Lecture 1 : Basic Riemannian Geometry (07/14/2021)

1.0. Conventions

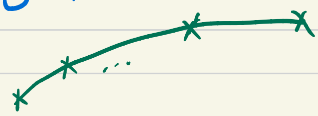
- $\mathcal{U} \subseteq \mathbb{R}^n$ domain : open, connected
- All the interested curves are piecewise C^k, $k \geq 1$
- All the metrics are default C^∞
- Einstein : $a^i b_i = \sum_{i=1}^{n} a^i b_i$

1.1. Riemannian Structure and geodesics

- Let $X = (x_1, \dots x_n)$, $Y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$.

Euclidean Distance

$$d_0(X,Y) \equiv \| X - Y \| = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}$$



Euclidean length

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a continuous curve.

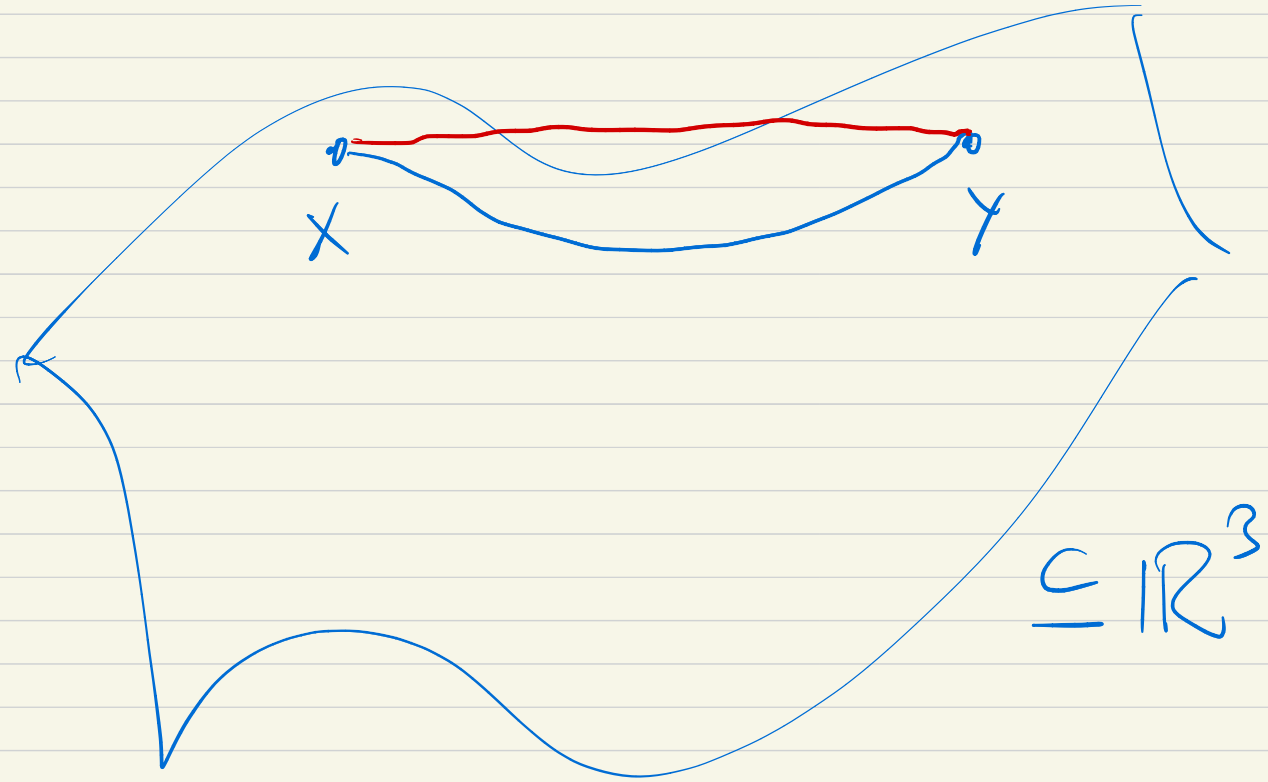
Then $L_0(\gamma) \equiv \sup_{P} \sum_{j=1}^{k} d_0(\gamma(t_{j-1}), \gamma(t_j))$,

where $P = \{ a < t_1 < t_2 < \dots < t_n = b \}$ partition.

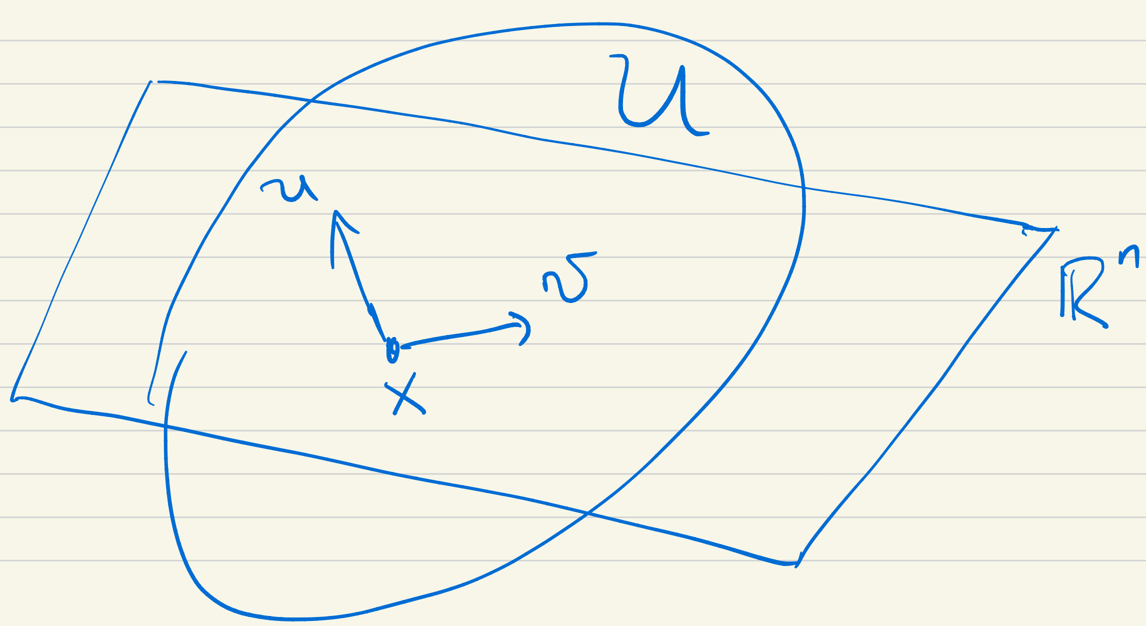
- We say γ is rectifiable if $L_0(\gamma) < \infty$.

- If $\gamma : [a,b] \to \mathbb{R}^n$ is C^1, then γ is rectifiable and

$$L_0(\gamma) = \int_0^1 \|\gamma'(t)\| \, dt$$



x
y
⊆ ℝ³



U
u
v
x
ℝⁿ

Riemanian Structure

A Riemannian C^k-structure

$$g: \mathcal{U} \to (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \text{ is a } C^k \text{ map}$$

that assigns each $x \in \mathcal{U}$ to a nonnegative symmetric bilinear function, i.e., $\forall x \in \mathcal{U}$, $\exists$ nonnegative symmetric $n \times n$ matrix G_x s.t.

$$g_x(u, v) = \langle u, G_x \cdot v \rangle, \ \forall u, v \in \mathbb{R}^n$$

Euclidean inner product

- Arc length in terms of g:

Let $\gamma: [a,b] \to \mathbb{R}^n$ be a C^1-curve.

Then

$$L_g(\gamma) \equiv \int_a^b g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{\frac{1}{2}} \, dt$$

- ## Intrinsic Distance

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a domain with a Riemannian structure g. Then

$$d_g(x,y) \equiv \inf \left\{ L_g(\gamma) \,\middle|\, \begin{array}{l} \gamma \text{ is a piecewise } C^1\text{-curve} \\ \text{connecting } x \text{ and } y \end{array} \right\}$$

- $\boxed{\text{Metric space}}$ A pair (X, d) :

(1) $d(p,q) \geq 0 \quad \forall\, p,q \in X$

"=" holds iff $p=q$

(2) $d(p,q) = d(q,p), \ \forall\ p,q \in X$.

(3) $d(p,q) \leq d(p,r) + d(r,q), \ \forall p,q,r \in X$.

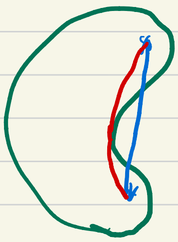
- $\boxed{\text{Intrinsic Distance and length space}}$

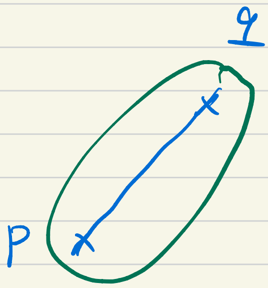
Let (X,d) be a metric space. Then we define the instrinsic distance $d^{\#}$ by

$$d^{\#}(x,y) \equiv \inf\left\{ L(\gamma) \;\middle|\; \begin{array}{l} \gamma:[0,1]\to X \text{ is a} \\ \text{curve connecting } x \text{ and } y \end{array} \right\}.$$

(X,d) is called a length space if

$d = d^{\#}$.

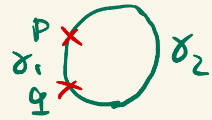




- Examples:

* $(\mathbb{R}^n, d_0)$ is a length space.

* $(\mathcal{U}, d_0)$ is a length space if $\mathcal{U}$ is convex

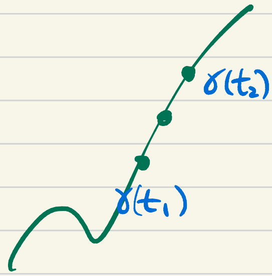


$\boxed{\text{Geodesic}}$ Let (X,d) be a metric space and let $\mathcal{U} \subseteq X$ be a domain.

A curve $\gamma : [a,b] \to \mathcal{U}$ is said to be a geodesic if $\forall t \in [a,b]$,

$$d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$$

$\forall t_1, t_2$ sufficiently closed to t

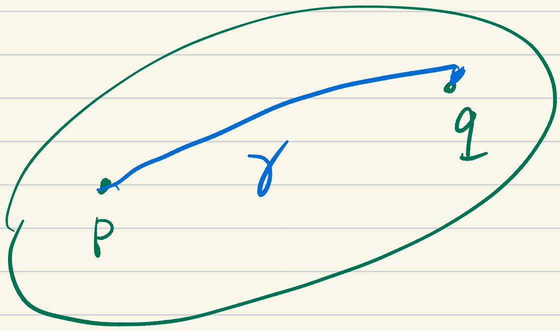


A geodesic is minimal if $d(\gamma(a), \gamma(b)) = L(\gamma)$

Note: any geodesic is locally minimal.

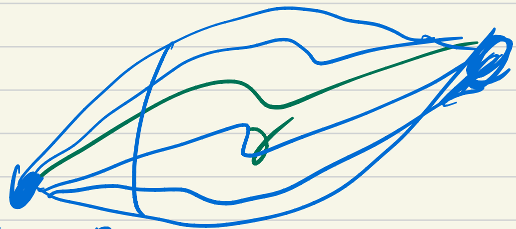
Theorem. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a domain equipped with a Riemannian structure g s.t. $(\mathcal{U}, d_g)$ is a complete metric space. Then $\forall p, q \in \mathcal{U}$, $\exists$ a minimal geodesic connecting p and q.

$L_g(\gamma) = d_g(p,q)$



1.2 Exponential map, connection, variation of arc length

- Variation of a curve.



Let $\gamma : [a,b] \to \mathcal{U}$ be a curve.

A one-parameter variation of γ is a map

$$V : [a,b] \times \underline{(-1,1)} \to \mathcal{U} \quad \text{s.t.} \quad V(t,0) = \gamma(t) \quad \forall t \in [a,b].$$

We denote $\underline{\gamma_s(\cdot) = V(\cdot, s)}$, $\underline{\sigma_t(\cdot) = V(t, \cdot)}$

$$L_s \equiv L_g(\gamma_s), \quad \underline{X(t)} = \left.\frac{\partial}{\partial s}\right|_{s=0} V(t,s)$$

<u>Lemma</u>. Let $\gamma_s : [0,1] \to \mathcal{U}$ be parametrized by <u>proportional to its arc length</u> with $l = L_g(\gamma)$. Then the following holds,

$$\boxed{\left.\frac{d}{ds}\right|_{s=0} L_s = \frac{1}{l} \int_0^1 g_{\gamma(t)}\left(\nabla_{\gamma'(t)} X(t), \gamma'(t)\right) dt,}$$

where $\nabla_{\gamma'(t)} X(t) \equiv \frac{d}{dt} X(t) + \Gamma_{\gamma(t)}(X(t), \gamma'(t))$,

$$\Gamma_x(u, v) \equiv \frac{1}{2} G_x^{-1} \underline{DG_x(u)} v.$$

<u>Pf.</u>

$$\frac{d}{ds} L_g(\gamma_s) = \frac{d}{ds} \int_0^1 g_{\gamma_s(t)}(\gamma_s'(t), \gamma_s'(t))^{\frac{1}{2}} dt$$

$$= \int_0^1 \frac{\partial}{\partial s} \underline{g_{\gamma_s(t)}(\gamma_s'(t), \gamma_s'(t))^{\frac{1}{2}}}\, dt$$

$$= \int_0^1 \frac{\partial}{\partial s} \langle \gamma_s'(t), \underline{G_{\gamma_s(t)}} \cdot \gamma_s'(t) \rangle^{\frac{1}{2}}\, dt$$

$$= \int_0^1 \frac{1}{2} \boxed{g_{\gamma_s(t)}(\gamma_s'(t), \gamma_s'(t))^{-\frac{1}{2}}} \frac{\partial}{\partial s} \langle \underline{\gamma_s'(t)}, G_{\gamma_s(t)} \cdot \gamma_s'(t) \rangle\, dt$$

$$= \frac{1}{2\|\gamma_s'\|_g} \int_0^1 \left(\left\langle \underline{\frac{\partial}{\partial s}\frac{\partial V}{\partial t}}, G_{\gamma_s(t)} \cdot \frac{\partial V}{\partial t} \right\rangle + \left\langle \frac{\partial V}{\partial t}, \frac{\partial}{\partial s}\left(\underline{G_{\gamma_s(t)} \cdot \frac{\partial V}{\partial t}}\right) \right\rangle \right) dt$$

$$= \frac{1}{2\|\gamma_s'\|_g} \int_0^1 \left(\left\langle \underbrace{\frac{\partial}{\partial t}\frac{\partial V}{\partial s}}_{\frac{dX}{dt}}, G_{\gamma_s} \cdot \underbrace{\frac{\partial V}{\partial t}}_{\gamma'} \right\rangle + \left\langle \frac{\partial V}{\partial t}, DG\left(\frac{\partial V}{\partial s}\right) \frac{\partial V}{\partial t} \right\rangle + \left\langle \underbrace{\frac{\partial V}{\partial t}}_{\gamma'}, G_{\gamma_s} \cdot \underbrace{\frac{\partial}{\partial s}\frac{\partial V}{\partial t}}_{\frac{dX}{dt}} \right\rangle \right)$$

Letting $s = 0$,

$$= \frac{1}{2l} \int_0^1 \left(2\left\langle \gamma', G \cdot \frac{dX}{dt} \right\rangle + \langle \gamma', G \cdot (G^{-1} DG(X) \gamma') \rangle \right) dt$$

$$= \frac{1}{l} \int_0^1 \langle \gamma', \nabla_{\gamma'} X \rangle\, dt$$

Covariant Derivative

For any smooth map Γ that assigns every $x \in \mathcal{U}$ to a bilinear map $\Gamma_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the corresponding covariant derivative X along γ is defined by

$$\nabla_{\gamma'(t)} X(t) = \frac{d}{dt} X(t) + \Gamma_{\gamma(t)}(\gamma'(t), X(t)).$$

- Basic properties:

(1) ($\mathbb{R}$-linearity)

$$\nabla_{\gamma'}(\alpha X + \beta Y) = \alpha \nabla_{\gamma'} X + \beta \nabla_{\gamma'} Y$$

(2) (Leibniz)

$$\nabla_{\gamma'}(fX) = f'(t) \cdot X + f \nabla_{\gamma'} X$$

(3) (g-parallel)

$$\frac{d}{dt} g_{\gamma(t)}(X, Y) = g_\gamma(\nabla_{\gamma'} X, Y) + g_\gamma(X, \nabla_{\gamma'} Y)$$

Check them!

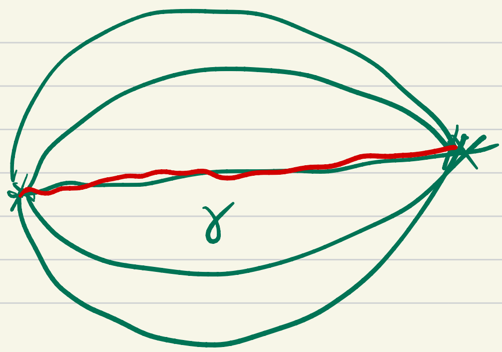
(4) (C^∞-linear)

$$\nabla_{f \cdot \gamma'(t)} Y = f \nabla_{\gamma'(t)} Y,$$

$$\forall f \in C^\infty(\mathcal{U}),$$

$$\frac{d}{ds}\Big|_{s=0} L_s = \int_0^{\ell} \left\langle \nabla_{\gamma'(t)} X(t), \gamma'(t) \right\rangle dt$$

$$= \int_0^{\ell} \left(\frac{d}{dt} \left\langle X(t), \gamma'(t) \right\rangle - \underline{\left\langle X(t), \nabla_{\gamma'(t)} \gamma'(t) \right\rangle} \right) dt$$



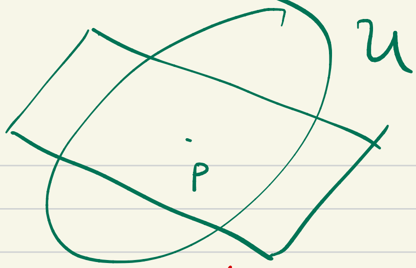
Corollary The first variation can be written as

$$\frac{d}{ds}\Big|_{s=0} L_s = \langle X(\ell), \gamma'(\ell)\rangle - \langle X(0), \gamma'(0)\rangle - \int_0^\ell \langle X(t), \nabla_{\gamma'(t)}\gamma'(t)\rangle \, dt.$$

In particular, if $V(0,s) \equiv \gamma(0)$, $V(\ell,s) = \gamma(\ell)$, then

$$\frac{d}{ds}\Big|_{s=0} L_s = -\int_0^\ell \langle X(t), \nabla_{\gamma'(t)}\gamma'(t)\rangle \, dt.$$

Corollary If $\gamma : [a,b] \to \mathcal{U}$ is a geodesic, then γ satisfies $\nabla_{\gamma'(t)}\gamma'(t) \equiv 0$ $\forall t \in [a,b]$, i.e., $\gamma''(t) + \Gamma_{\gamma(t)}(\gamma'(t), \gamma'(t)) = 0$.



- A remark on the tangent vector:

A tangent vector $v \in T_p U$ is identified with the operator of directional derivative

$$D_v(f) \equiv \frac{df}{dt} \quad \forall\, \gamma : [0,1] \to U \text{ with } \gamma(0) = p,\ \gamma'(0) = v.$$

The above gives an intrinsic description of tangent vectors.

We always denote $v(f) \equiv D_v(f)$.

Check:

(1) $v(\alpha f + \beta g) = \alpha v(f) + \beta v(g)$,

(2) $v(1) = 0$,

(3) $v(fg) = v(f) g + f v(g)$.

Levi-Civita Connection

An operator $\nabla : \mathfrak{X}(\mathcal{U}) \times \mathfrak{X}(\mathcal{U}) \to \mathfrak{X}(\mathcal{U})$ is called a linear (or affine) connection if

(1) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$

(2) $\nabla_{fX} Y = f \nabla_X Y$

(3) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

(4) $\nabla_X (fY) = X(f) Y + f \nabla_X Y$,

where $f \in C^\infty(\mathcal{U})$, $X, Y, Z \in \mathfrak{X}(\mathcal{U})$.

Furthermore, a linear connection ∇ is called a Levi-Civita (Riemannian) connection if it satifies

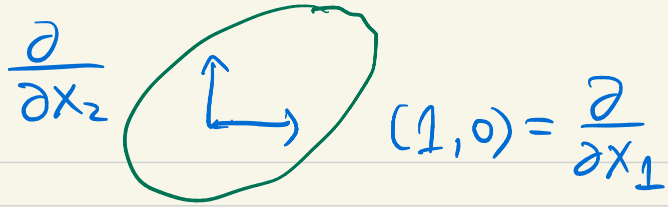
$(XY - YX)(f)$

(5) [torsion-free or symmetric]

$$\nabla_X Y - \nabla_Y X - \underbrace{[X,Y]}_{= XY - YX} = 0$$

(6) [g-parallel]

$$X\, g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$



Local representation

Denote by $\partial_i = \partial/\partial x_i$ the coordinate frames

We write $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$

Using (6),

$$\partial_i g_{jk} = g(\nabla_{\partial_i}\partial_j, \partial_k) + g(\partial_j, \nabla_{\partial_i}\partial_k).$$

Permuting the indices and applying (5), (6),

then

$$g(\nabla_{\partial_i}\partial_j, \partial_k) = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j \partial_{ik} - \partial_k g_{ij}\right)$$

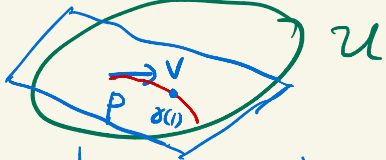
Finally,

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

Christoffel Symbol

- Geodesic equation in coordinates $\gamma(t) = (x_m(t))$

$$\frac{d^2 x_m(t)}{dt^2} + \Gamma^m_{ij}(t) \cdot \frac{dx_i(t)}{dt} \cdot \frac{dx_j(t)}{dt} = 0$$



Exponential map Let $\mathcal{U}\subseteq\mathbb{R}^n$ be a domain equipped with a Riemannian metric g.

Then for any $p\in\mathcal{U}$, the exponential map at p is defined by

$$\mathrm{Exp}_p(v)=\gamma(1), \quad v\in T_p\mathcal{U}\equiv\mathbb{R}^n,$$

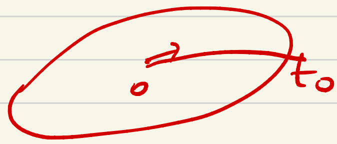
where γ is the geodesic s.t. $\gamma(0)=p$, $\gamma'(0)=v$.

Why called "exponential" ???

<u>Lemma</u>. Let $p\in\mathcal{U}$, $\mathcal{O}_p\subset T_p\mathcal{U}\equiv\mathbb{R}^n$ <u>open</u>.

Assume that $\mathrm{Exp}_p:\mathcal{O}_p\to\mathcal{U}$ is well-defined.

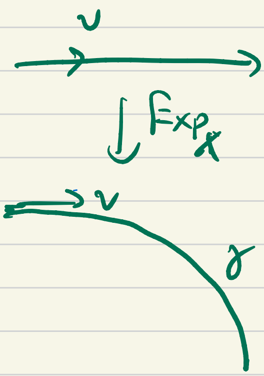
Then $\gamma(t)\equiv\mathrm{Exp}_p(tv)$, $t\in[0,t_0]$ is a geodesic with $\gamma(0)=p$, $\gamma'(0)=v$ if $tv\in\mathcal{O}_p$ $\forall t\in[0,t_0]$.



<u>Lemma</u>. Let $\mathcal{E}(x,v)\equiv(x,\mathrm{Exp}_x(v))$. Then

$\mathrm{Exp}_x(0)=x$

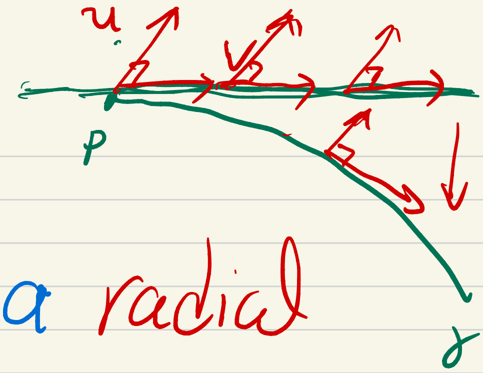
$$D(\mathcal{E})\Big|_{(x,0)}=\begin{array}{cc}
\overbrace{\begin{bmatrix}\mathrm{Id} & 0\\ \mathrm{Id} & \mathrm{Id}\end{bmatrix}}^{(x,v)\mapsto x} & \begin{array}{c} n\\ n\end{array}\\
\begin{array}{cc} n & n\end{array} &
\end{array}$$



In particular,

$\mathrm{Exp}_p:\mathcal{O}_p\to\mathcal{U}$ is a local diffeomorphism.

Lemma (Gauß Lemma)



The exponential map is a radial isometry, i.e.,

$$g_{\mathrm{Exp}_p(v)}\big(D\mathrm{Exp}_p(v), D\mathrm{Exp}_p(u)\big) = g_p(v,u)$$

What about other directions?

Proof. Consider the variation

$$V(t,s) = \mathrm{Exp}_p(t(v+su)).$$

Then $\gamma_s(\cdot) = V(\cdot, s)$ is a geodesic $\forall\ s\in(-\epsilon,\epsilon)$.

$$X(t) = \frac{\partial}{\partial s}\Big|_{s=0} V(t,s) = (D\mathrm{Exp})_{tv}(tu)$$

$$= t(D\mathrm{Exp})_{tv}(u)$$

$X(1) = \underline{(D\mathrm{Exp})_v(u)}$, $\quad \gamma_0'(1) = (D\mathrm{Exp})_v(v)$

$X(0) = 0$, $\quad \gamma_0'(0) = v$

Applying the first variation formula,

$$\left.\frac{d}{ds}\right|_{s=0} L_s = g(X(1), \gamma_0'(1)) - g(X(0), \gamma_0'(0))$$

$$= \|v\|_g^{-1} \cdot g((DExp)_v(u), (DExp)_v(v)).$$

On the other hand,

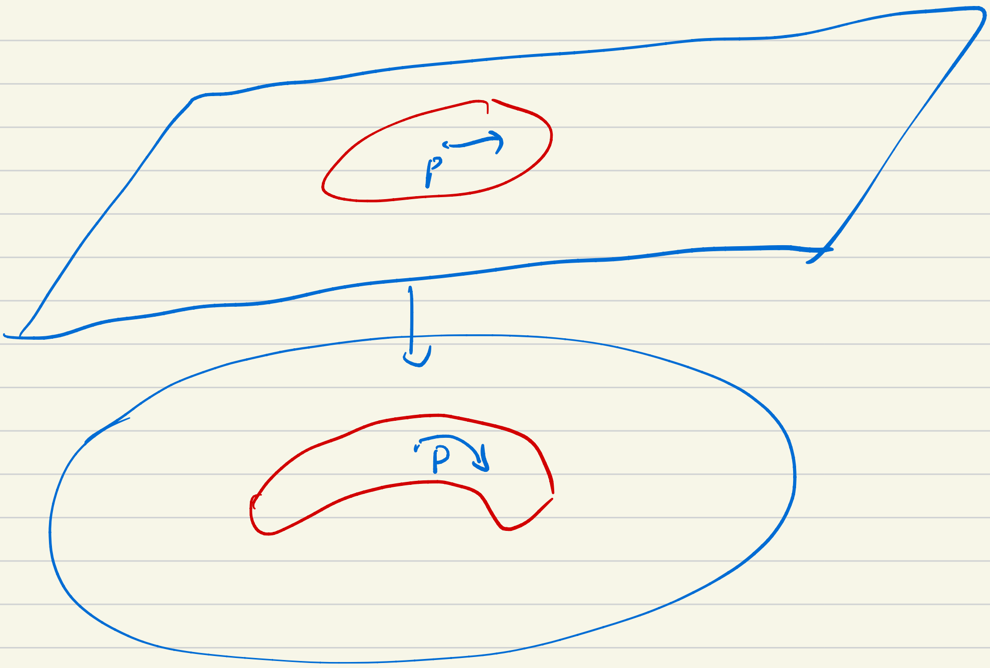
$$L_s = \|v + su\|_g = g(v+su, v+su)^{\frac{1}{2}}$$

Then

$$\left.\frac{d}{ds}\right|_{s=0} L_s = \|v\|_g^{-1} \cdot g(u, v).$$

$$\Longrightarrow g((DExp)_v(u), (DExp)_v(v)) = g(u, v).$$

$\square$



$$B_\delta(0^n) = \{ x \in \mathbb{R}^n \mid d_0(x, 0^n) < \delta \}$$

$$B_\delta(p) = \{ x \in U \mid d_g(p, x) < \delta \}$$