kappa(B_R(0^*))}. \tag{5.23}$$

Letting $\epsilon \to 0$, the desired comparison immediately follows.

For the rigidity part, we assume $\frac{\mathrm{Area}(\partial B_R(p))}{\mathrm{Area}_\kappa(\partial B_R(0^*))} = \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_\kappa(B_R(0^*))}$ and let

$$V(r) \equiv \mathrm{Vol}(B_r(p)), \quad V_\kappa(r) \equiv \mathrm{Vol}_\kappa(B_r(0^*)) \tag{5.24}$$

and

$$S(r) \equiv \mathrm{Area}(\partial B_r(p)), \quad S_\kappa(r) \equiv \mathrm{Area}_\kappa(\partial B_r(0^*)). \tag{5.25}$$

Then

$$\frac{V(R)}{S(R)} = \frac{\int_0^R S(t)dt}{S(R)} = \int_0^R \frac{S(t)}{S(R)} dt. \tag{5.26}$$

By the relative area comparison, it holds that for every $0 < t \leq R$,

$$\frac{S(t)}{S(R)} \geq \frac{S_\kappa(t)}{S_\kappa(R)}. \tag{5.27}$$

Therefore,

$$\frac{V(R)}{S(R)} \geq \int_0^R \frac{S_\kappa(t)}{S_\kappa(R)} = \frac{\int_0^R S_\kappa(t)dt}{S_\kappa(R)} = \frac{V_\kappa(R)}{S_\kappa(R)}. \tag{5.28}$$

The assumption $\frac{V(R)}{S(R)} = \frac{V_\kappa(R)}{S_\kappa(R)}$ and the above inequality imply that for every $0 < t \leq R$,

$$\frac{S(R)}{S_\kappa(R)} \equiv \frac{S(t)}{S_\kappa(t)}. \tag{5.29}$$

Since $\lim_{t\to 0} \frac{S(t)}{S_\kappa(t)} = 1$, so for every $0 < t < R$, $S(t) = S_\kappa(t)$ and hence $V(t) = V_\kappa(t)$. Then $B_R(p)$ is isometric to $B_R(0^n) \subset S^n_\kappa$.

□

Theorem 5.8 (Bishop-Gromov's relative volume comparison)**.** *Let (M^n, g) be complete and satisfy* $\mathrm{Ric}_g \geq (n-1)\kappa$. *Then for any $x \in M^n$, the quantity*

$$Q_x(r) \equiv \frac{\mathrm{Vol}(B_r(p))}{V_k(r)} \tag{5.30}$$

is monotone decreasing with $\lim_{r\to 0} Q_x(x) = 1$, where $V_k(r) \equiv \mathrm{Vol}_\kappa(B_r(0^*))$ and $B_r(0^*) \subset S^n_\kappa$. Moreover, if $Q_x(R) = Q_x(r)$ for some $r \leq R$, then $B_R(x)$ is isometric to $B_R(0^*)$.

Proof. For any $R > r > 0$, applying Theorem 5.6,

$$\frac{\mathrm{Vol}(B_R(p)) - \mathrm{Vol}(B_r(p))}{\mathrm{Vol}_\kappa(B_R(0^*)) - \mathrm{Vol}_\kappa(B_r(0^*))} \leq \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_\kappa(B_r(0^*))}. \tag{5.31}$$

Straightforward computations imply

$$\frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_\kappa(B_R(0^*))} \leq \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_\kappa(B_r(0^*))}. \tag{5.32}$$

Now let us prove the rigidity part. First, by volume comparison, for any $\epsilon > 0$,

$$\frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_k(B_R(0^n))} \leq \frac{\mathrm{Vol}(B_{R-\epsilon}(p))}{\mathrm{Vol}_k(B_{R-\epsilon}(0^n))} \leq \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_k(B_r(0^n))}. \tag{5.33}$$

If $\frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_k(B_R(0^n))} = \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_k(B_r(0^n))}$ holds for some $0 < r < R$, we have that for every $\epsilon > 0$,

$$\frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_k(B_R(0^n))} = \frac{\mathrm{Vol}(B_{R-\epsilon}(p))}{\mathrm{Vol}_k(B_{R-\epsilon}(0^n))} = \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_k(B_r(0^n))}, \tag{5.34}$$

and hence for every $\epsilon > 0$

$$\frac{\mathrm{Vol}(B_R(p)) - \mathrm{Vol}(B_{R-\epsilon}(p))}{\mathrm{Vol}_k(B_R(0^n)) - \mathrm{Vol}_k(B_{R-\epsilon}(0^n))} = \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_k(B_r(0^n))}. \tag{5.35}$$

Let $\epsilon \to 0$,

$$\frac{\mathrm{Vol}(\partial B_R(p))}{\mathrm{Vol}_k(\partial B_R(0^n))} = \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_k(B_r(0^n))} = \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_k(B_R(0^n))}. \tag{5.36}$$

Then the isometric rigidity follows from Corollary 5.7. □

Theorem 5.9 (Bonnet-Myers)**.** *Let* (M^n, g) *satisfy* $\mathrm{Ric}_g \geq n-1$*. Then* $\mathrm{diam}_g(M^n) \leq \pi$*.*

Proof. We prove it by contradiction. Suppose there exists $\epsilon > 0$ such that $\mathrm{diam}_g(M^n) = \pi+\epsilon$. Let $\gamma : [0, \pi+\epsilon] \to M^n$ be a minimal geodesic connecting $p, q \in M^n$ such that $L(\gamma) = \pi + \epsilon$.

We denote $r(x) \equiv d(x,p)$. Then r is smooth at $\gamma(t)$ for $t \in (0, \pi+\epsilon)$. By Laplacian comparison,

$$(\Delta r)(\gamma(t)) \leq (n-1) \cdot \frac{\cos t}{\sin t} \to -\infty, \text{ as } t \to \pi. \tag{5.37}$$

Contradiction. □

Theorem 5.10 (Cheng's Maximal Diameter Theorem)**.** *Let* (M^n, g) *satisfy* $\mathrm{Ric}_g \geq n-1$ *and* $\mathrm{diam}_g(M^n) = \pi$*. Then* (M^n, g) *must be isometric to the round sphere* $\mathbb{S}^n$ *of curvature* $+1$*.*

Proof. We will apply the relative volume comparison theorem to prove this rigidity. Let $p, q \in M^n$ satisfy $d_g(p,q) = \pi$ Let us denote $B_{\frac{\pi}{2}}(0^*) \equiv \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n : x_{n+1} > 0\}$. Notice that

$$B_\pi(p) = B_\pi(q) = M^n, \tag{5.38}$$

and hence

$$\mathrm{Vol}_g(B_\pi(p)) = \mathrm{Vol}_g(B_\pi(q)) = \mathrm{Vol}_g(M^n). \tag{5.39}$$

Applying volume comparison,

$$\frac{\mathrm{Vol}(B_{\frac{\pi}{2}}(p))}{\mathrm{Vol}_{g_1}(B_{\frac{\pi}{2}}(0^*))} \geq \frac{\mathrm{Vol}_\pi(p)}{\mathrm{Vol}(\mathbb{S}^n)}. \tag{5.40}$$

Immediately,

$$\frac{\mathrm{Vol}(B_{\frac{\pi}{2}}(p))}{\mathrm{Vol}(M^n)} \geq \frac{\mathrm{Vol}_{g_1}(B_{\frac{\pi}{2}}(0^*))}{\mathrm{Vol}(\mathbb{S}^n)} = \frac{1}{2}. \tag{5.41}$$

"=' holds iff $B_\pi(p)$ is isometric to $\mathbb{S}^n$. Similarly, $\mathrm{Vol}(B_{\frac{\pi}{2}}(q)) \geq \frac{1}{2}\mathrm{Vol}(M^n)$. On the other hand, $\dot{(}p,q) = \pi$ implies $B_{\frac{\pi}{2}}(p) = B_{\frac{\pi}{2}}(q) = \emptyset$. Therefore,

$$\mathrm{Vol}(M^n) \geq \mathrm{Vol}(B_{\frac{\pi}{2}}(p)) + \mathrm{Vol}(B_{\frac{\pi}{2}}(q)) \geq \frac{1}{2}\mathrm{Vol}(M^n) + \frac{1}{2}\mathrm{Vol}(M^n) = \mathrm{Vol}(M^n). \tag{5.42}$$

Therefore,

$$\mathrm{Vol}_g(B_{\frac{\pi}{2}}(p)) = \mathrm{Vol}_g(B_{\frac{\pi}{2}}(q)) = \frac{1}{2}\mathrm{Vol}(M^n). \tag{5.43}$$

Applying Bishop-Gromov's volume comparison, M^n is isometric to $\mathbb{S}^n$. □

5.2. Compact subsets in $(\mathcal{M}et, d_{GH})$.

Definition 5.11 (Gromov-Hausdorff approximation)**.** Let $(X, d_X), (Y, d_Y) \in \mathcal{M}et$. A map $f : X \to Y$ is called an ϵ-Gromov-Hausdorff approximation (ϵ-GHA) if the following properties hold:

(1) (ϵ-isometric embedding) $|d_Y(f(p), f(q)) - d_X(p,q)| < \epsilon$ for all $p, q \in X$.
(2) (ϵ-onto) $T_\epsilon(f(X)) = Y$.

Definition 5.12. Let $(X, d_X), (Y, d_Y) \in \mathcal{M}et$. Then we define

$$\hat{d}_{GH}((X, d_X), (Y, d_Y)) \equiv \inf\left\{\epsilon > 0 | \exists\ \epsilon - \text{GHAs } \phi : X \to Y \text{ and } \psi : Y \to X\right\}. \tag{5.44}$$

Lemma 5.13. $\frac{2}{3}d_{GH} \leq \hat{d}_{GH} \leq 2d_{GH}$.

Proof. The first step is to prove that $d_{GH}(X,Y) \leq \frac{3}{2}\hat{d}_{GH}(X,Y)$ for any two compact metric spaces (X, d_X) and (Y, d_Y). Let $\hat{d}_{GH}(X,Y) = r$. For any $r' > r$, let $f : X \to Y$ and $h : Y \to X$ be r'-GHAs. We will construct an admissible metric $\bar{d}$ on $X \sqcup Y$ such that $Y \subset T_{\frac{3r'}{2}}(X)$ and $X \subset T_{\frac{3r'}{2}}(Y)$.

First, we define an admissible metric $\bar{d}_f$ on $X \sqcup Y$:

$$\bar{d}_f(x,y) \equiv \inf_{z \in X}\left\{d(x,z) + d(f(x), y)\right\} + \frac{r'}{2}, \quad x \in X,\ y \in Y. \tag{5.45}$$

We will verify that $\bar{d}_f$ satisfies the triangle inequality. Given any $\epsilon > 0$, let $z, w \in X$ such that

$$\inf_{x' \in X}\left\{d_X(x_1, x') + d_Y(f(x'), y)\right\} \geq d_X(x_1, z) + d_Y(f(z), y) - \epsilon, \tag{5.46}$$

$$\inf_{x' \in X}\left\{d_X(x_2, x') + d_Y(f(x'), y)\right\} \geq d_X(x_2, w) + d_Y(f(w), y) - \epsilon. \tag{5.47}$$

Then

$$
\begin{aligned}
& \bar{d}_f(x_1, y) + \bar{d}_f(x_2, y) \\
= & \inf_{x' \in X} \{d_X(x_1, x') + d_Y(f(x'), y)\} + \inf_{x' \in X} \{d_X(x_2, x') + d_Y(f(x'), y)\} + r' \\
\geq & \, d_X(x_1, z) + d_Y(f(z), y) + d_X(x_2, w) + d_Y(f(w), y) + r' - 2\epsilon \\
\geq & \, d_X(x_1, z) + d_Y(f(z), f(w)) + d_X(x_2, w) + r' - 2\epsilon \\
\geq & \, d_X(x_1, z) + d_X(z, w) + d_X(x_2, w) - 2\epsilon \\
\geq & \, d_X(x_1, x_2) - 2\epsilon.
\end{aligned}
$$

Therefore, $d_X(x_1, x_2) \leq \bar{d}_f(x_1, y) + \bar{d}_f(x_2, y)$. Similarly, one can prove $\bar{d}_f(x, y_1) + \bar{d}_f(x, y_2) \geq d(x, y)$.

Taking any $x \in X$,

$$
\bar{d}_f(x, f(x)) = \inf_{z \in X} \{d_X(x, z) + d_Y(f(z), f(x))\} + \frac{r'}{2} = \frac{r'}{2}. \tag{5.48}
$$

Therefore, $X \subset T_{r'/2}(Y)$. On the other hand, for any $y \in Y$, there exists some $x \in X$ such that $d_Y(f(x), y) < r$. So it follows that

$$
\bar{d}_f(x, y) \leq \bar{d}_f(x, f(x)) + \bar{d}_f(f(x), y) < \frac{r'}{2} + r < \frac{3r'}{2}, \tag{5.49}
$$

which implies $Y \subset T_{3r'/2}(X)$. Finally, we conclude $d_{GH}(X, Y) \leq \frac{3}{2}\hat{d}_{GH}(X, Y)$.

Next, we will prove $\hat{d}_{GH} \leq 2d_{GH}$. Let (X, d_X) and (Y, d_Y) be compact metric spaces. Denote $r \equiv d_{GH}(X, Y)$. For any $\epsilon > 0$, we will construct $(2r + \epsilon)$-GHAs $f : X \to Y$ and $h : Y \to X$. Let $\bar{d}$ an admissible metric one $X \sqcup Y$ such that

$$
\bar{d}_H(X, Y) \leq r + \frac{\epsilon}{2}, \tag{5.50}
$$

which implies that for any $x \in X$, there exists some $y \in Y$ such that $\bar{d}(x, y) \leq r + \epsilon$. Then we define $f : X \to Y$ by $f(x) \equiv y$. By triangle inequality,

$$
\begin{aligned}
|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| & \leq |\bar{d}(x_1, f(x_1)) + \bar{d}(x_2, f(x_2))| \\
& \leq 2r + \epsilon.
\end{aligned} \tag{5.51}
$$

Therefore, $f : X \to Y$ is an $(2r + \epsilon)$-GHA. □

Lemma 5.14. *Let $f : X \to Y$ be an ϵ-GHA. Then there exists an 3ϵ-GHA, $h : Y \to X$ such that $h \circ f : X \to X$ is an 2ϵ-GHA on X. Moreover, $d_Y(h \circ f(x), \mathrm{Id}_X(x)) < \epsilon$ and $d_X(f \circ h(y), \mathrm{Id}_Y(y))$ for any $x \in X$ and $y \in Y$.*

Proof. Exercise. □

Definition 5.15 (ϵ-net)**.** Let (X, d) be a metric space. An ϵ-dense subset $A \subset X$ is called an ϵ-net if $d(x_i, x_j) \geq \epsilon$ for any $x_i, x_j \in A$.

Exercise 5.16. *Let (X, d) be a compact metric space. Show that for any $\epsilon > 0$, there is a finite ϵ-net $A \subset X$.*

Lemma 5.17. *Let $(X_j, d_j) \xrightarrow{GH} (X_\infty, d_\infty)$. Then the following properties hold:*

(1) $\mathrm{diam}(X_j) \to \mathrm{diam}(X_\infty) < \infty$.

(2) *For any $\epsilon > 0$, there exists some number $N = N(\epsilon) > 0$ such that X_j has an ϵ-net $X_j(\epsilon)$ with $|X_j(\epsilon)| \leq N(\epsilon)$ for all j.*

Proof. We only prove item (2). For any $\epsilon > 0$, there is a $\frac{\epsilon}{10}$-net $X(\frac{\epsilon}{10})$ on X with $|X(\frac{\epsilon}{10})| = N(\epsilon)$. Let $X_j(\epsilon) = \{x_i^j\}$ be an ϵ-net in X_j. Taking any admissible metrics d_j such that $(d_j)_H(X_j, X) \to 0$. Then passing to a subsequence, we can take $x_i \in X$ such that $\bar{d}_j(x_i^j, x) \to 0$ for $i = 1, 2, \ldots$. Then for any such limits, it holds that $d(x_i, x_k) \geq \epsilon$ for $i \neq k$. We define a map $\phi_j : X_j(\epsilon) \to X(\frac{\epsilon}{10})$ by $\phi(x_i^j) = y_k$ such that $d(x_i, y_k) < \frac{\epsilon}{4}$. For j sufficiently large, ϕ must be injective (otherwise, it contracts the triangle inequality). Therefore, $|X_j(\epsilon)| \leq |X(\frac{\epsilon}{10})| = N(\epsilon)$. $\square$

Example 5.18. *In the following examples, the metric spaces diverge in the Gromov-Hausdorff distance.*

(1) *(Infinite diameter) Let $X = \{p, q\}$ and $d_j(p, q) = j$.*
(2) *($N(\epsilon)$ is not uniformly bounded) Let S^1 be the unit circle. We denote $\widehat{Q} \equiv (\mathbb{Q} \times \mathbb{Q}) \cap S^1$. Then $\widehat{Q} \equiv \{q_j\}_{j=1}^{\infty}$ is countable and dense in S^1. Let ℓ_j be the segment from $(0,0)$ to q_j. Then we define a sequence of metric spaces as follows:*

$$X_j \equiv \bigcup_{k=1}^{j} \ell_k, \tag{5.52}$$

and the metric on ℓ_j is the inherits from the Euclidean metric. Then obviously $N(\epsilon) \to \infty$ as $\epsilon \to 0$.

Theorem 5.19 (Gromov)**.** *If a subset $\mathcal{C} \subset \mathcal{M}et$ is precompact if the following conditions hold:*

(1) *There exists some $D > 0$ such that $\operatorname{diam}(X) \leq D$ for any $X \in \mathcal{C}$.*
(2) *For any $\epsilon > 0$, there exists $N_0 = N_0(\epsilon) > 0$ such that for any $\epsilon > 0$, X has an ϵ-net $X(\epsilon)$ such that $|X(\epsilon)| \leq N_0(\epsilon)$.*

Proof. Given any sequence $\{X_j\} \subset \mathcal{C}$, we will select a Cauchy subsequence $\{X_{j_k}\}$ such that for any $\epsilon > 0$, there exists some $N > 0$ such that for all $j_k, j_\ell \geq N$,

$$d_{GH}(X_{j_k}, X_{j_\ell}) < \epsilon. \tag{5.53}$$

Let $X_{j_k}(\epsilon)$, $X_{j_\ell}(\epsilon)$ denote some ϵ-nets, respectively. By triangle inequality,

$$d_{GH}(X_{j_k}, X_{j_\ell}) \leq d_{GH}(X_{j_k}(\epsilon), X_{j_\ell}(\epsilon)) + 2\epsilon. \tag{5.54}$$

So it suffices to find a subsequence of $\{X_{j_k}\}$ such that $X_{j_k}(\epsilon)$ converges.

First, let $\epsilon_j \to 0$ be a monotone sequence. We take an ϵ_1-net $X_j(\epsilon_1) \equiv \{x_i^1, \ldots, x_i^{s_1}\}$ of X_j. By assumption, $s_1 \leq N_0(\epsilon)$. Passing to a subsequence, we can just assume $s = s_j$ for all j. Let $d_j \equiv d_{X_j}\Big|_{X_j(\epsilon_1)}$. The matrix $(d_j(x_k^j, x_\ell^j))$ can be viewed as a point in $\mathbb{R}^{s^2}$ and

$$|(d_j(x_k^j, x_\ell^j)) - 0^{s^2}| = \sum_{k,\ell=0}^{s} d_j(x_k^j, x_\ell^j)^2 \leq s \cdot D. \tag{5.55}$$

Then there exists a convergent subsequence of the above matrix sequence such that

$$|(d_{j_1}(x_k^{j_1}, x_\ell^{j_1})) - (d_{j_1'}(x_k^{j_1'}, x_\ell^{j_1'}))| < \epsilon_1. \tag{5.56}$$

Therefore, $d_{GH}(X_{j_1}(\epsilon), (X_{j_1'}(\epsilon)) < \epsilon_1$ for all j_1, j_1'.

For $\epsilon_2 > 0$ and $\{X_{j_1}\}$, repeating the above, we have a subsequence $X_{j_2}(\epsilon)$ such that $d_{GH}(X_{j_2}(\epsilon), (X_{j_2'}(\epsilon)) < \epsilon_1$ for all j_2, j_2'. Iterating the above and applying the standard diagonal argument, we can find a subsequence $\{X_\ell\}$ which is a Cauchy sequence.

Finally, for any $\epsilon > 0$, we pick $\epsilon_j < \frac{\epsilon}{3}$. By our construction,

$$d_{GH}(X_k(\epsilon_j), X_\ell(\epsilon_j)) < \epsilon_{\min\{k,\ell\}} < \epsilon_j. \tag{5.57}$$

Then the conclusion follows from the triangle inequality. $\square$

Corollary 5.20 (Gromov). *We denote*

$$\mathcal{M}(n, \kappa, D) \equiv \{(M^n, g) : \operatorname{diam}(M^n) \le D, \quad \operatorname{Ric}_g \ge (n-1)\kappa\}. \tag{5.58}$$

Then $\mathcal{M}(n, \kappa, D)$ is precompact in $(\mathcal{M}et, d_{GH})$.

Proof. Let $\{p_i\}_{i=1}^N$ be an ϵ-dense subset in M^n such that $B_{\epsilon/5}(p_i) \cap B_{\epsilon/5}(p_j) = \emptyset$ for any $i \neq j$. We take $i_0 \in \{1, \ldots, N\}$ such that

$$\operatorname{Vol}(B_{\frac{\epsilon}{5}}(p_{i_0})) = \min\left\{\operatorname{Vol}(B_{\frac{\epsilon}{5}}(p_i)) : i = 1, \ldots, N\right\}. \tag{5.59}$$

Since the above $\epsilon/5$-balls are disjoint,

$$\operatorname{Vol}(B_D(p_{i_0})) = \operatorname{Vol}(M^n) \ge \sum_{i=1}^N \operatorname{Vol}(B_{\epsilon/5}(p_i)) \ge N \cdot \operatorname{Vol}(B_{\epsilon/5}(i_0)). \tag{5.60}$$

Applying Bishop-Gromov's volume comparison, we obtain a uniform bound of N,

$$N \le \frac{\operatorname{Vol}(B_D(p_{i_0}))}{\operatorname{Vol}(B_{\epsilon/5}(i_0))} \le \frac{V_\kappa(D)}{V_\kappa(\epsilon/5)} = N(\epsilon, n, D, \kappa). \tag{5.61}$$

Finally, by Theorem 5.19 we conclude the precompactness of $\mathcal{M}(n, \kappa, D)$. $\square$

6. Further Reading

- Standard textbooks
  (1) Knowledge about manifolds: [Lee13] (Chapters 1-5)
  (2) Elementary Riemannian geometry: [dC92], [Gro99]
  (3) More advanced Riemannian geometry (with more technical tools): [Pet16]
  (4) Comparison geometry: [CE08]
- Modern topics in metric Riemannian geometry
  (1) Survey on metric geometry: [Fuk06], [Ron10]
  (2) Geometry of the Ricci curvature: [Che01]
- Textbooks of Elliptic PDEs
  (1) Fast-going [HL11]
  (2) Thick book [GT01]

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