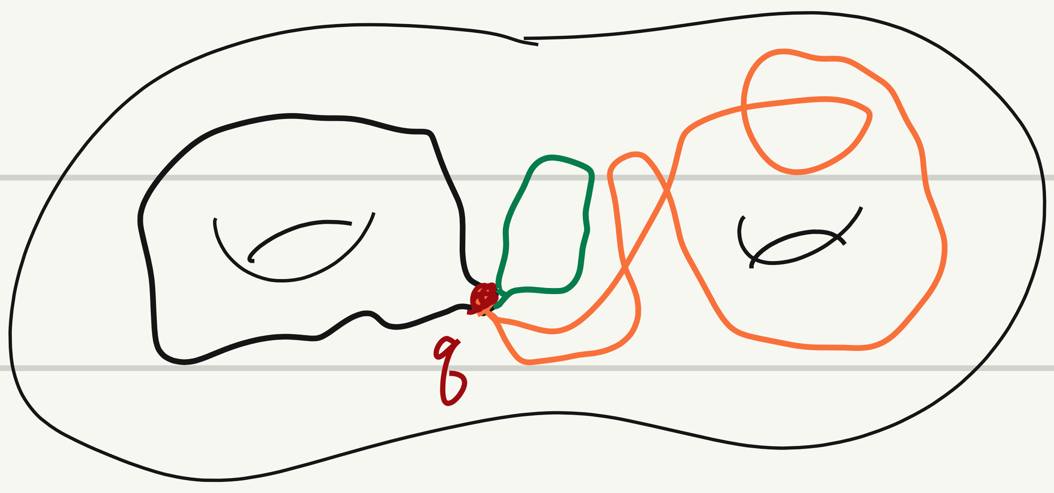
VI Surface Topology:

1. Loops on hyp sf $S = \mathbb{H}/\Gamma$

- A loop (based at q, parametrized) on S is the image of
  $\gamma : [0,1] \to S$ continuous s.t. $\gamma(0) = \gamma(1) = q$.



$L(S,q) = \{\gamma \mid \text{loops based } q\}$

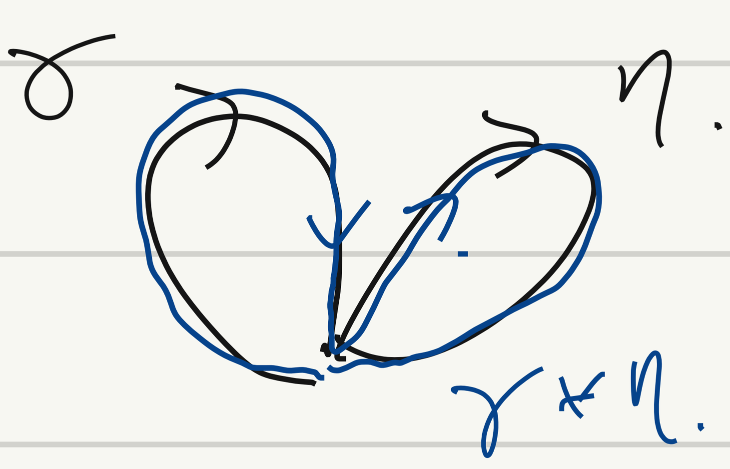
- $\gamma, \gamma' \in L(S,q)$.
  $\gamma \sim_q \gamma'$ homotopic to each othe relative to q if.

$\exists$: $H : [0,1] \times [0,1] \to S$ continuous s.t.
$(t, s) \mapsto H(t,s)$

$H(t,0) = \gamma(t)$ $H(t,1) = \gamma'(t)$.
$\forall s$ $H(0,s) = H(1,s) = q$.

- $\gamma, \eta \in L(S,q)$.

$$\gamma * \eta(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \eta(2t-1) & \frac{1}{2} < t \le 1 \end{cases} \qquad \gamma(2 \times \tfrac{1}{2}) = \eta(2 \times \tfrac{1}{2} - 1) = q$$



Prop: If $\gamma \sim_q \gamma'$, $\eta \sim_q \eta'$ then $\gamma * \eta \sim \gamma' * \eta'$

Prop: "$\sim_q$" is an equi relation among γ's in $L(S,q)$

$[\gamma]_q := \{\gamma' \in L(S,q) \mid \gamma' \sim_q \gamma\}$

Def: $\pi_1(S,q)$ the fund. group of S rel to q.

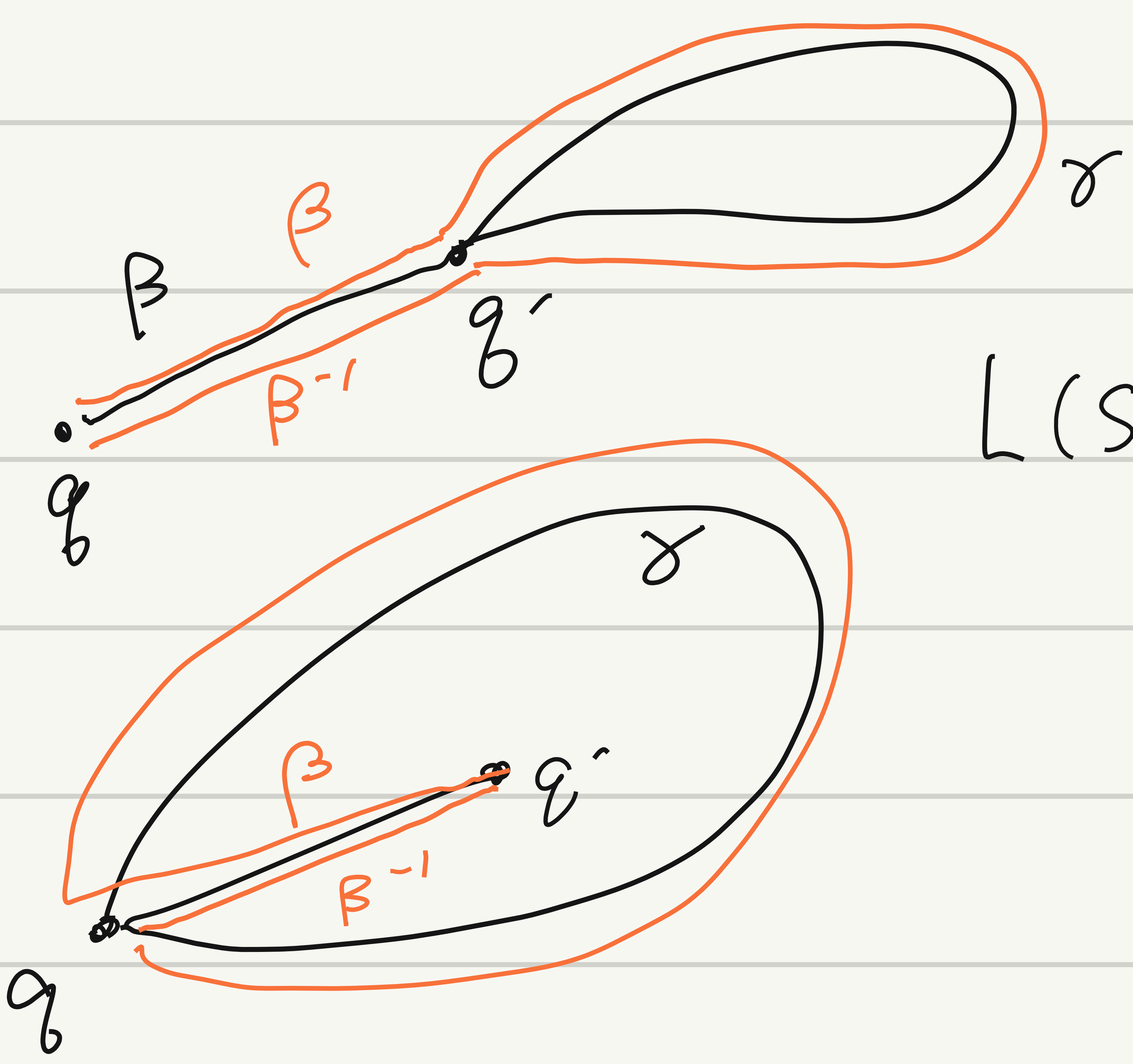
$$\pi_1(S,q) := \{ [\gamma]_p \mid \gamma \text{ a loop based at } q \}.$$

"$*$" $\rightsquigarrow$ group structure.

- $[\gamma \equiv q]_q = id$
- $[\gamma]_q^{-1} = [-\gamma]_q$
- $[\gamma]_q * [\eta]_q = [\gamma * \eta]_q$.

$$-\gamma : [0,1] \to S$$
$$t \mapsto \gamma(1-t).$$

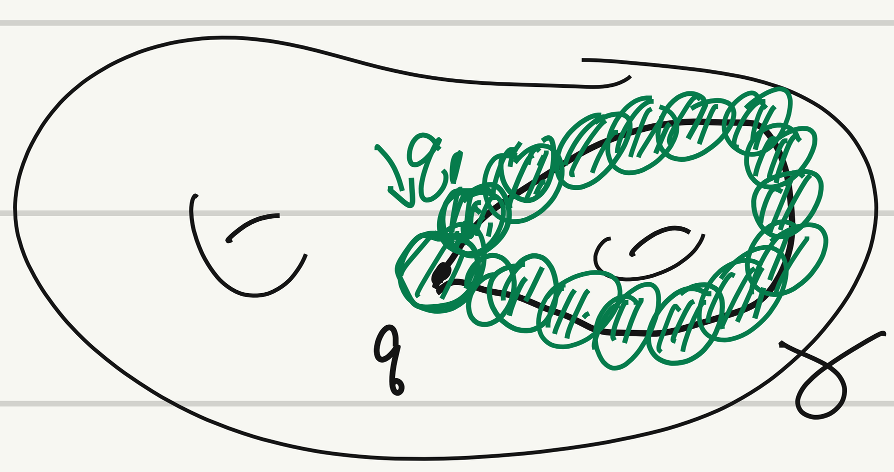
Let $q, q' \in S$. $\quad \pi_1(S,q) \quad \pi_1(S,q')$

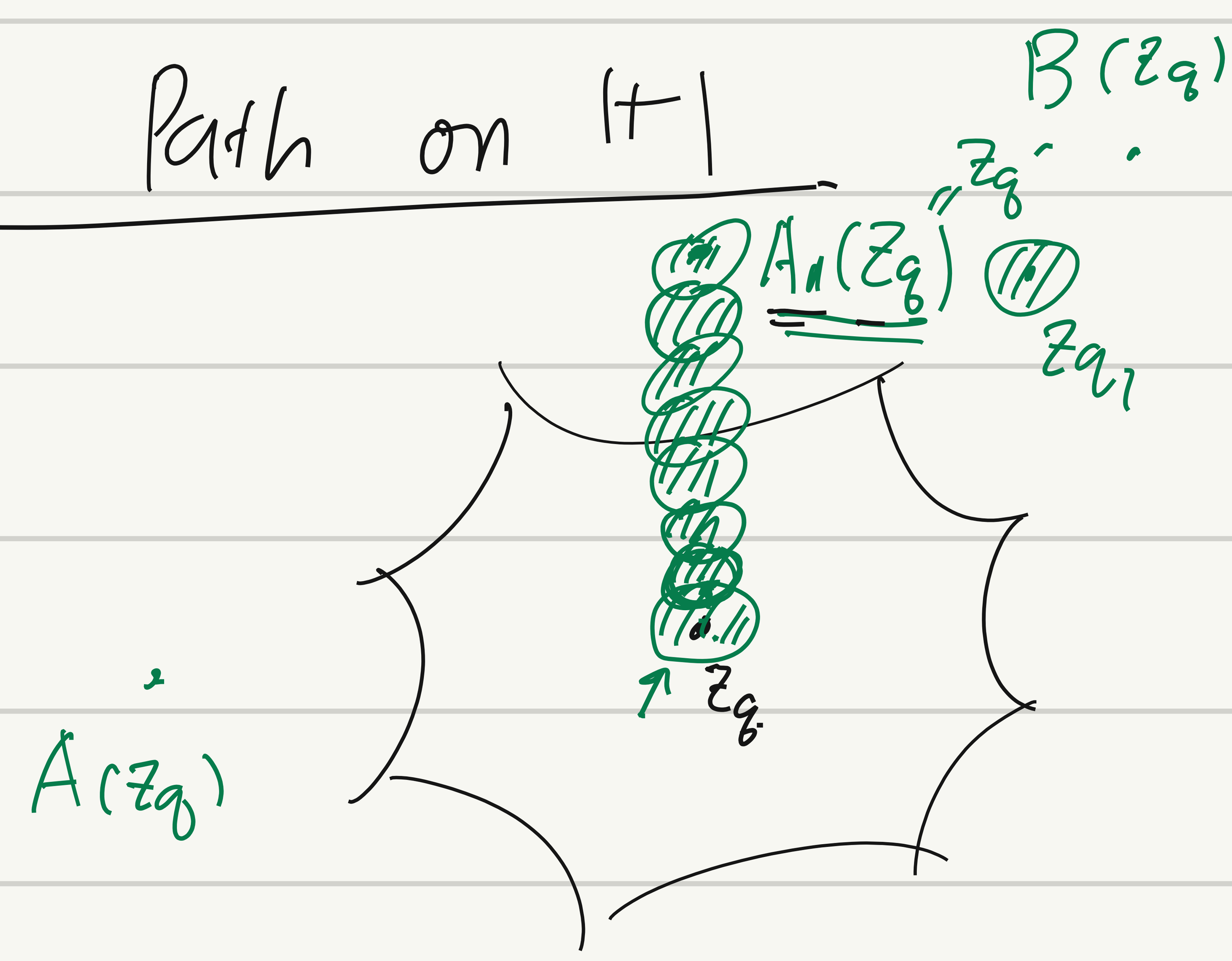


$$L(S,q') \xrightarrow{\text{bij.}} L(S,q).$$
$$\gamma \mapsto \underline{\beta * \gamma * \beta^{-1}}$$

Prop: $\pi_1(S,q) \cong \pi_1(S,q')$.

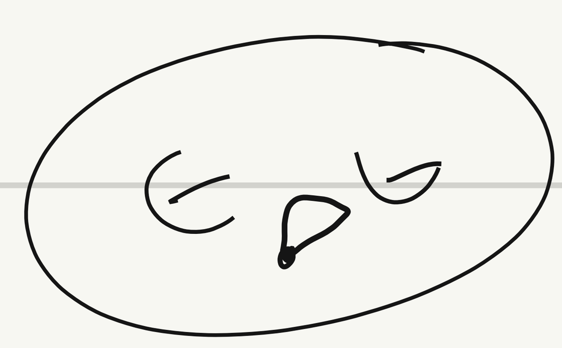
2. Loop in S vs Path on $\mathbb{H}$

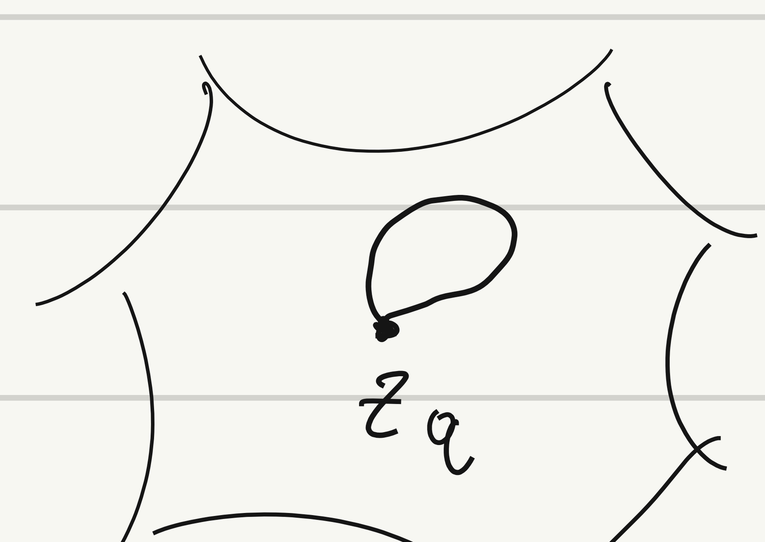




$\mathbb{H}$

$P(z_q) = q \in S$





Prop: $[\gamma]_q = id. \iff z_q' = z_q.$

Prop: $\pi_1(S) \cong \Gamma$ $\qquad P : \pi_1(S) \to PSL(2,\mathbb{R})$

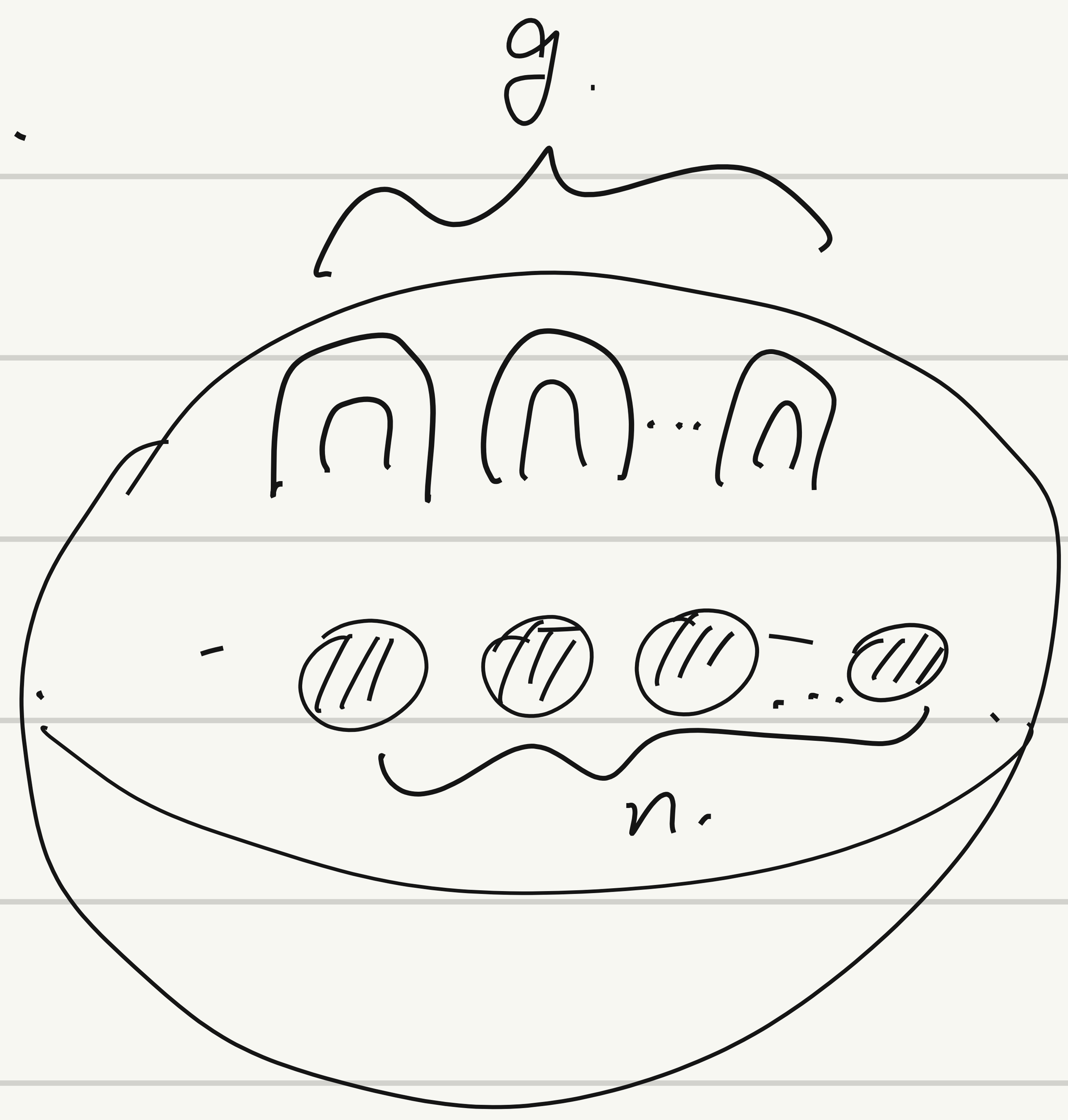
inj homomorphism.

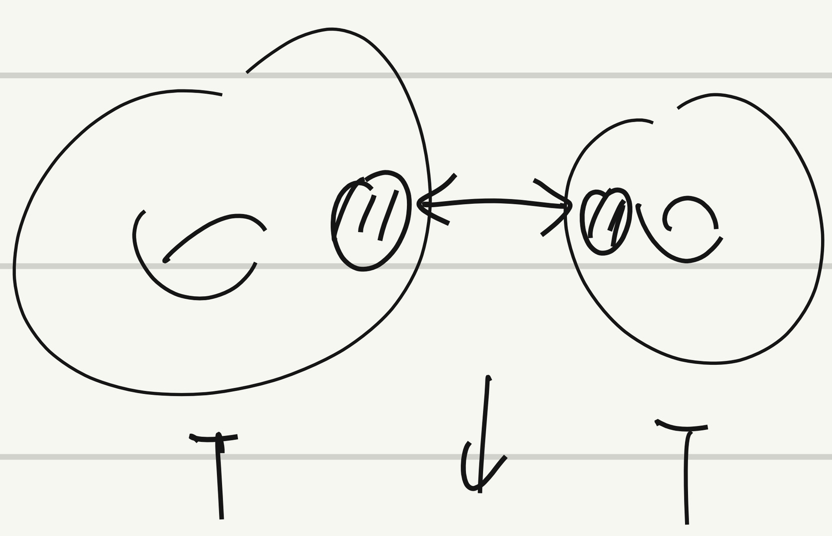
3. Classification of surface.

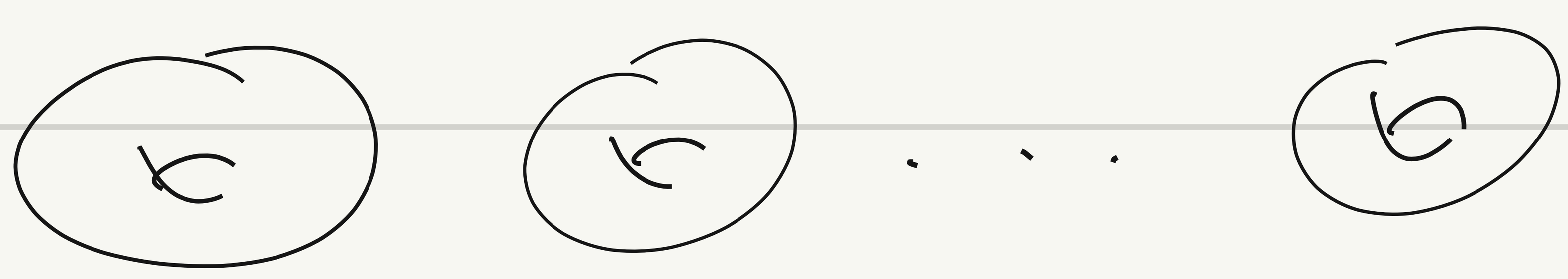
g : genus

n : # of ∂ compomponent.

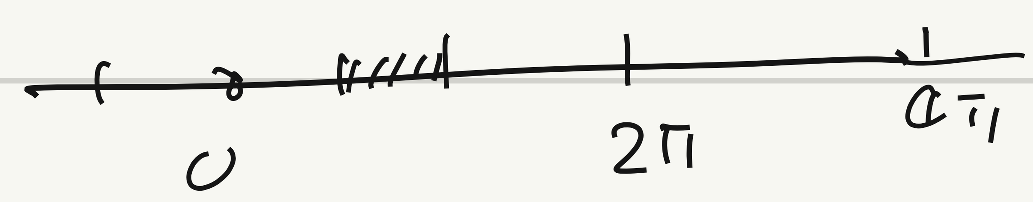
$S_{g,n}$

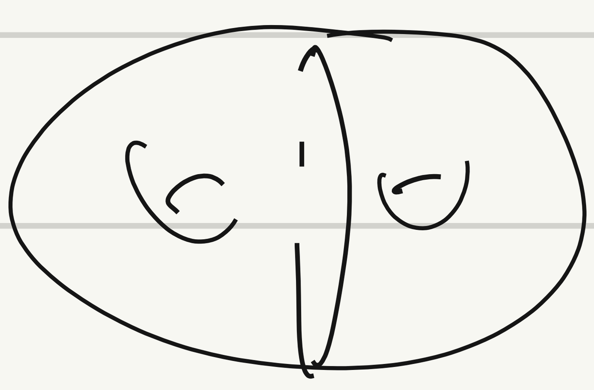




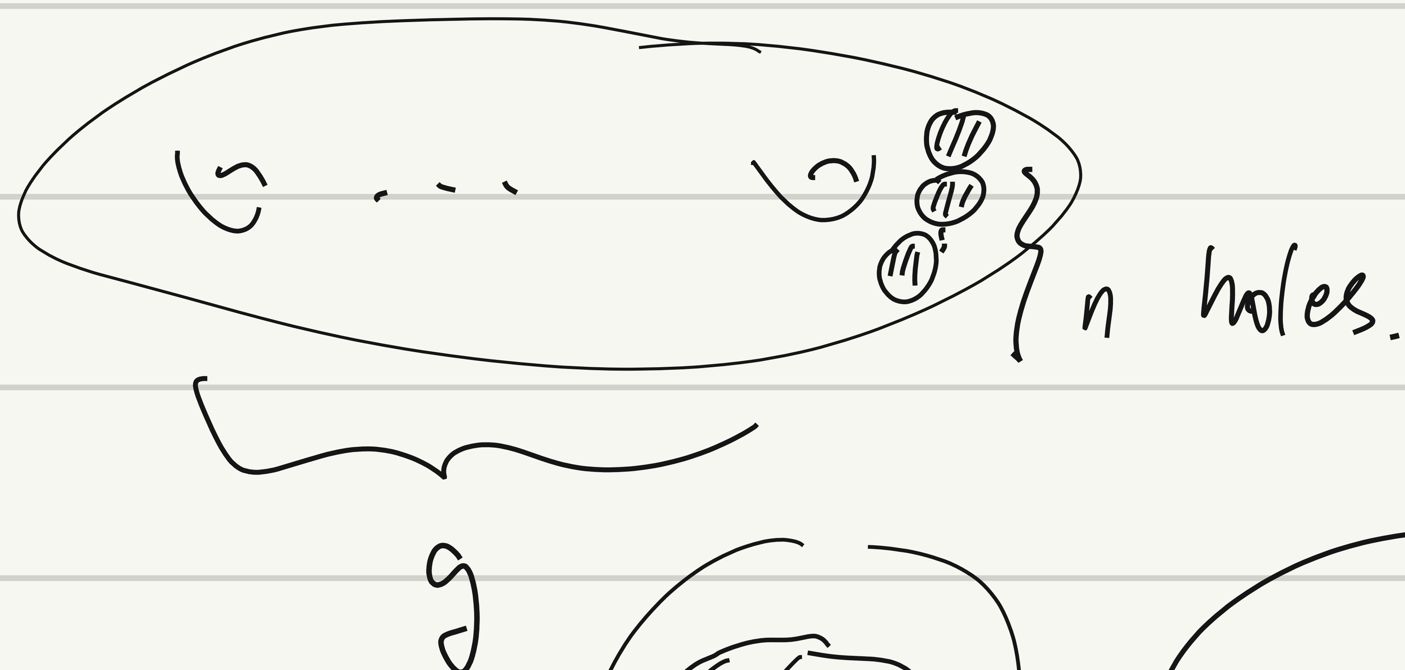


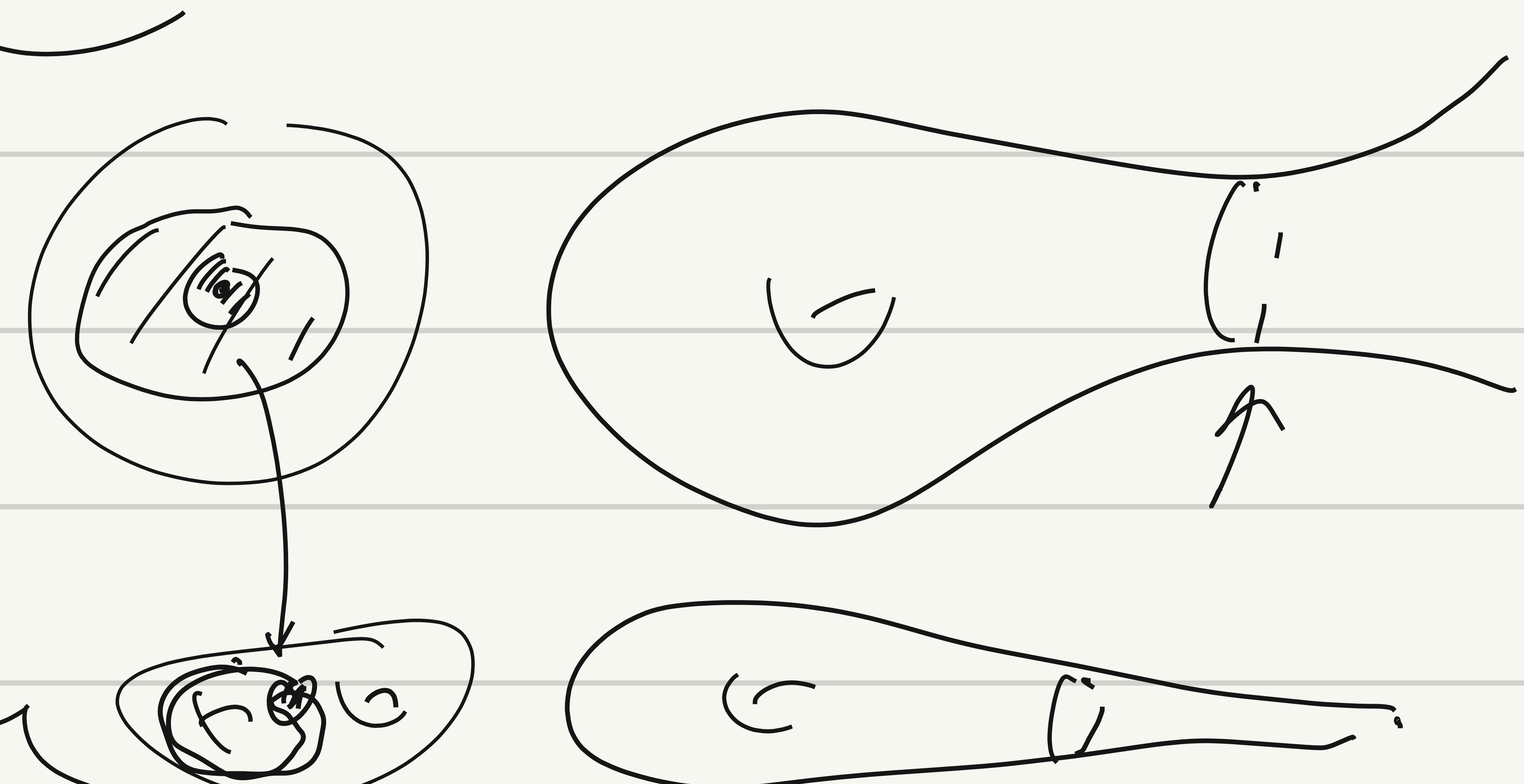
$\underbrace{T \# T \# \cdots \# T}_{g}$



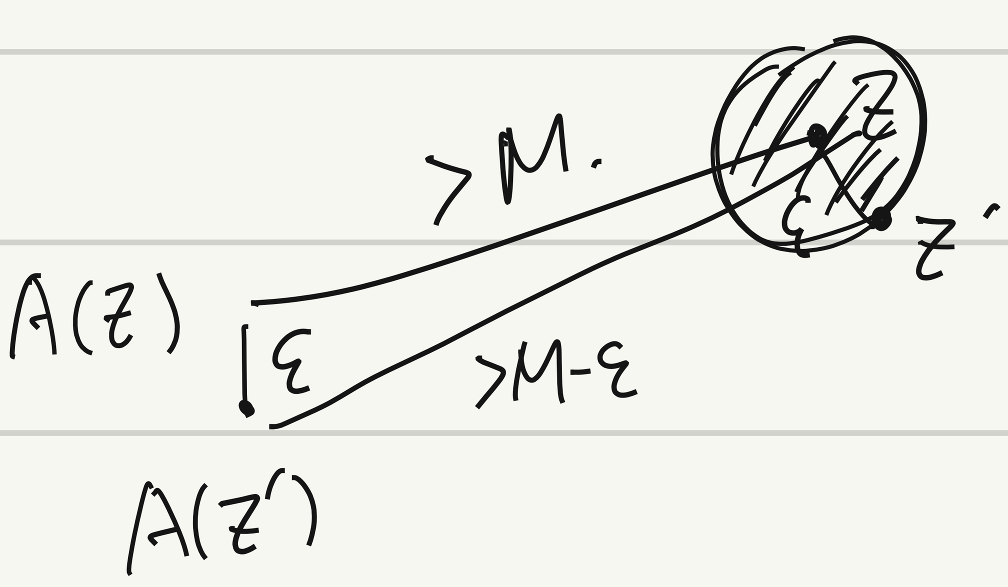


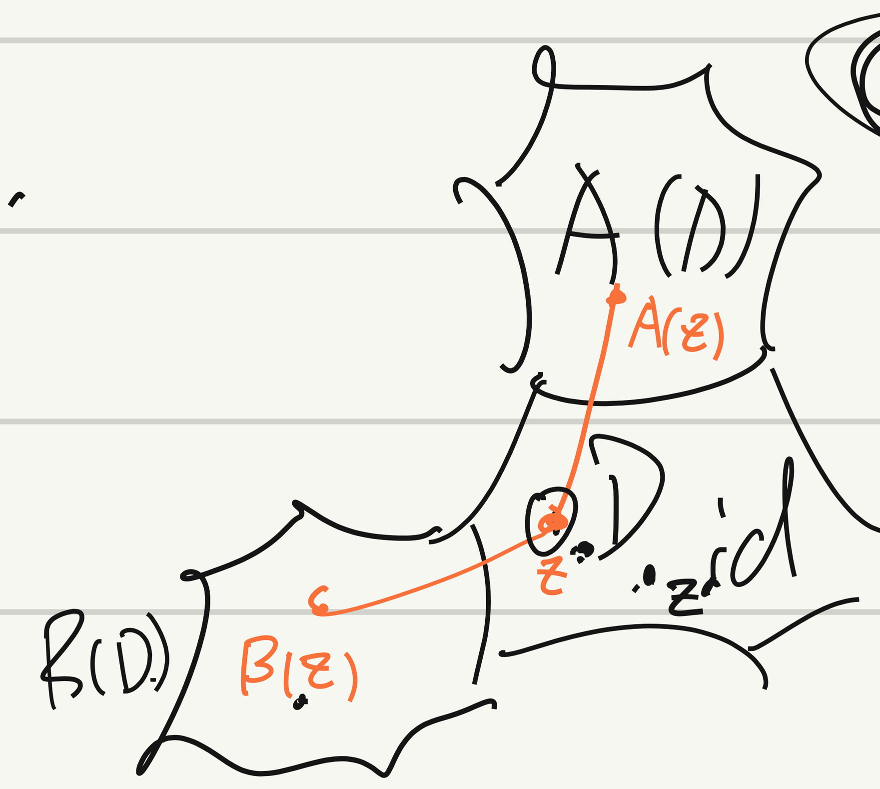
$S = T \# T$

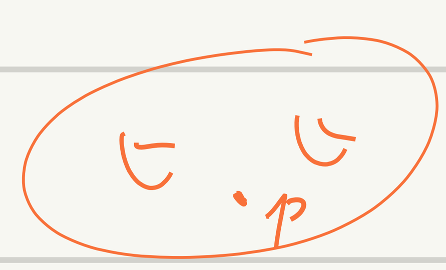




$$\chi(S) = 2 - 2g - n$$





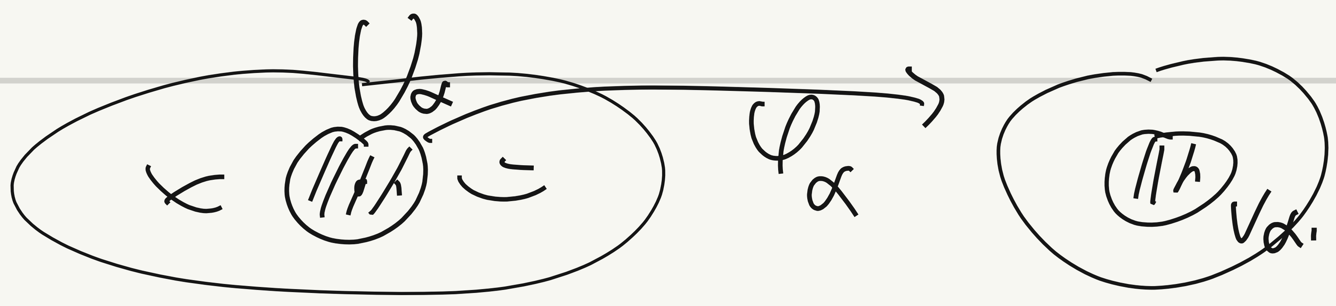


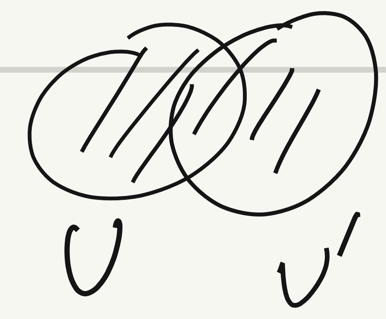
$$d_{\mathbb{H}}(A(w), B(w')) = d_{\mathbb{H}}(w, A^{-1}B(w'))$$

$P : \mathbb{H} \to \mathbb{H}/P$

local isometry

$\mathbb{H}/P \quad \mathrm{dist}([z]_P, [z']_P) := \inf\{ d_{\mathbb{H}}(w, w') \mid w \in [z]_P,\ w' \in [z']_P \}$





$$\varphi_U \circ \varphi_{U'}^{-1}\Big|_{\varphi_{U'}(U \cap U')} = A\big|_{\varphi(U \cap U')} \quad A \in \mathrm{Isom}^+(\mathbb{H})$$

Universal cover

$$
\begin{tikzcd}
\mathbb{H} \arrow[r, "p'"] \arrow[dr, "p"'] & S' \arrow[d, "p_1"] \\
 & S
\end{tikzcd}
$$

k-cover.

$p = p_1 \circ p'$

$$
\begin{tikzcd}
\mathbb{H} \arrow[r, "p'"] \arrow[dr, "p"'] & \mathbb{H}/\Gamma' \arrow[d, "p_1"] \\
 & \mathbb{H}/\Gamma
\end{tikzcd}
$$

k-cover

$\boxed{\Gamma' \triangleleft \Gamma \text{ index } k.}$