Introduction to hyperbolic surfaces

Exercises VI

1. (Easy) Let $\theta = \alpha\pi$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Use definition to show that the group generated by

$$\rho_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

ρ_θ does not act properly discontinuously on $\mathbb{H}$.

2. Consider the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ given by

$$\begin{aligned} (m,n) : \mathbb{R}^2 &\to \mathbb{R}^2, \\ (x,y) &\mapsto (x+m, y+n). \end{aligned}$$

Let T be the flat torus $\mathbb{R}^2/\mathbb{Z}^2$. Let D be the unit square determined by $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. Let $l(x,y)$ be the Euclidean line in $\mathbb{R}^2$ passing $(0,0)$ and (x,y).

a) (Normal) Using $\mathbb{Z}^2$ action to send all points in $l(1,2)$ to D. Draw the image.

(Hint: Consider cut $l(1,2)$ using the lines defined by $x = m$ and $y = n$, then translate each segment back to D using $\mathbb{Z}^2$-action.)

b) (Normal) Using $\mathbb{Z}^2$ action to send all points in $l(3,2)$ to D. Draw the image.

c) (Hard) What is the algorithm to draw the image of $l(p,q)$ with $\gcd(p,q) = 1$ in D?

d) (Hard) What could we say about the image of the line $l(1,\sqrt{2})$.

3. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Let $\Gamma = \langle A, B \rangle$.

a) (Normal) Show that Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, and conclude that it acts properly discontinuously on $\mathbb{H}$.

b) (Normal) Find a fundamental domain of for the Γ-action.

(Hint: Consider the Dirichlet domain centered at i.)

c) (Normal) Compute the area of the surface $S = \mathbb{H}/\Gamma$.

4. Let $\gamma(x,x')$ be a complete geodesic in $\mathbb{H}$ with end point x and x' and oriented from x to x'.

a) (Normal) Find a pair of matrices A and B such that A sends $\gamma(0,1)$ to $\gamma(\infty,2)$, and B sends $\gamma(0,\infty)$ to $\gamma(1,2)$.

(Hint: To send a geodesic to another, it is enough to send the end points of the former to those of the latter.)

b) (Normal) Describe all solutions of a) using parameter(s). How many parameters are needed?

c) (Easy) For any (A, B) a solution of a), let $\Gamma(A, B)$ be the subgroup of $\mathrm{SL}(2, \mathbb{R})$ generated by A and B. Let

$$S(A, B) = \mathbb{H} / \Gamma(A, B)$$

Compute the area of $S(A, B)$ for any solution (A, B) of a).