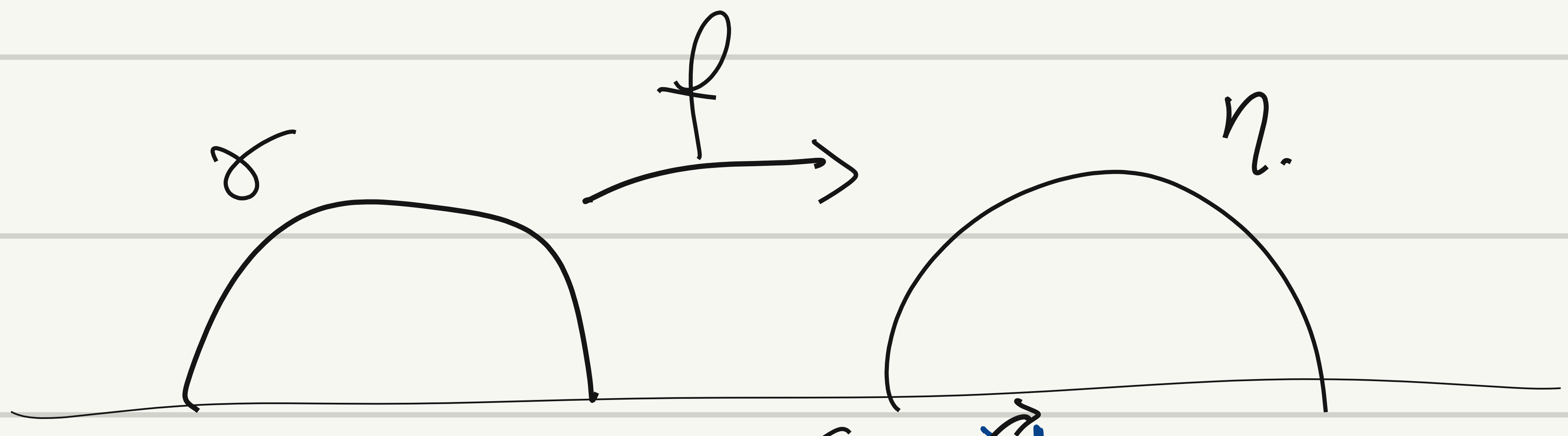
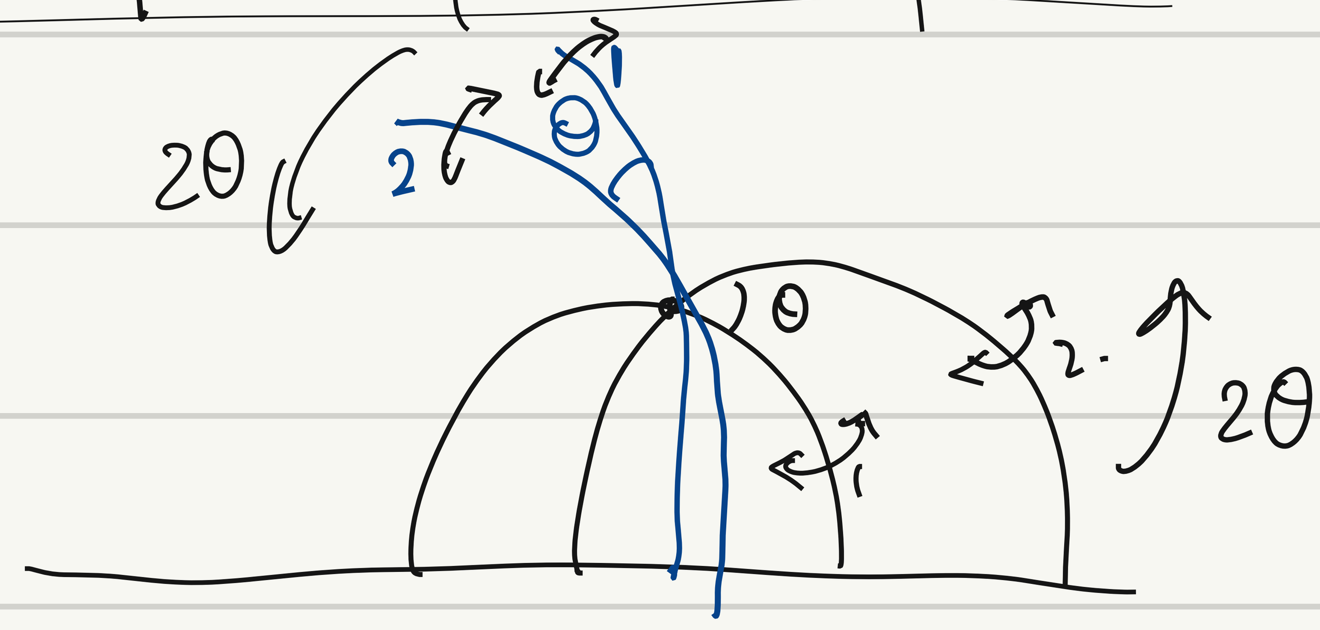
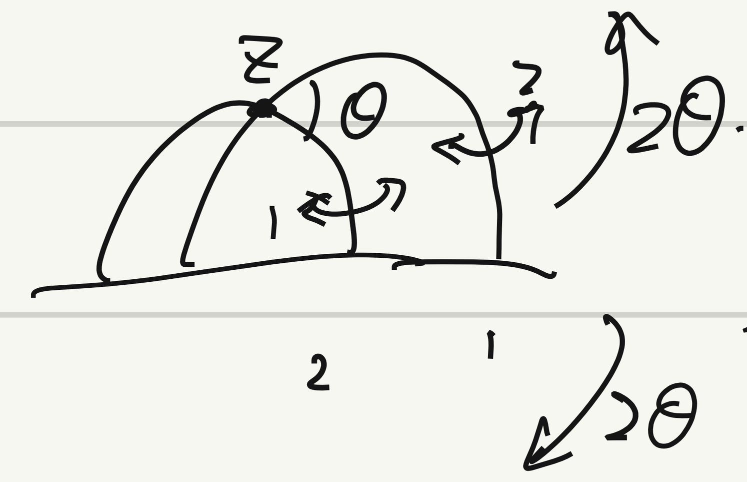
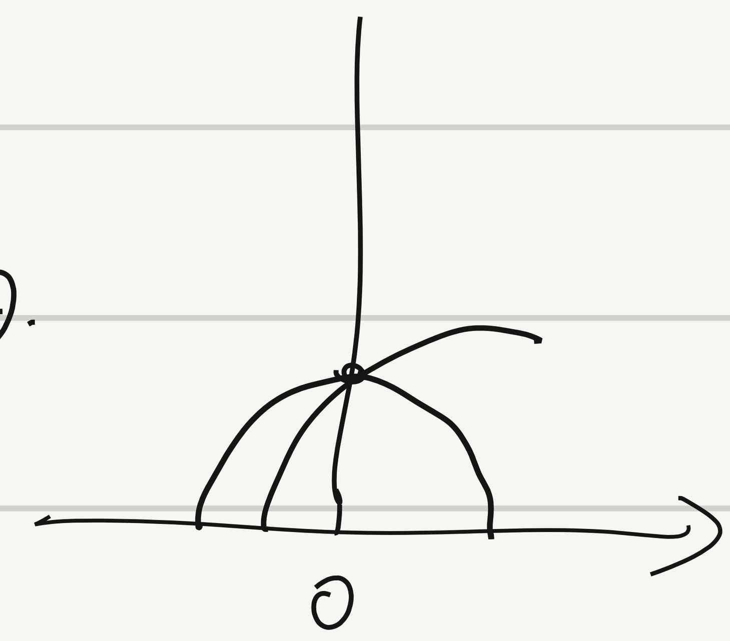




$\tau_\eta = f\circ\tau_\gamma\circ f^{-1}$





4. Determine an isometry:

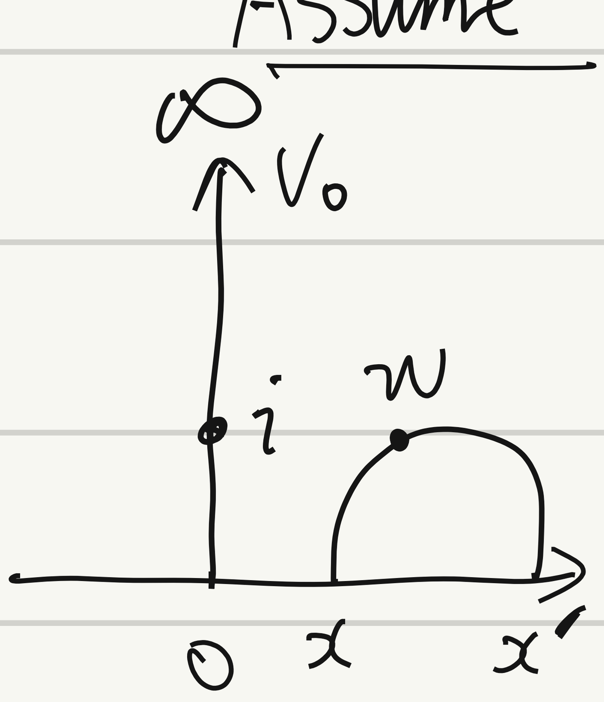
Let $f: \mathbb{H} \to \mathbb{H}$ be an isometry. We consider its extension to $\overline{\mathbb{H}}$

$\tilde{f}: \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ continuous.

Assume:

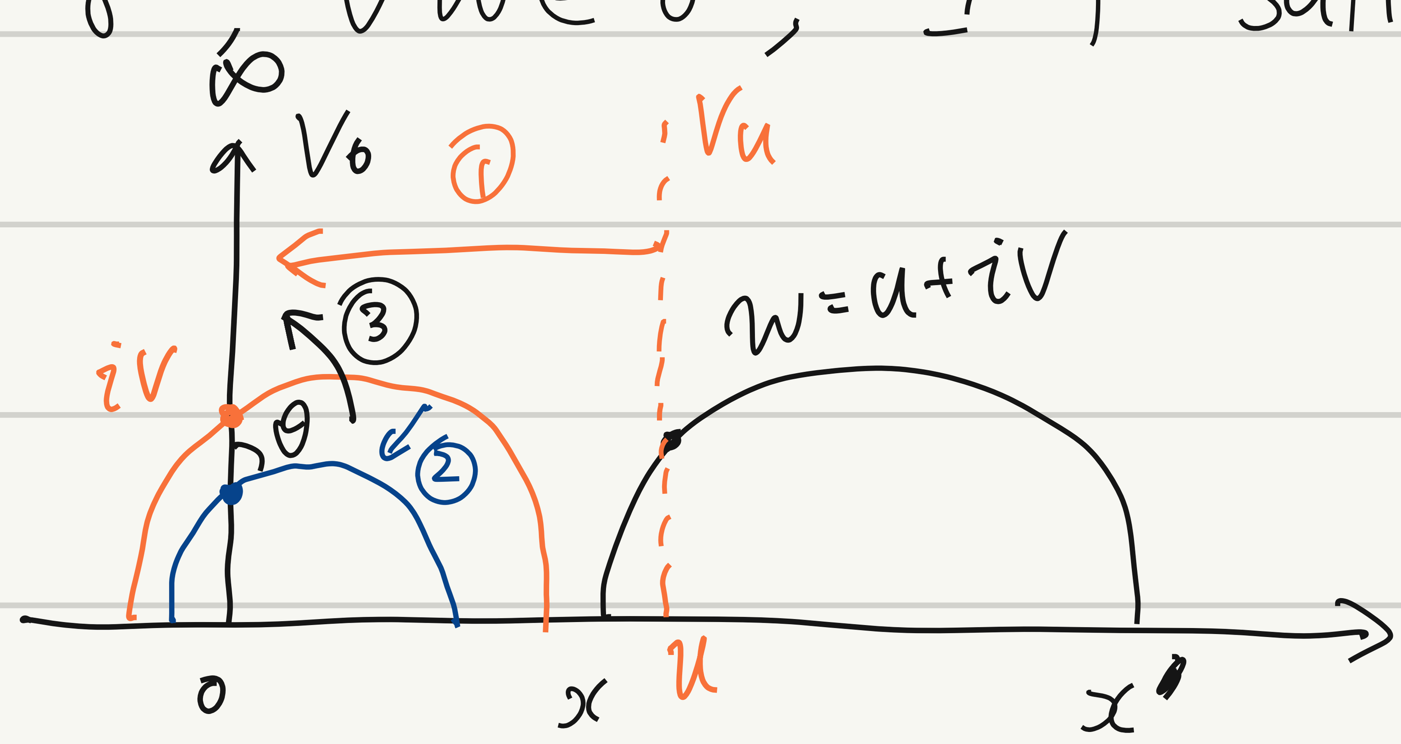
$$\begin{cases} f(0) = x \\ f(\infty) = x' \\ f(i) = w \in \gamma \end{cases}$$

end points x and x'.



Prop: $\forall \gamma$, $\forall w \in \gamma$, $\exists f$ satisfies the above conditions.

Proof:



Construct f^{-1}

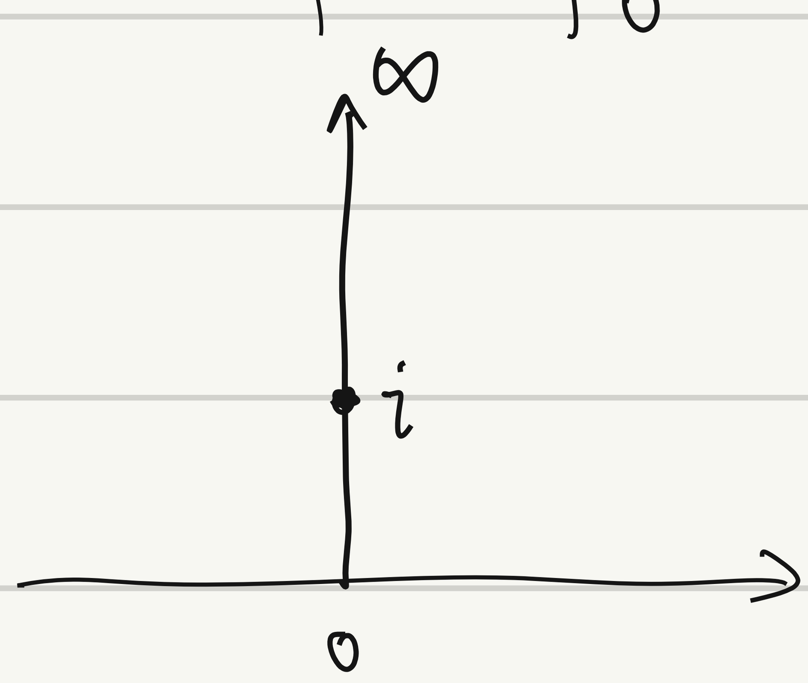
① $T_{-u}(z) = z - u$

② $\phi_{v^{-1}}(z) = v^{-1} z$

③ $\rho_{\theta/2}(z) = \dfrac{\cos\frac{\theta}{2}\, z + \sin\frac{\theta}{2}}{-\sin\frac{\theta}{2}\, z + \cos\frac{\theta}{2}}$.

$$f_0 = \rho_{\theta/2} \circ \phi_{v^{-1}} \circ T_{-u} \quad \begin{cases} f_0(x') = \infty \\ f_0(x) = 0 \\ f_0(w) = i \end{cases}$$

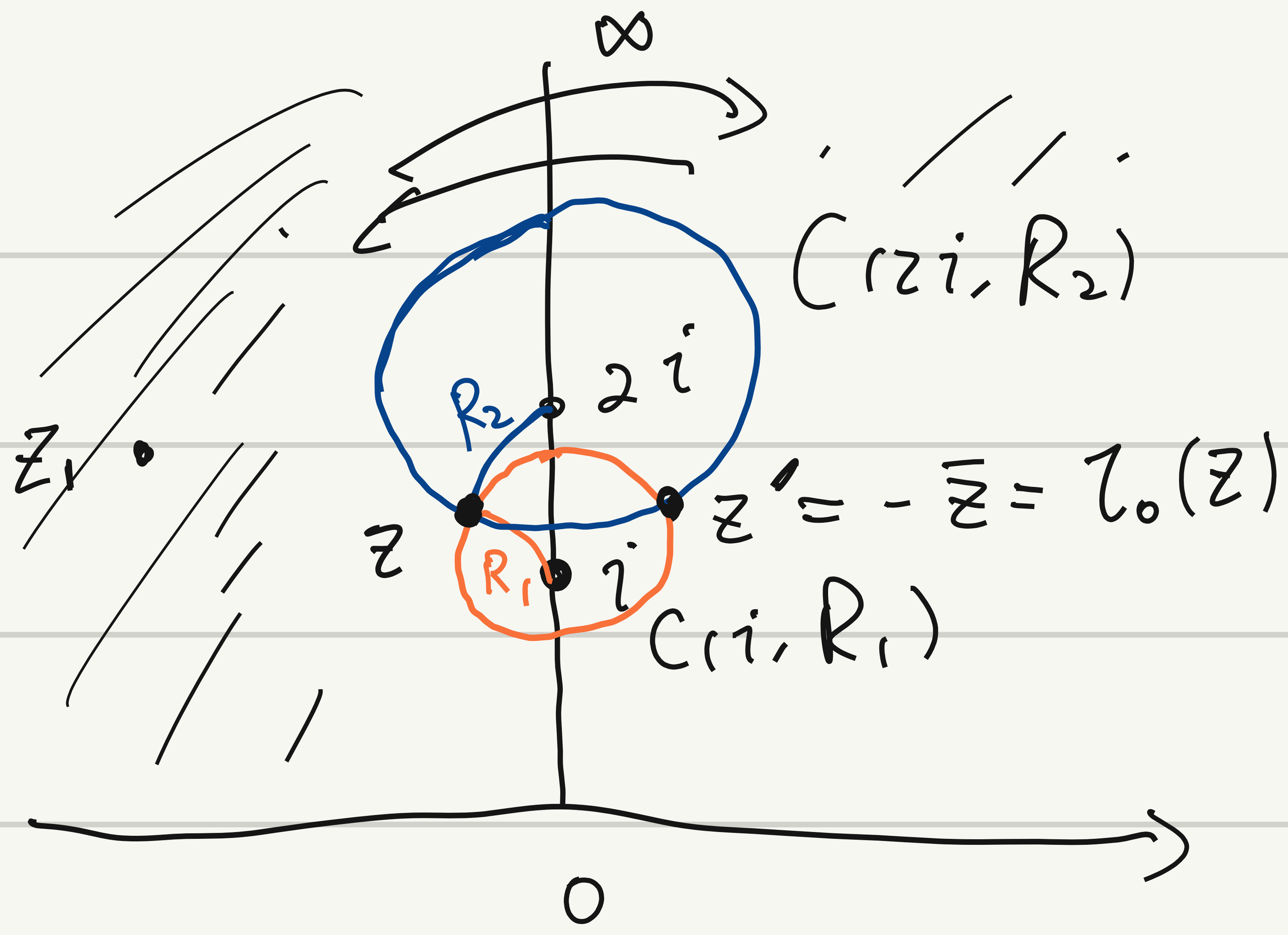
$f = f_0^{-1}$ ✓ ∎



Prop: There are only 2 such f.

Proof: it is enough to study f

$$\begin{cases} f(0) = 0 \\ f(\infty) = \infty \\ f(i) = i \end{cases}$$



$\forall z \in \mathbb{H}$, $R_1 = d_{\mathbb{H}}(z, i)$

$R_2 = d_{\mathbb{H}}(z, 2i)$

$d_{\mathbb{H}}(f(z), \underbrace{f(i)}_{= i}) = d_{\mathbb{H}}(z, i)$

$\Rightarrow f(z) \in C(i, R_1)$

$C(i, R_1) \cap C(2i, R_2)$

$d_{\mathbb{H}}(f(z), \underbrace{f(2i)}_{= 2i}) = d_{\mathbb{H}}(z, 2i)$

$\Rightarrow f(z) \in C(2i, R_2)$

$\forall z$ $\quad f(z) = z$ or $f(z) = z' = -\bar{z}$.

$\quad f(z_1) = z_1$ or $f(z_1) = z_1' = -\bar{z}_1$

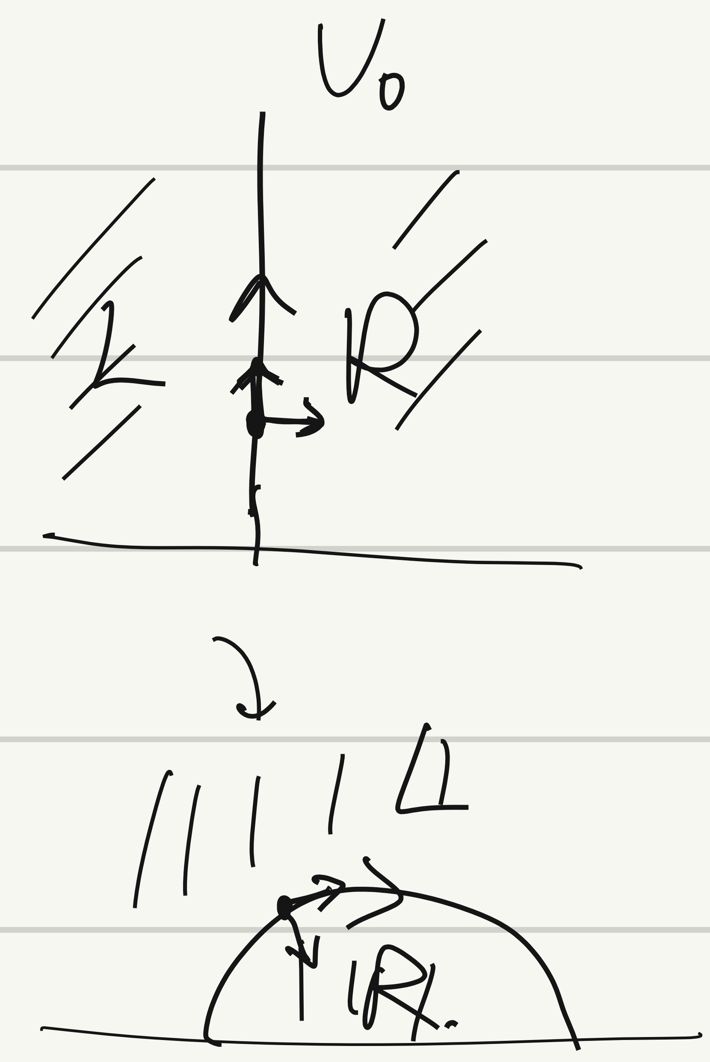
$\Rightarrow$ $\quad \underline{f = id}$ or $\underline{f = \tau_0}$ reflection along V_0. ∎

Prop: $\forall \gamma$ endpoint x, x', $\forall w \in \gamma$

$\exists$ only $\underline{2}$ isometries s.t. $x \mapsto 0$

$x' \mapsto \infty$

$w \mapsto i$



If $x = 0$ $\quad x' = \infty$ $\quad w = i$

$f(0) = 0$
$f(\infty) = \infty$
$f(i) = i$

id preserves the orientation of $\mathbb{H}$

τ_0 reverse the orientation of $\mathbb{H}$.

① $f(x), f(x') \sim \underline{f(\gamma)}$

② $f(w)$ $w \in \gamma$ $\quad$ can determine $\underline{f}$.

③ orientation.

Orientation preserving $\quad$ orientation reversing

$\underline{T_t, \phi_\lambda, \rho_\theta}$ $\quad$ $\underline{T_t, \phi_\lambda, \rho_\theta}$ τ

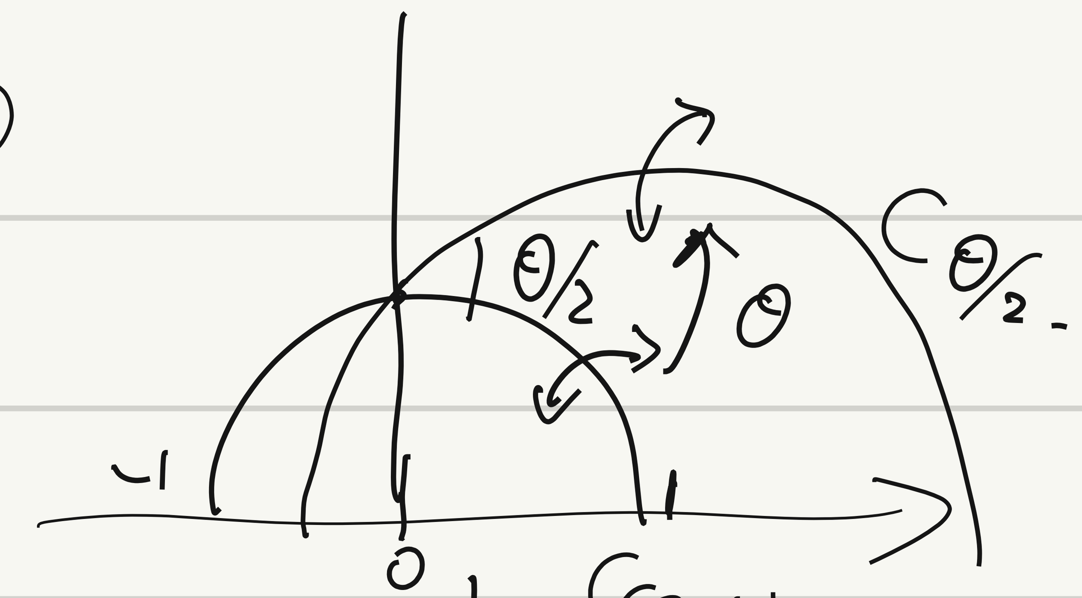
Cor: $\forall f$ isometry, f can be written as a compositions of reflections of $\mathbb{H}$.

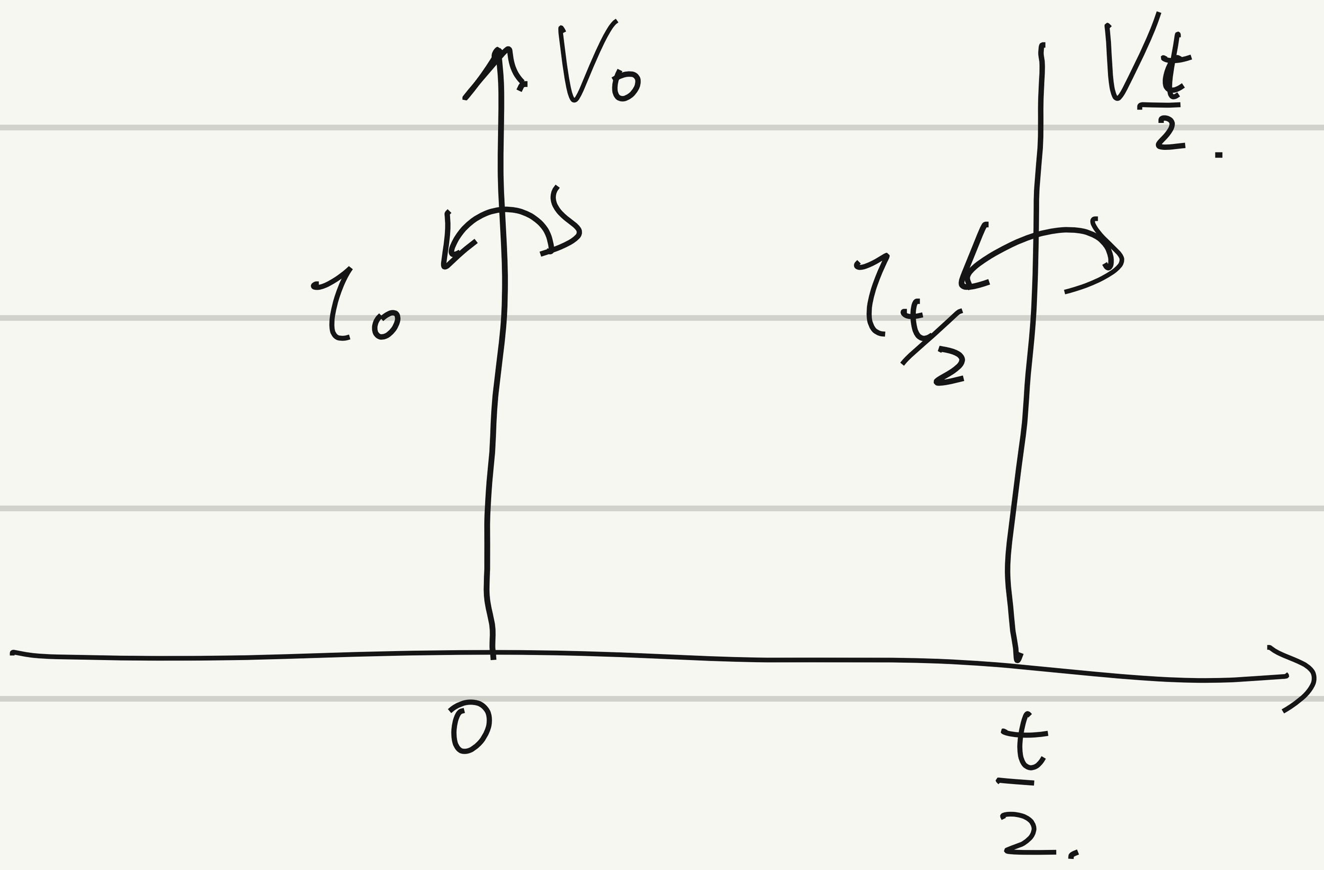
Proof:

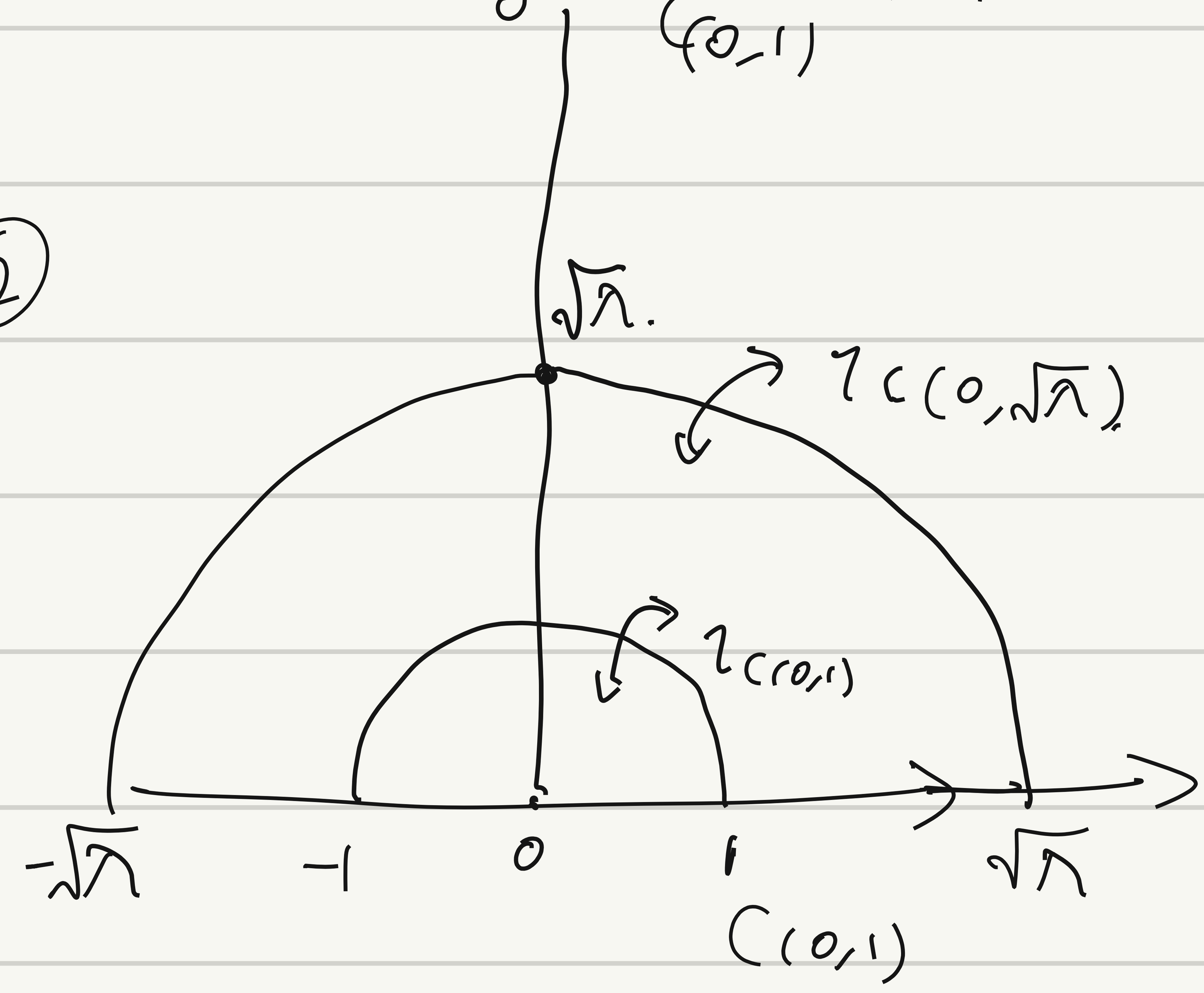
① $T_t = \iota_{\frac{t}{2}} \circ \iota_0$

② $\phi_\lambda = \iota_{C(0,\sqrt{\lambda})} \circ \iota_{C(0,1)}$

③ $\rho_\theta = \iota_{\theta/2} \circ \iota_{C(0,1)}$

③ 

① 

② 

$\iota_0(z) = -\bar{z}$

$\iota_{\frac{t}{2}}(z) = -\bar{z} + t$

$T_t(z) = \iota_{\frac{t}{2}} \circ \iota_0(z) = -(\overline{-\bar{z}}) + t = z + t$

$\iota_{C(0,1)}(z) = \frac{1}{\bar{z}}$

$\iota_{C(0,\sqrt{\lambda})}(z) = \frac{0 \cdot \bar{z} + \sqrt{\lambda}^2 - 0}{\bar{z} - 0} = \frac{\lambda}{\bar{z}}$

$\phi_\lambda(z) = \iota_{C(0,\sqrt{\lambda})} \circ \iota_{C(0,1)}(z) = \frac{\lambda}{\overline{\left(\frac{1}{\bar{z}}\right)}} = \lambda z.$

Composition of 2 reflections:

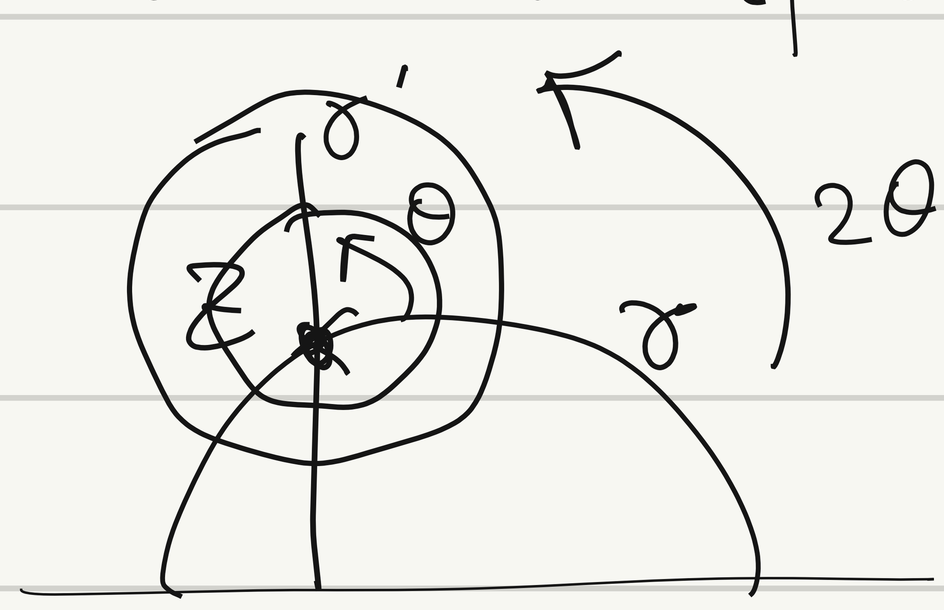
Let γ, γ' be two geodesic in $\mathbb{H}$

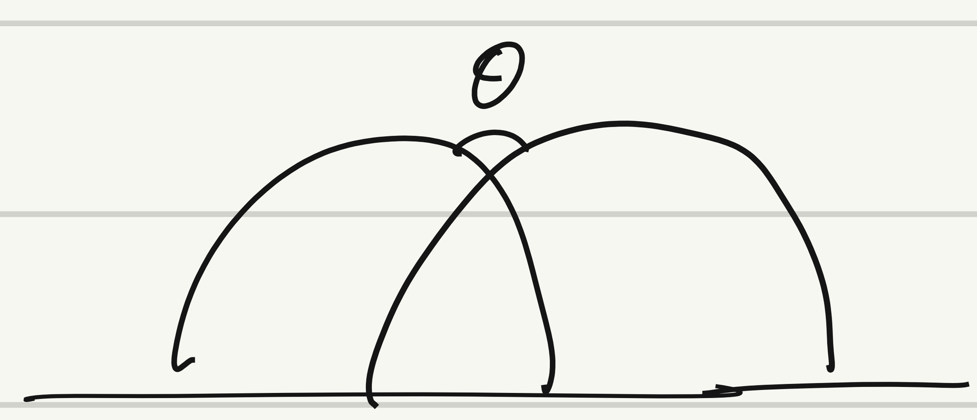
ι, ι' be the associated reflections of $\mathbb{H}$.

3 Case:

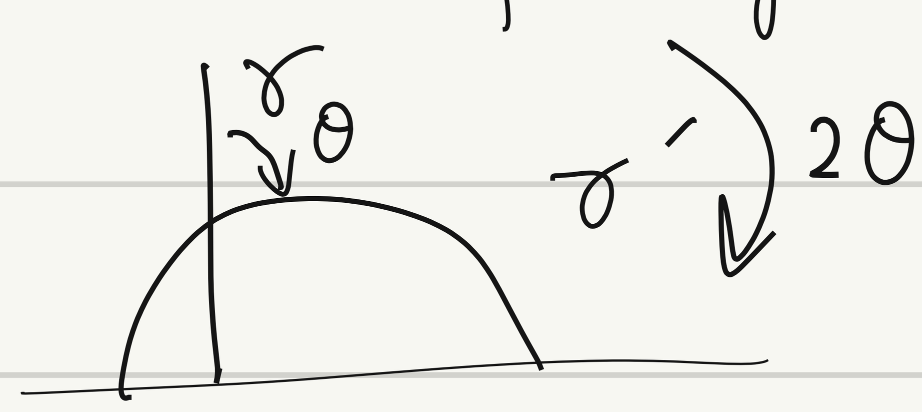
① $\boxed{\text{Intersecting}}$:

$\gamma \cap \gamma' = \{z\}$



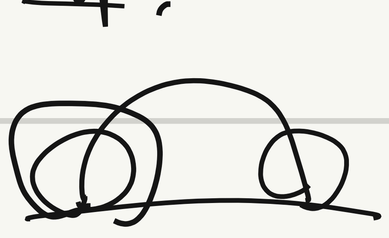


$\underline{\underline{\iota' \circ \iota}}$ rotation at z of angle 2θ.



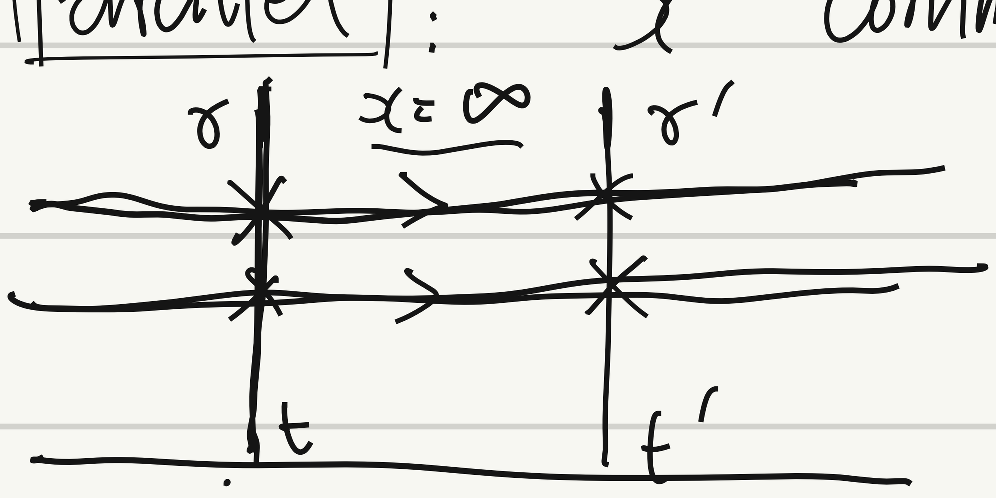
② $\boxed{\text{Parallel}}$: x common end point.

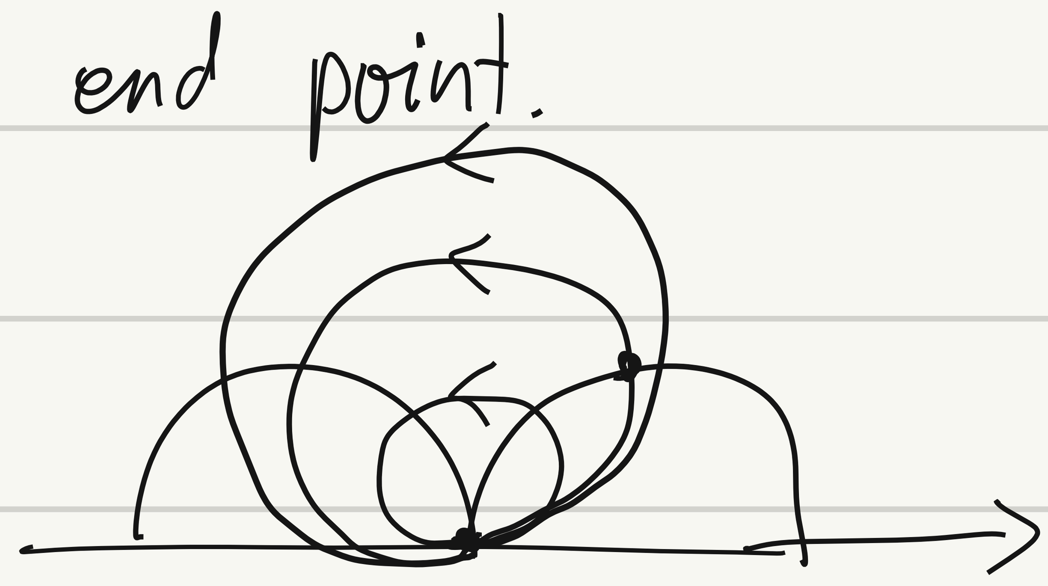
Prop:



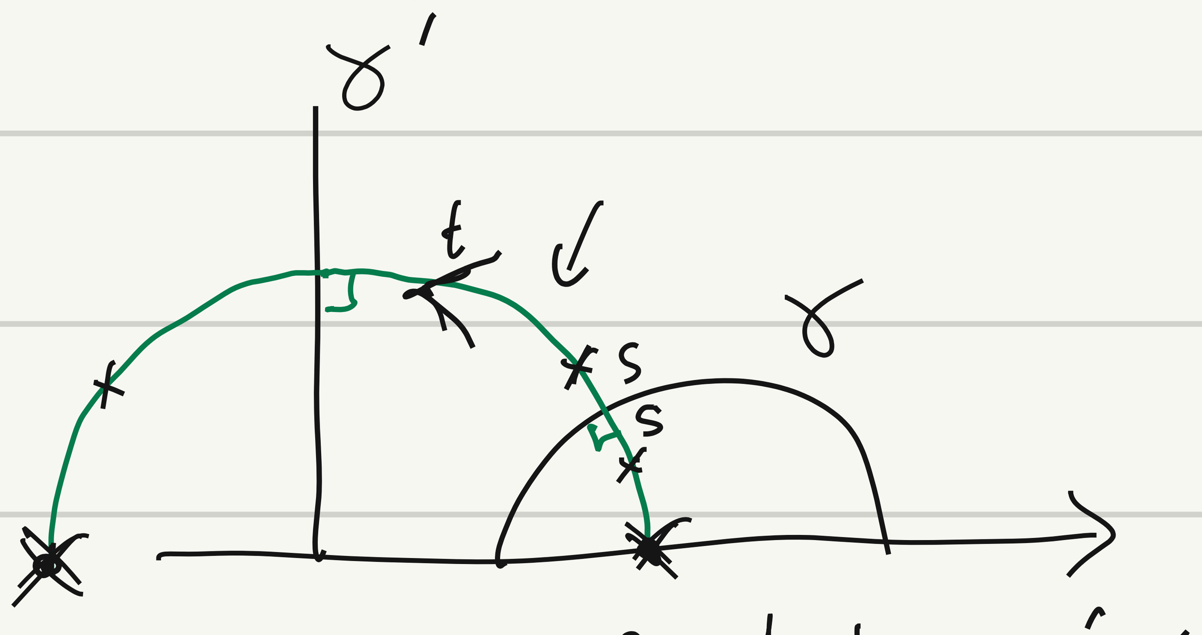
ι reflection.

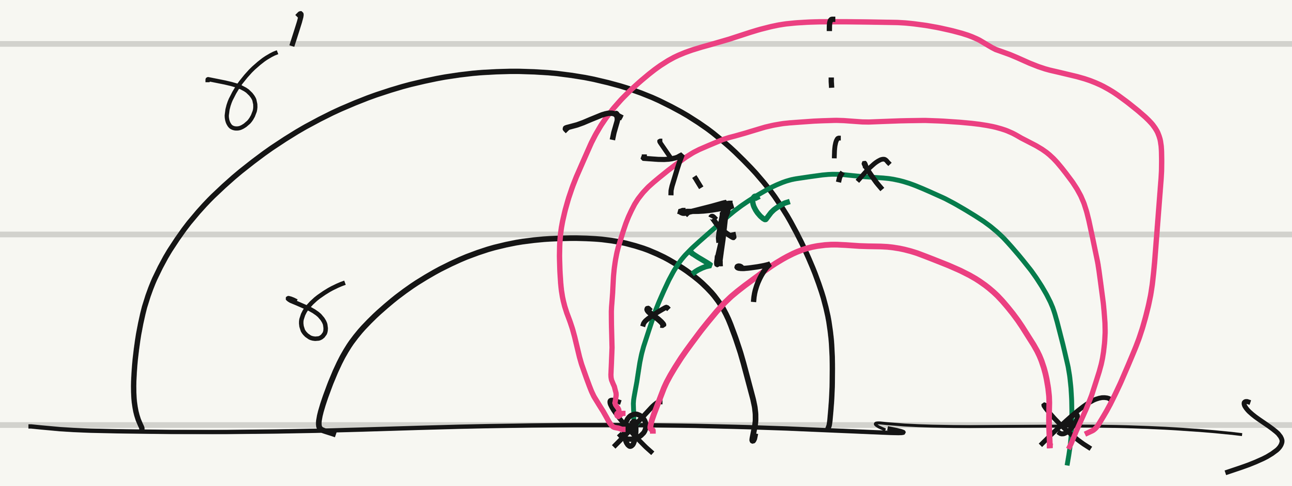
preserve horocycles center at x or x'

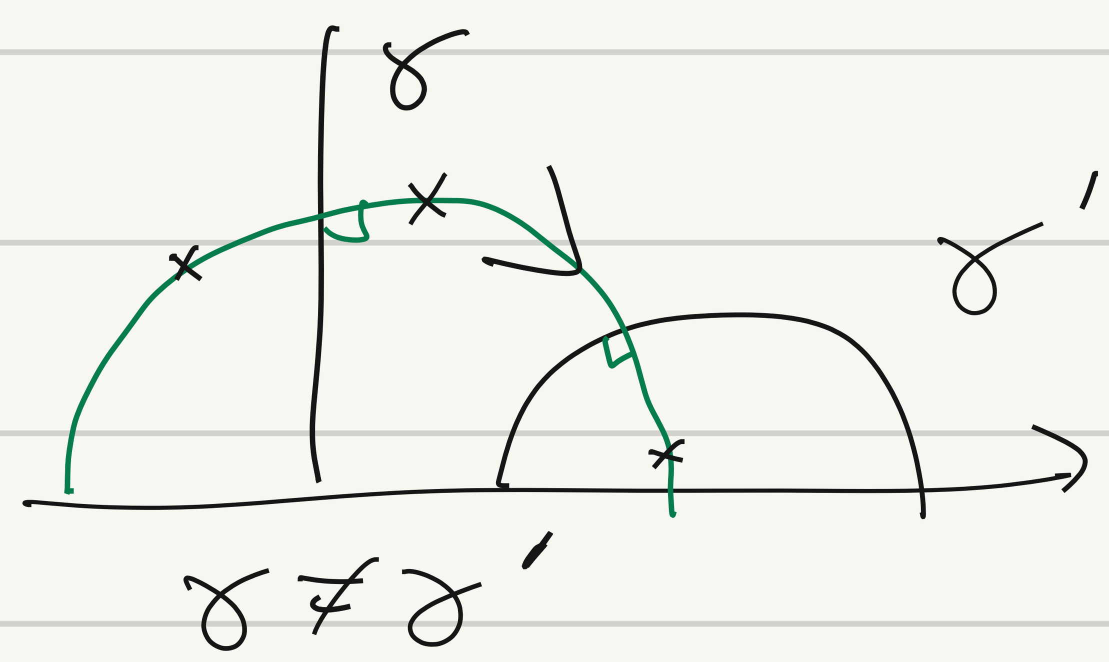




③ disjoint



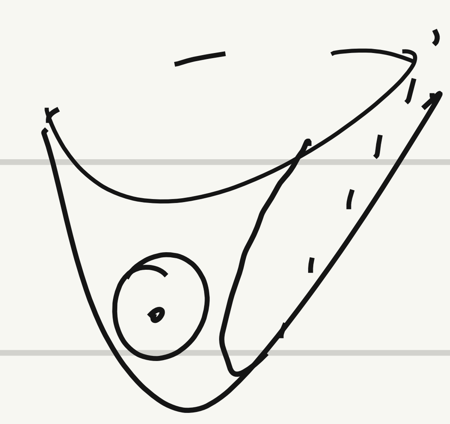




Def: $\gamma \cap \gamma' = \{z\}$, $\tau'\circ\tau$ is elliptic.

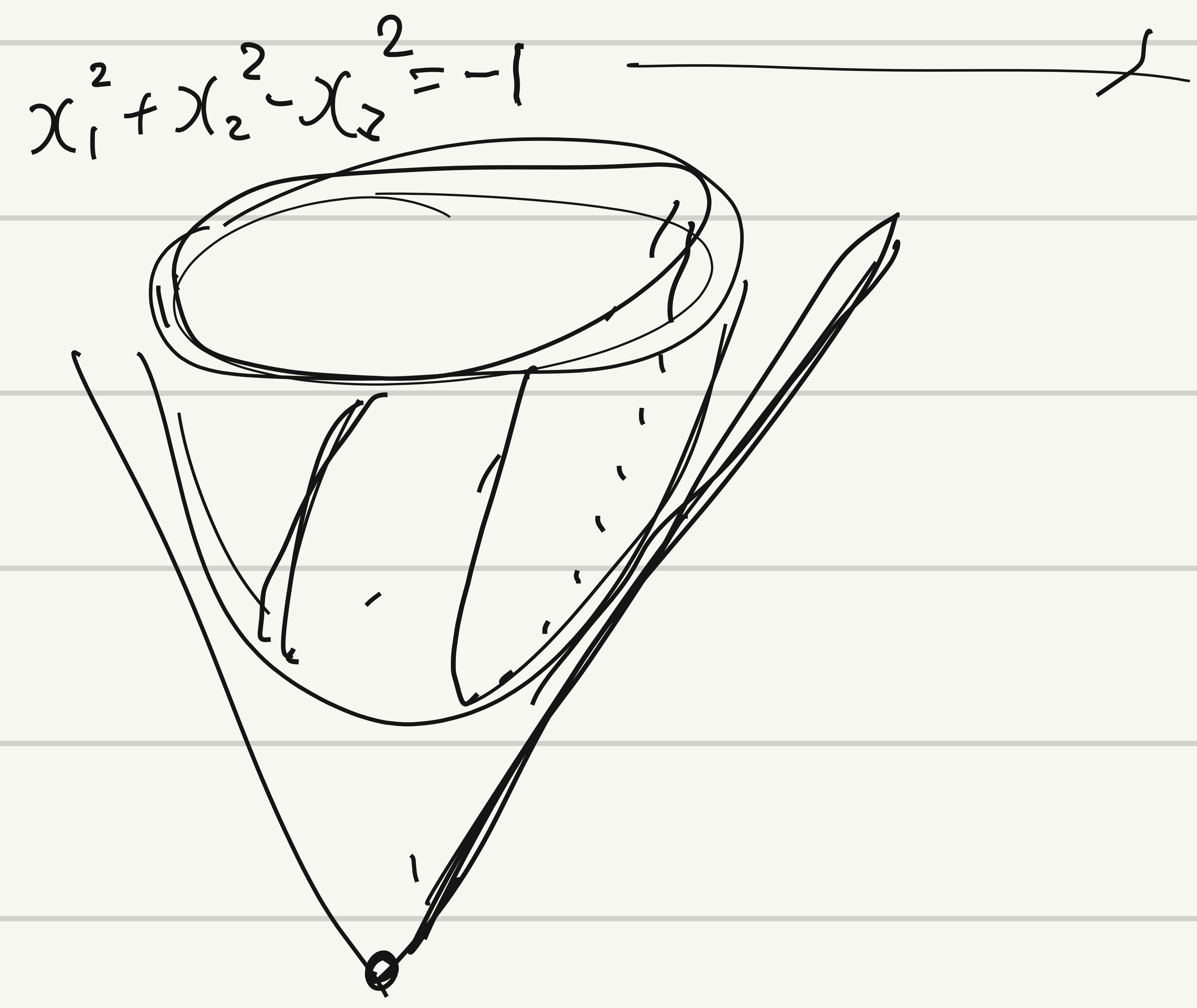
γ, γ' parallel, $\tau'\circ\tau$ is parabolic.

γ, γ' disjoint, $\tau'\circ\tau$ is hyperbolic.



γ axis of τ_γ

$\tau'\circ\tau$ hyperbolic.



common perpendicular geodesic of γ, γ' is called the axis of $\tau'\circ\tau$.

Prop: Every $f \overset{\neq id}{\in} \mathrm{Isom}(\mathbb{H})$ can be written as a composition of

— 2 reflections if orientation preserving

— 3 reflections if ———— reversing.

Proof: $\forall f \in \mathrm{Isom}(\mathbb{H})$, $\exists \hat{f}: \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ continuous.

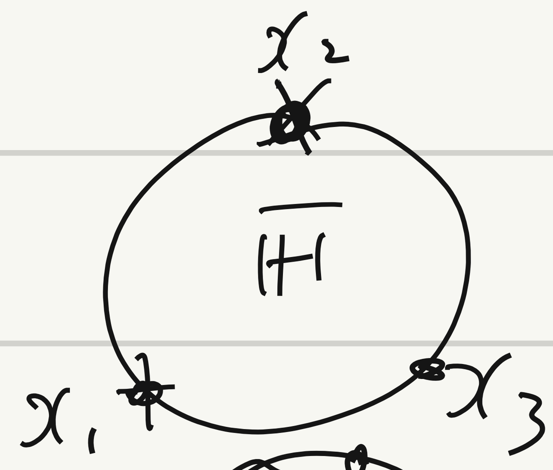
$\overline{\mathbb{H}} \cong D$

Brouwer fixed point theorem

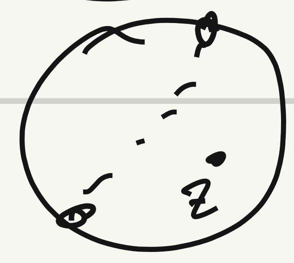
$\forall f: D \to D$ continuous, $\mathrm{Fix}(f)$ is non empty.

$\mathrm{Fix}(f) = \{p \in D \mid f(p) = p\}$.

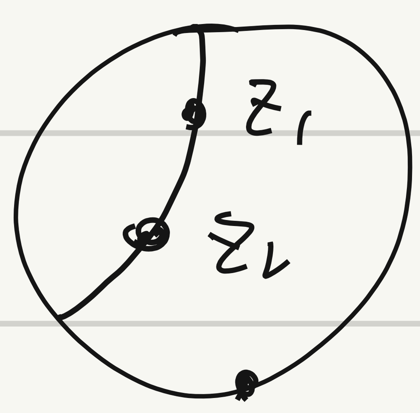
Lemma $f \in \mathrm{Isom}(\mathbb{H})$ is determined by.



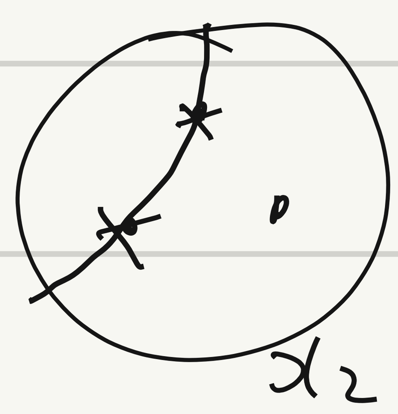
either $f(x_1)\ f(x_2)\ f(x_3)$ $\quad x_1 \neq x_2 \neq x_3 \in \partial\mathbb{H}$



or $f(x_1)\ f(x_2)\ f(z)$ $\quad x_1 \neq x_2 \in \partial\mathbb{H},\ z \in \mathbb{H}$, $z \notin \gamma(x_1, x_2)$

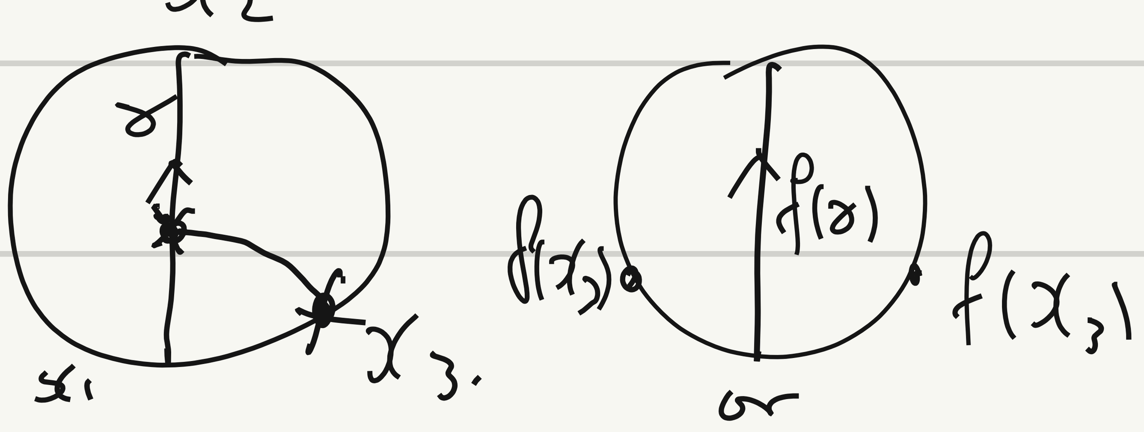


or $f(x),\ f(z_1)\ f(z_2)$ $\quad x \in \partial\mathbb{H}\ \ z_1 \neq z_2 \in \mathbb{H}$, x is not end port of γ passing z_1, z_2



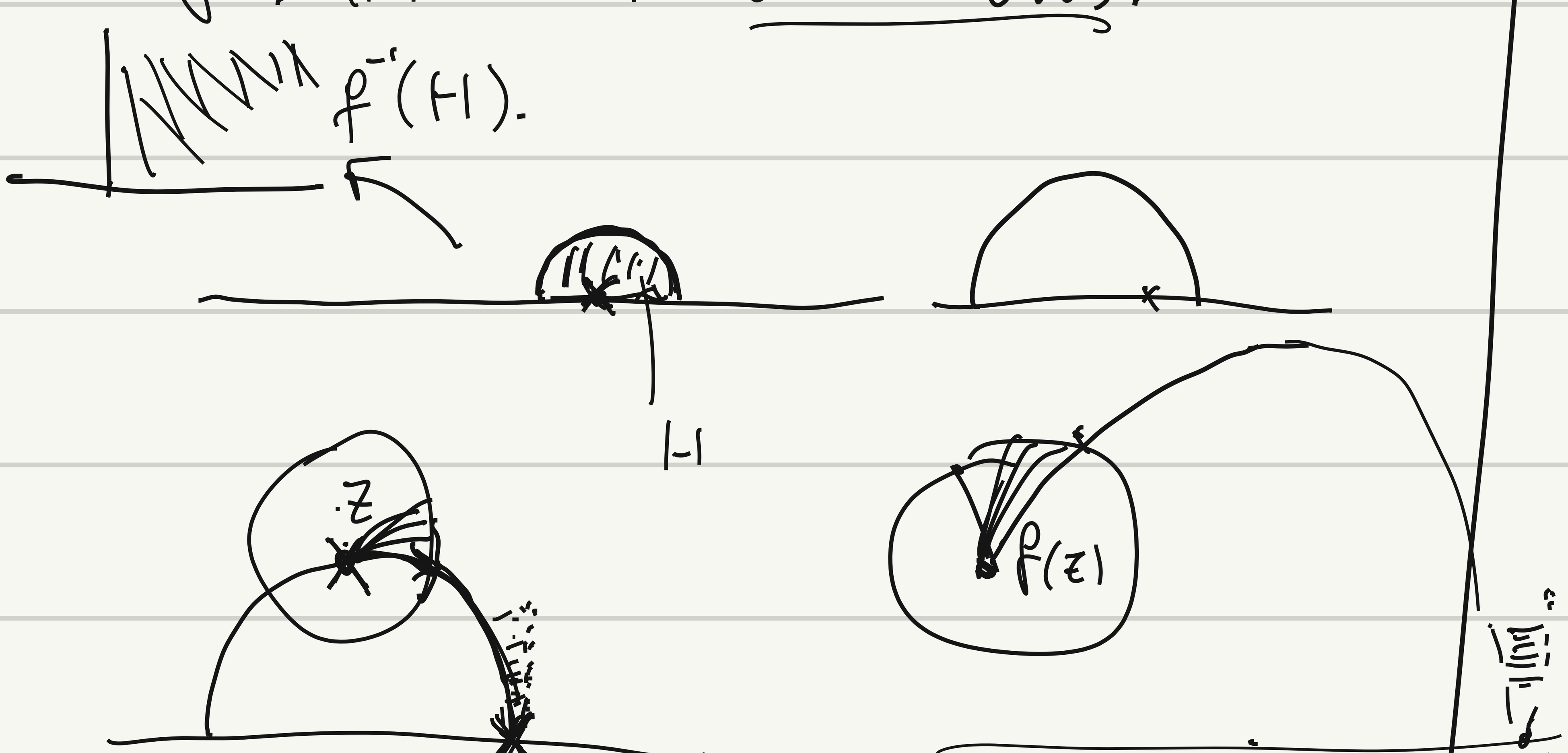
or $f(z_1)\ f(z_2)\ f(z_3)$ $\quad z_1 \neq z_2 \neq z_3 \in \mathbb{H}$ non-colinear.

Proof:



$f: \mathbb{H} \to \mathbb{H}$ isometry.

$\Rightarrow \exists\, \tilde{f}: \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ continuous.



$\boxed{\forall z_n \to x \quad f(z_n) \to f(x),\ n \to \infty}$.

① $\mathrm{Fix}(f) \neq \emptyset$

② Lemma.

$\Rightarrow |\mathrm{Fix}(f)| = 1$ or 2.

$\Rightarrow$ if $|\mathrm{Fix}(f)| \geq 3$ then $f = id$. ξ

$\times$. $\quad \cdot\ \mathrm{Fix}(f) =$

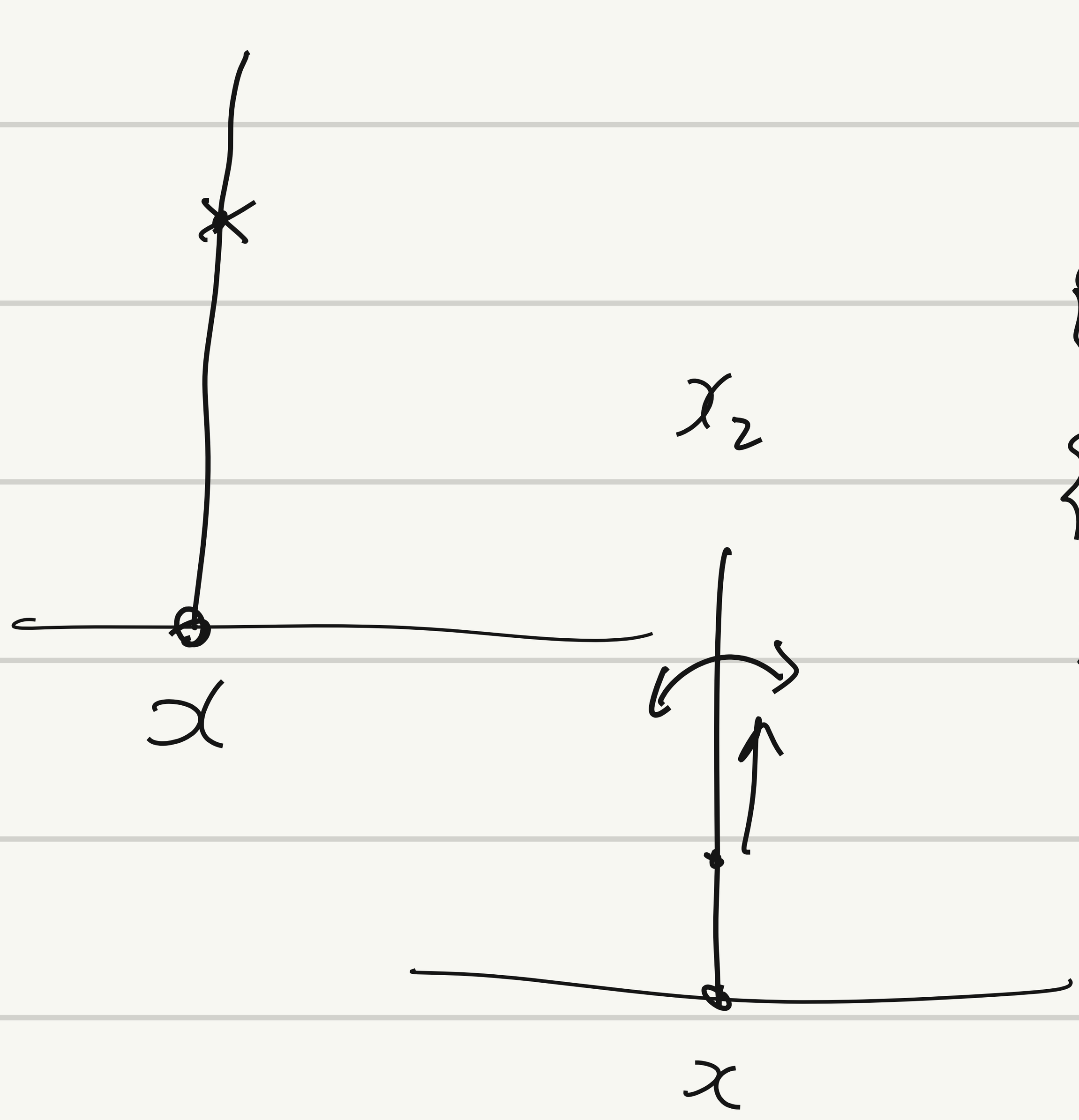
$\{z\}$ $\quad z \in \mathbb{H} \rightsquigarrow$ preserve $C(z, R) \Rightarrow$ elliptic 2 / elliptic + ref. 3.

$\{z_1, z_2\}$ fix $\gamma(z_1, z_2) \rightsquigarrow f$ reflection 1

$\{x, z\}$ fix $\gamma(x, z) \rightsquigarrow f$ reflection 1

$\{x, x_2\}$ preserve $\gamma(x, x_2) \rightsquigarrow f$. hyp 2 / hyp + reflection 3

$\{x\}$ preserve all $H(x, z)$ horocycles $\Rightarrow$ Para 2 / Para + reflection 3



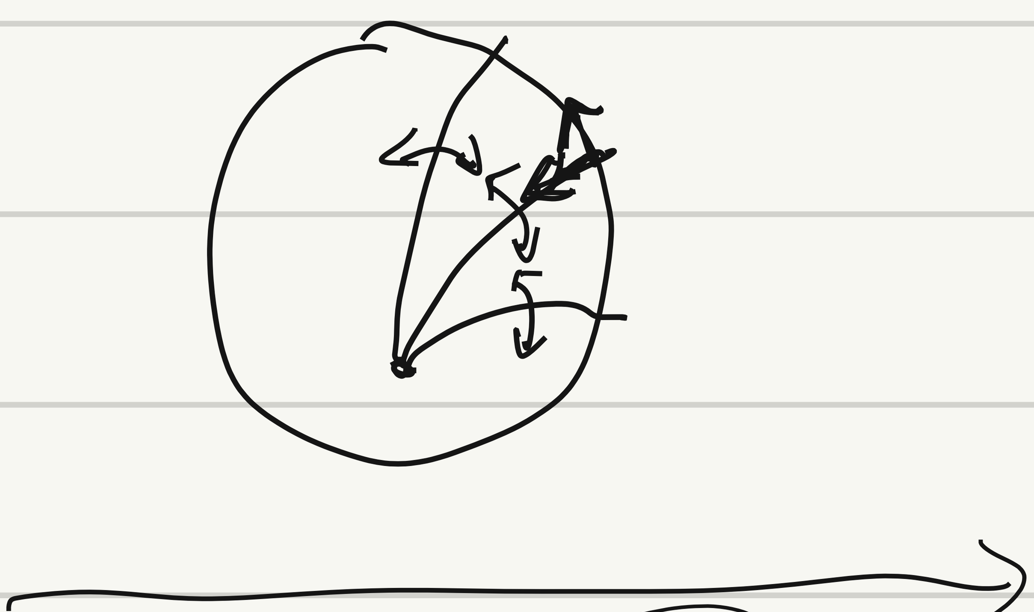
$\mathrm{Fix}(f) = \{z\}$ $\quad z \in \mathbb{H} \rightsquigarrow$ preserve $C(z,R) \Rightarrow \begin{cases} \text{elliptic} & 2 \\ \text{elliptic + ref} & 3 \end{cases}$ ✓

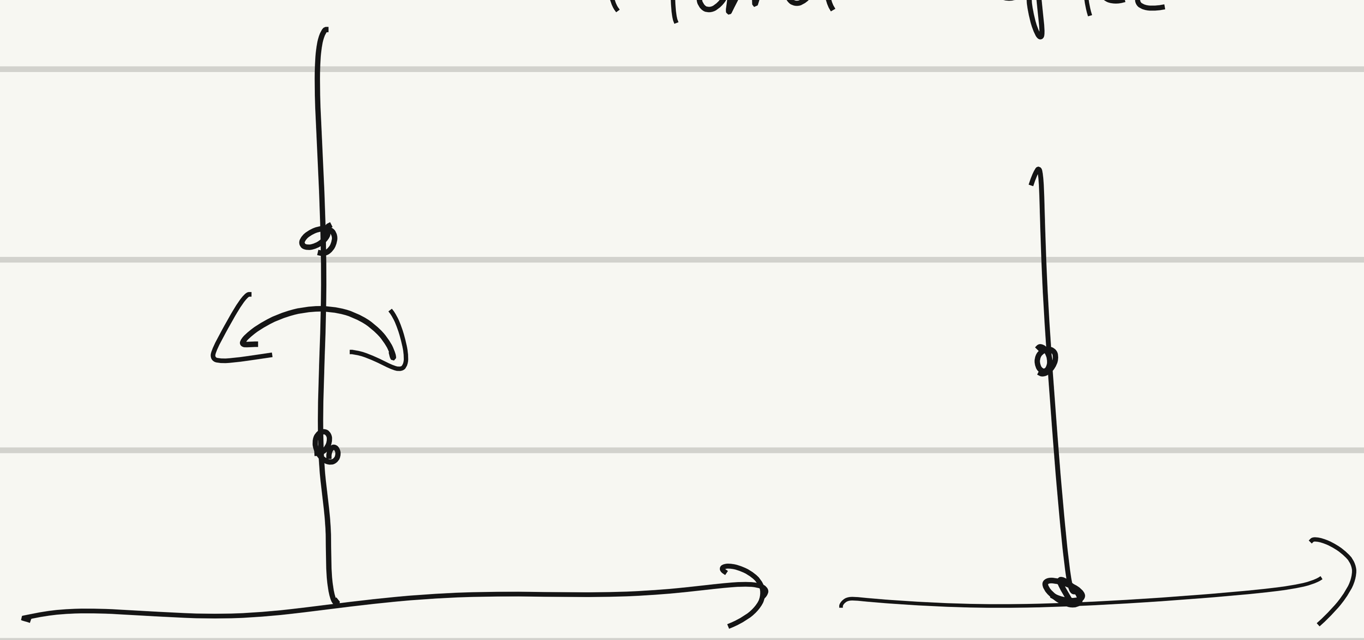
$\{z_1, z_2\}$ fix $\gamma(z_1,z_2) \rightsquigarrow f$ reflection 1

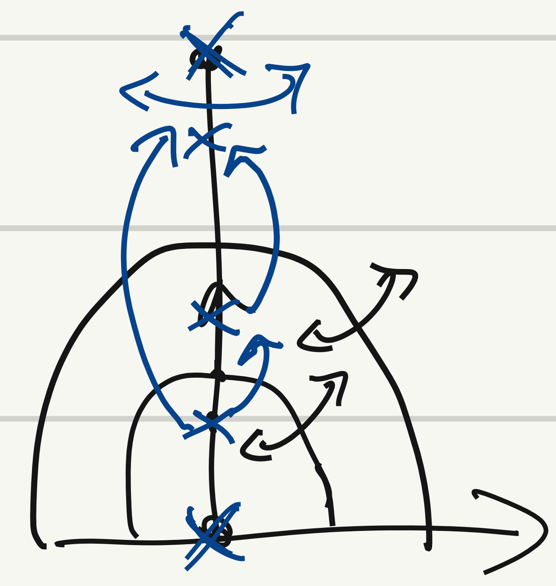
$\{x, z\}$ fix $\gamma(x,z) \rightsquigarrow f$ reflection 1

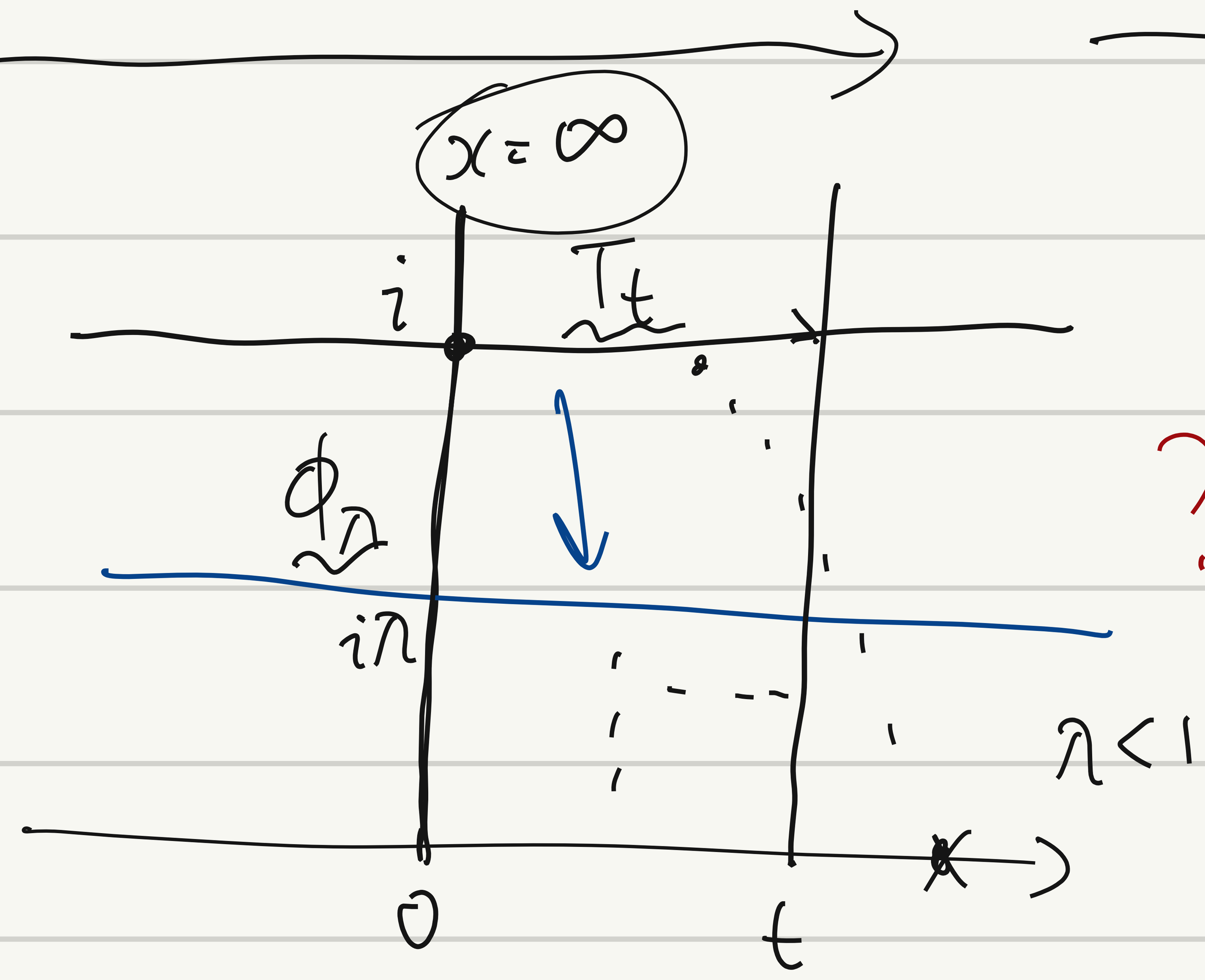
$\{x, x_2\}$ preserve $\gamma(x,x_2) \rightsquigarrow f \begin{cases} \text{hyp} & 2 \\ \text{hyp + reflection} & 3 \end{cases}$

$\{x\}$ preserve all $H(x,z)$ horocycles $\Rightarrow \begin{cases} \text{Para} & 2 \\ \text{Para + reflection} & 3 \end{cases}$









$\lambda < 1$

$$f(z) = \phi_\lambda \circ T_t(z) = \lambda(z+t).$$

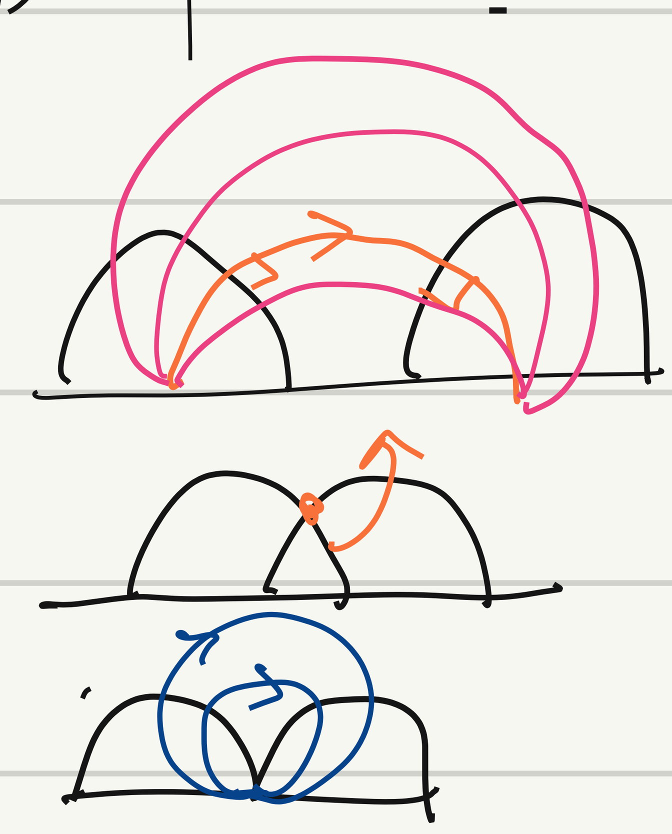
$$\lim_{n\to\infty} f^n(z) = \frac{t}{1-\lambda} \Rightarrow f\left(\frac{t}{1-\lambda}\right) = \frac{t}{1-\lambda}$$

5. Classification of isometries of $\mathbb{H}$

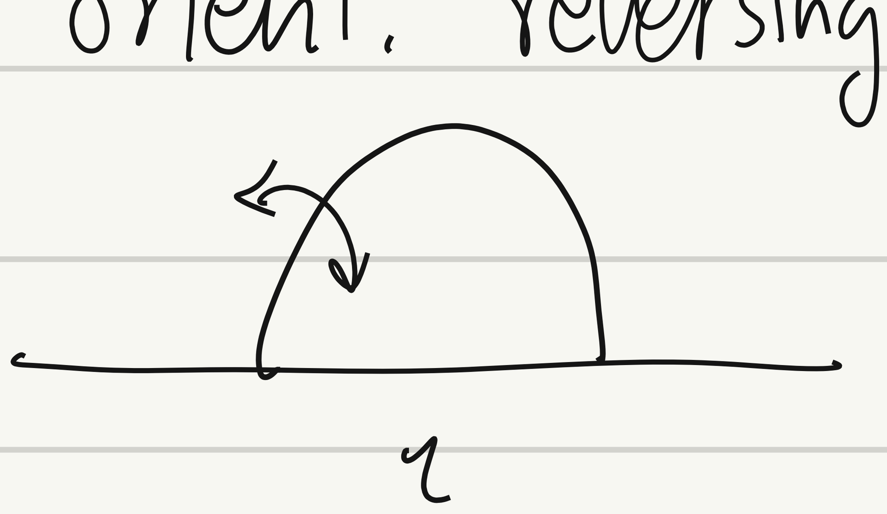
$\mathrm{Isom}(\mathbb{H})$ = isometry group of $\mathbb{H}$

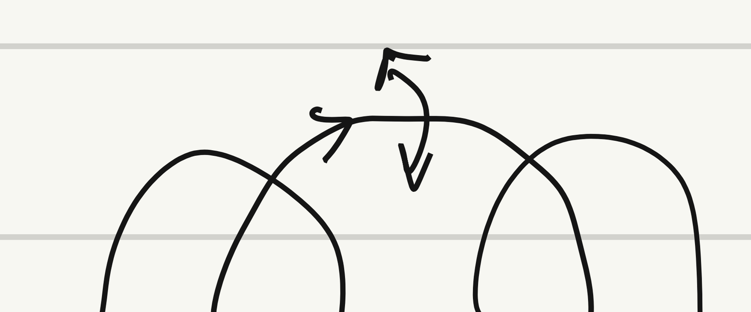
$\mathrm{Isom}^+(\mathbb{H})$ = orient. pres. isometry group of $\mathbb{H}$.

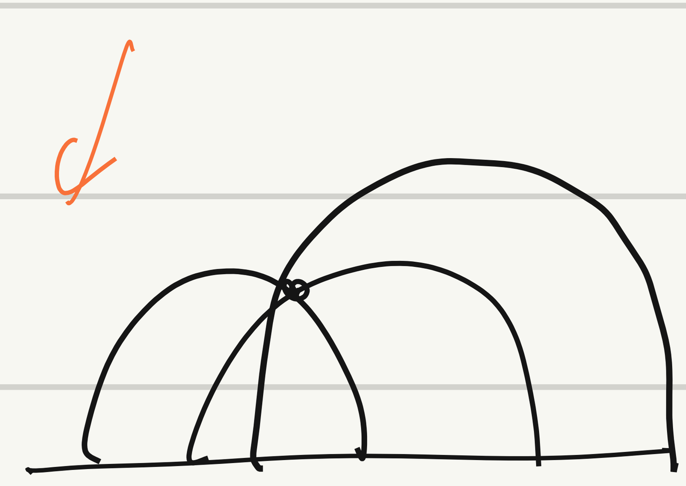
- $f \neq id \in \mathrm{Isom}^+(\mathbb{H})$
  - hyp
  - ellipitic
  - para.

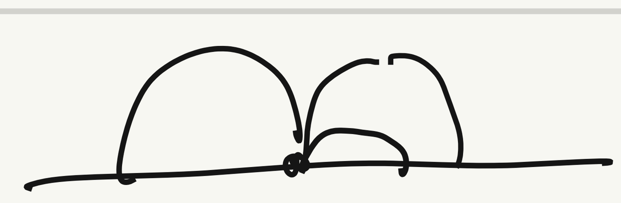


- $f \neq id$ orient. reversing.









Def: $f : \mathbb{H} \to \mathbb{H}$ isometry.

Translation distance of f

$$\ell(f) := \inf \{ d_{\mathbb{H}}(z, f(z)) \mid z \in \mathbb{H} \}$$

$\ell(f)$ is realizable if $\exists\, z \in \mathbb{H}$ s.t. $\ell(f) = d_{\mathbb{H}}(z, f(z))$.

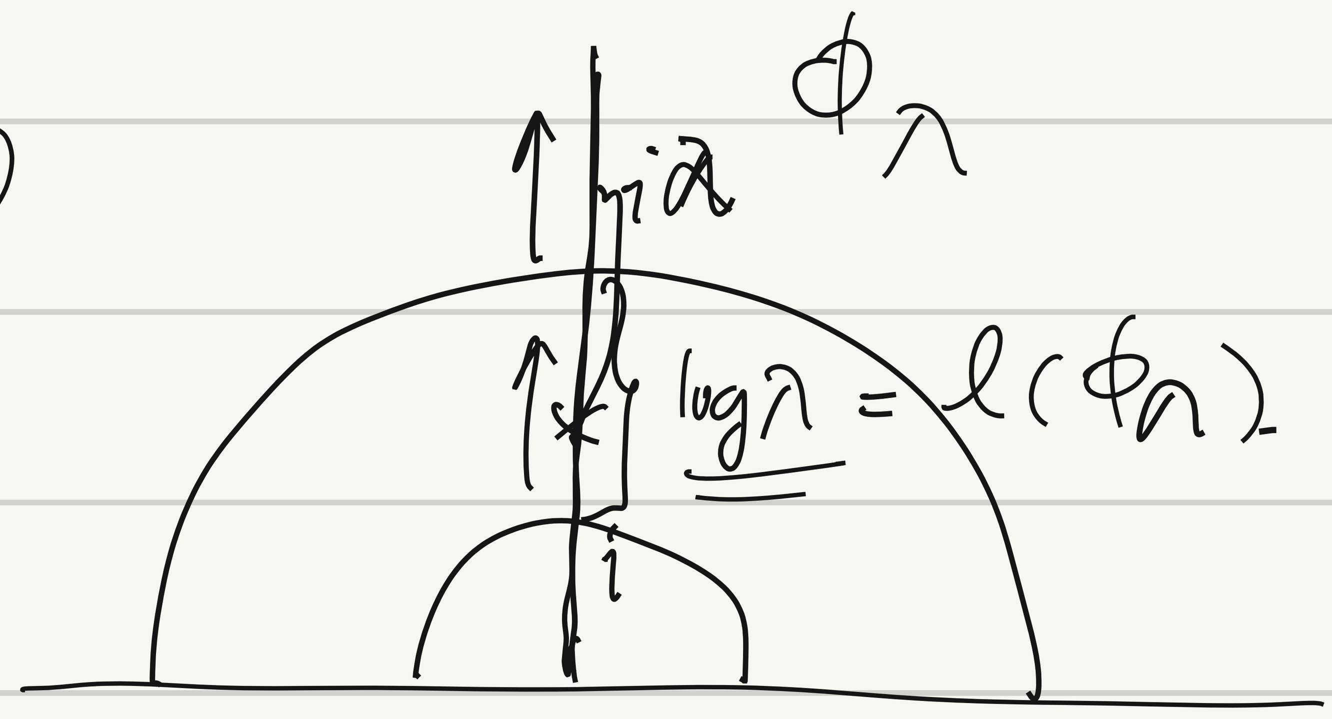
Prop: $f \in \mathrm{Isom}^+(\mathbb{H})$

① $\ell(f) > 0$ iff f hyp (realized by pts on axis of f.

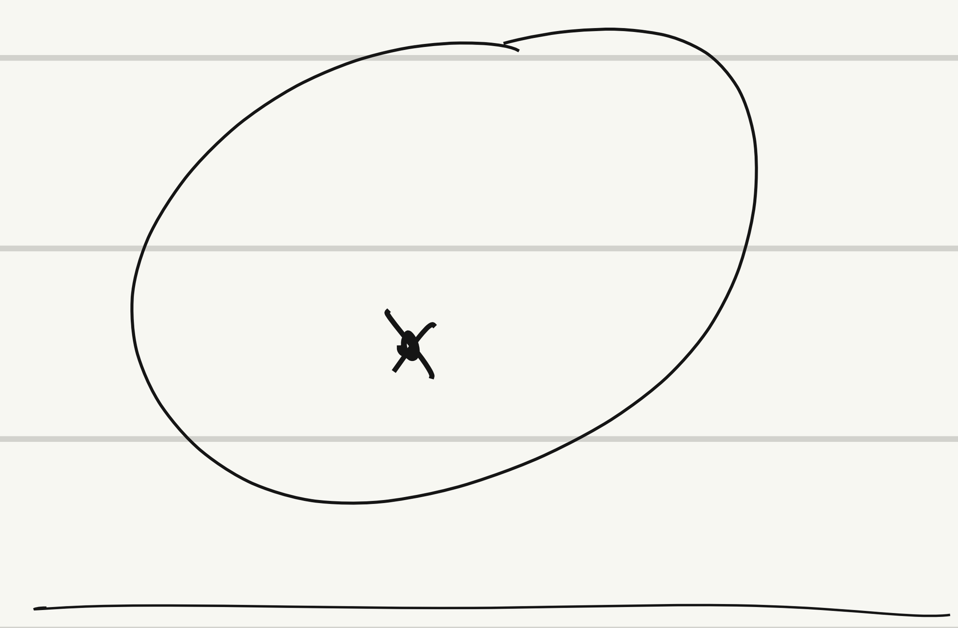
② $\ell(f) = 0$ realizable iff f ellipitic or id
$\exists! z$ $\qquad$ $\forall\infty$

③ $\ell(f) = 0$ not realizable iff parabolic.

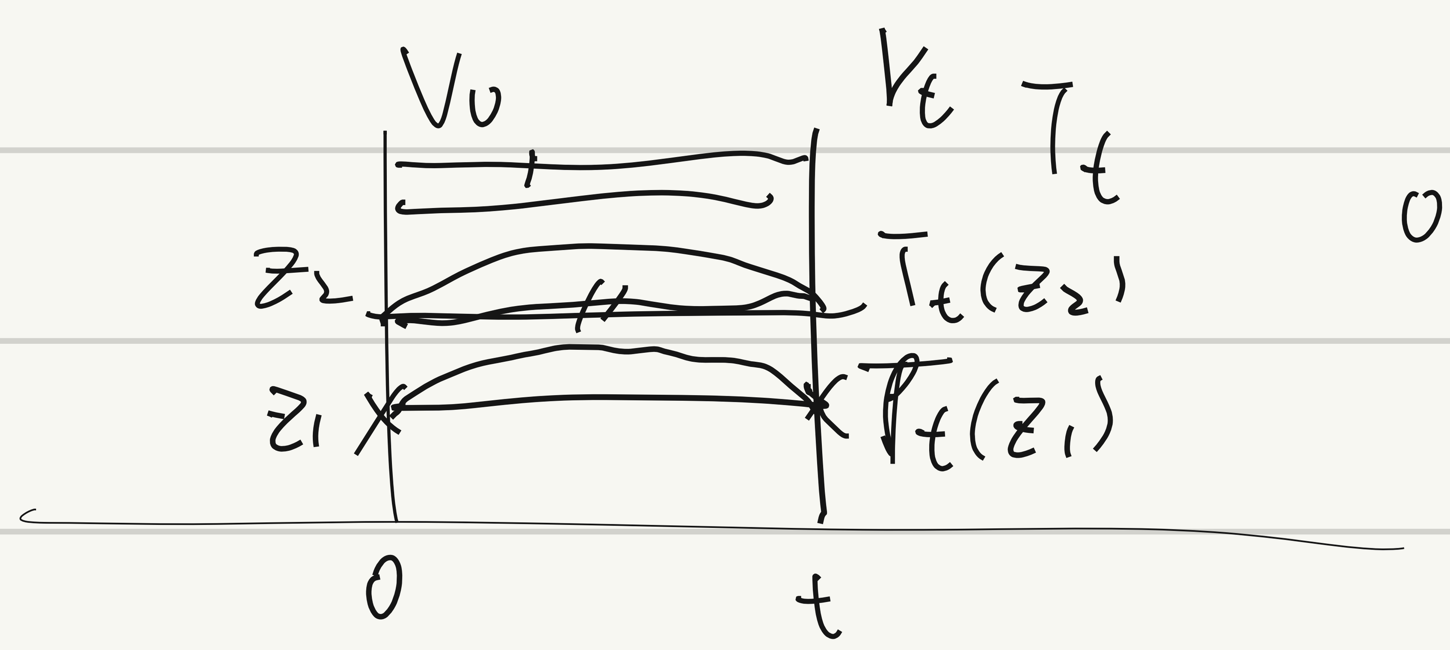
①



②



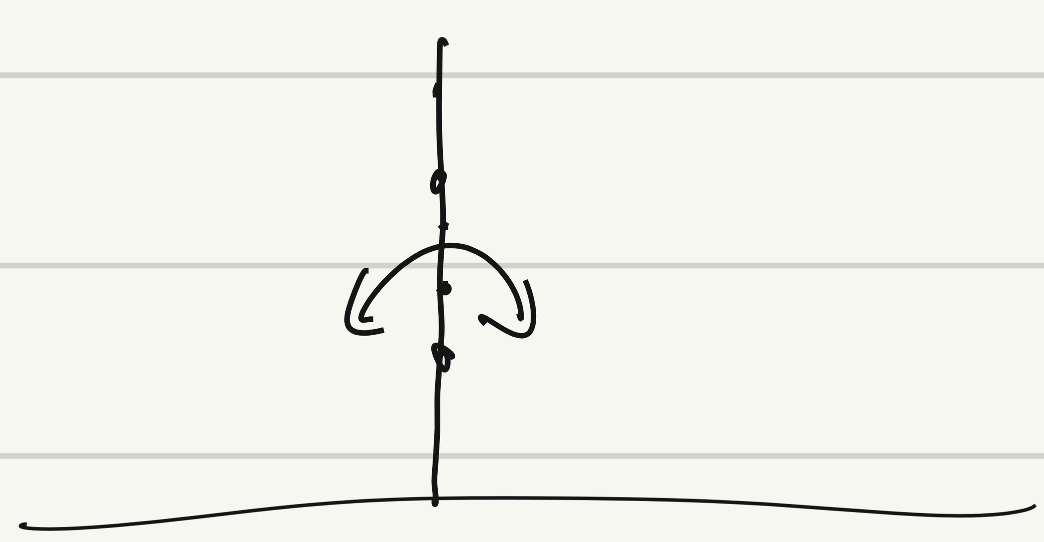
③



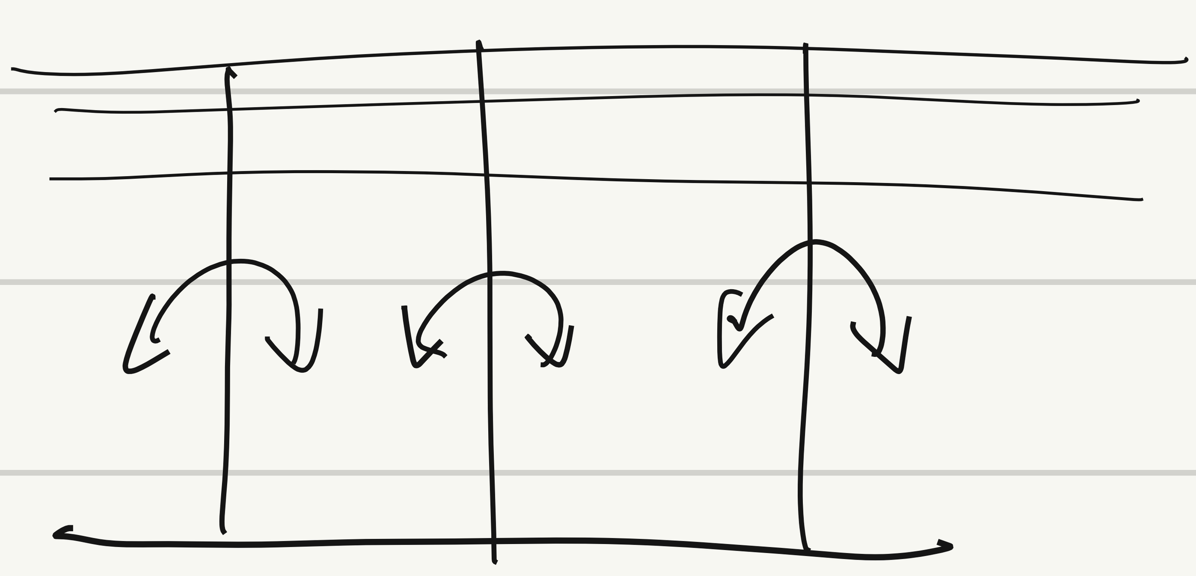
$0 \leq a_n < b_n \quad \forall n.$

$\lim_{n\to\infty} b_n = 0 \Rightarrow \lim_{n\to\infty} a_n = 0.$

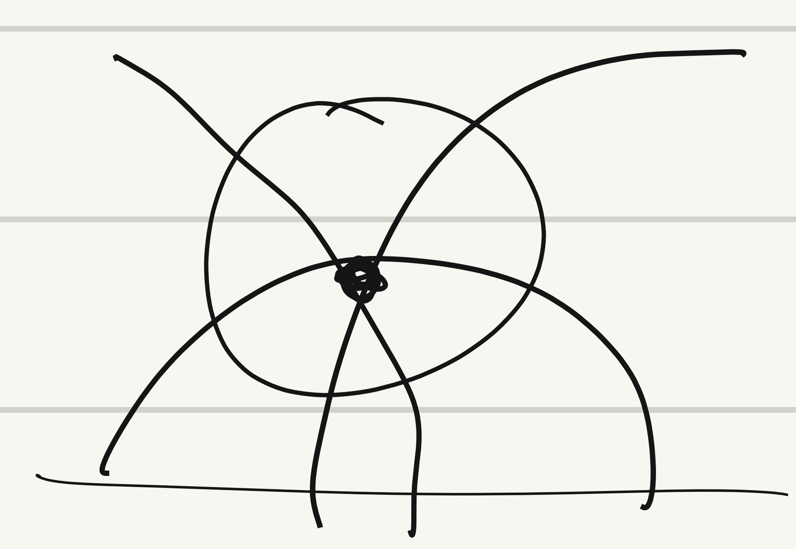
Orient. reversing



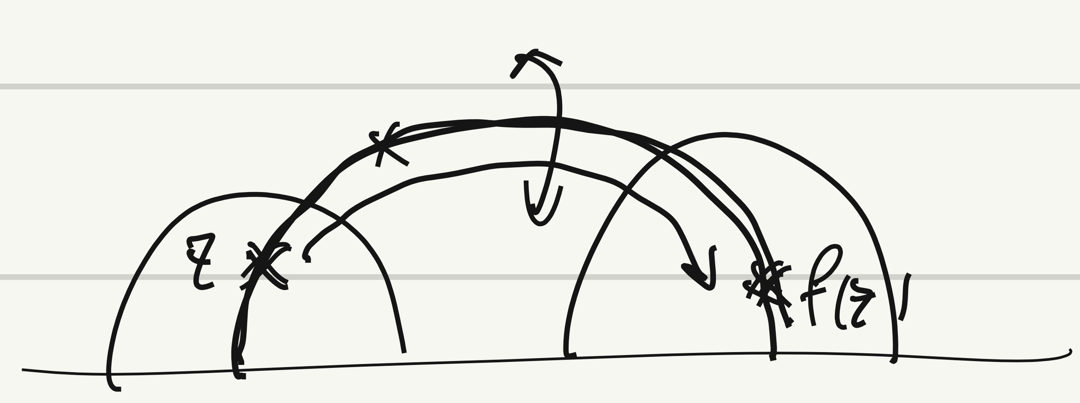
$\ell(f) = 0$



$\ell(f) = 0$



$\ell(f) = 0$



$\ell(f) > 0.$