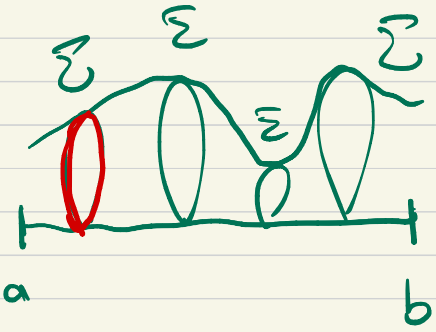
Lecture 3: More examples of Metric structures

3.1 Geometry of warped products

- (Σ, h) compact Riemannian manifold
- Warped product:



  * $[a,b] \times \Sigma$
  * $g = dr^2 + f^2(r) h$

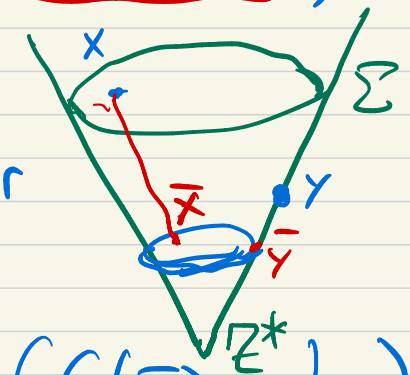
  for some $f = f(r) \geq 0$

- $R_g(X, \partial_r, \partial_r Y)$

- $R_g(X, Y, Z, W)$

- $R_g(X, Y, Z, \partial_r)$

Definition (Metric Cone (Euclidean cone))



Let (Σ, d_Σ) be a compact metric space with diameter $\le \pi$.

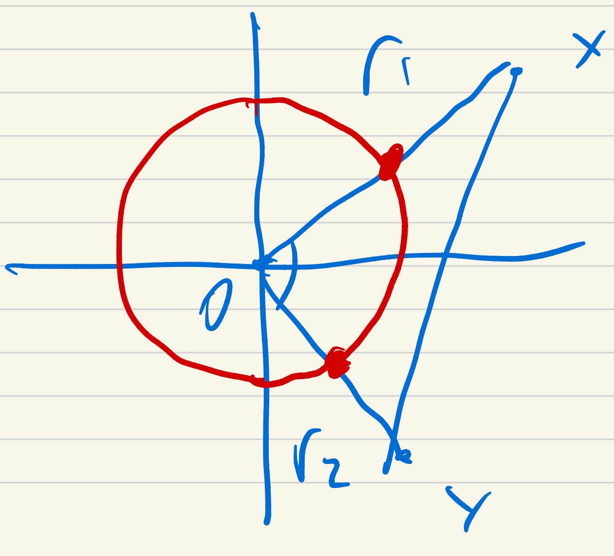
A metric space $(Z, d_C) \equiv (C(\Sigma), d_C)$ is the metric cone over the cross-section Σ, denoted by $C(\Sigma) = C_o(\Sigma)$, if

(1) Z homeomorphic to $(\Sigma \times [0,\infty)) / \Sigma \times \{0\}$

(2) d_C is given by the Euclidean law of cosine, i.e.,

$$d_C^2(x,y) = d^2(x, Z^*) + d^2(y, Z^*)^2 - 2\, d(x, Z^*)\, d(y, Z^*) \cos\left(d_\Sigma(\bar{x},\bar{y})\right).$$

Example ($\mathbb{R}^2$)



Definition [Rescaling]

- Let (X, d) be a metric space. For any $r > 0$.

  $d_r(p,q) := r \cdot d(p,q)$.

- Let (M^n, g) be a Riemannian manifold.

  $\forall r > 0$

  $g_r(v,w) = r^2 \cdot g(v,w)$.

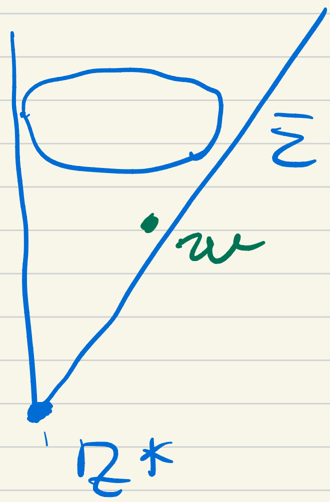
Lemma. Any metric cone is scaling invariant at z^*.

Any metric cone $Z = C(\Sigma)$ is equipped with a natural coordinate system

$p = (r, x)$, where $r = d_c(p, z^*)$.

Example. Let (Σ, h) be a Riemannian manifold with $\mathrm{diam}_h(\Sigma) \le \pi$.

$$g_c = dr^2 + r^2 h.$$



Example:

A metric cone $C(\Sigma)$ is smooth everywhere iff $C(\Sigma)$ is flat

$\iff$ Σ is the round sphere of curvature $+1$.

Examples:

(1) $\mathbb{R}^n \equiv C(S^{n-1})$

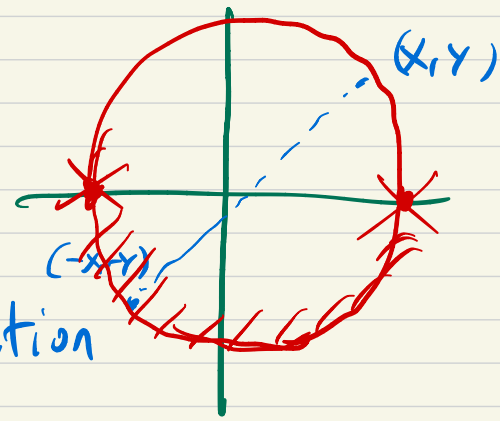
(2) $\mathbb{R}_+ \equiv C(pt)$

(3) $\mathbb{R} \times \mathbb{R}_+ = C([0,\pi])$

$$= \mathbb{R}^2/\mathbb{Z}_2$$

$\mathbb{Z}_2$ generated by $(x,y) \mapsto (x,-y)$

(4) $\mathbb{R}^2/\mathbb{Z}_2 = C(S^1_{\frac{1}{2}})$



$\mathbb{Z}_2$ generated by the involution

$(x,y) \mapsto (-x,-y)$

(5) $\mathbb{R}^4/\mathbb{Z}_2 = C(\mathbb{R}P^3)$

$\mathbb{R}^4 = C(S^3)$.

$\mathbb{Z}_2$ generated by the involution

$(x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, -x_3, -x_4)$

<u>Ex</u> $(C(\Sigma), g_C)$ of dimension n

- $\mathrm{Ric}_{g_C} \equiv 0$ iff $\underline{\mathrm{Ric}_\Sigma \equiv (n-2)\, g_\Sigma}$

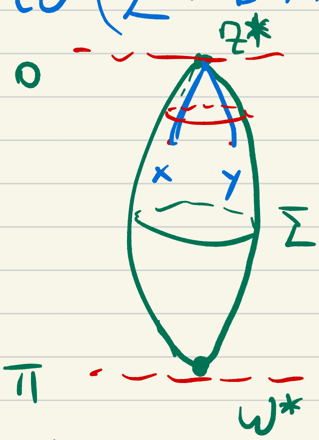
  Einstein

- $\sec_{g_C} \equiv 0$ iff $\underline{\sec_\Sigma \equiv +1}$

  spherical space form S^{n-1}/Γ

Definition (Spherical suspension)

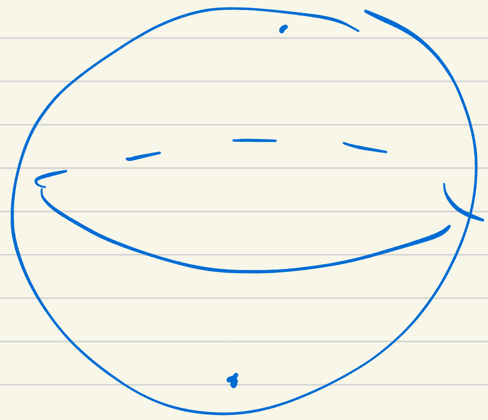
Let (Σ, d_Σ) be a metric space with diameter $\leq \pi$. A metric space (Z, d) is called the spherical suspension over Σ, denoted by $\mathrm{Susp}(\Sigma) = C_{+1}(\Sigma)$, if Z is homeomorphic to $(\Sigma \times [0,\pi]) / (\Sigma \times \{0,\pi\})$



and d is given by the spherical law of cosine

$$\cos d(x,y) = \cos d(z^*, x) \cos d(z^*, w) + \sin d(z^*, x) \sin d(z^*, y) \cos d_\Sigma(\bar{x}, \bar{y}).$$

Example S^2 is spherical suspension of S^1.

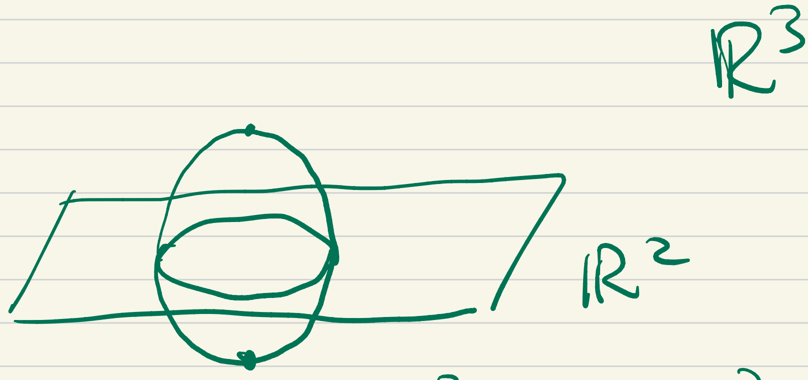


$$g_{S^2} = dr^2 + \sin^2(r)\, g_{S^1}$$

Theorem Let $C(\Sigma)$ be a metric cone over a compact Riemannian manifold (Σ, h).

Assume that $C(\Sigma) = \mathbb{R} \times C(W)$.

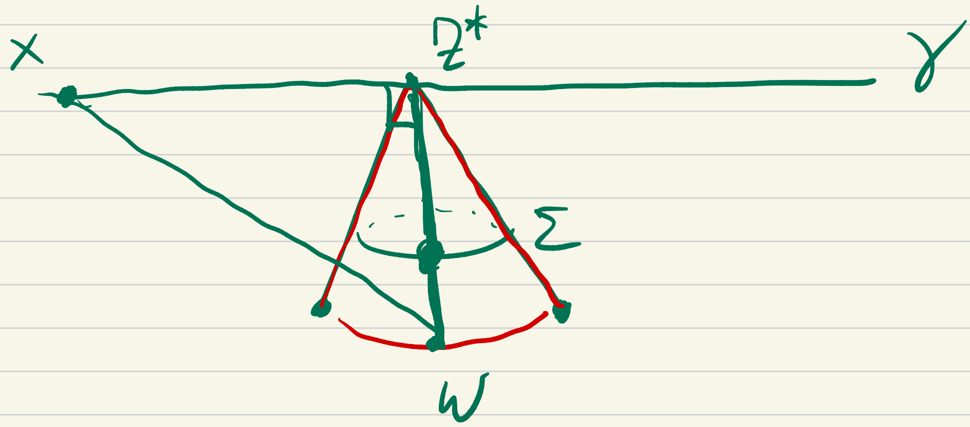
Then both Σ and W are round spheres of curvature $+1$. Moreover, Σ is the spherical suspension of W.



$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$.

Lemma. Let $C(\Sigma)$ be a metric cone over a compact metric space. If $C(\Sigma)$ admits a line γ. Then

$$C(\Sigma) \equiv \mathbb{R} \times C(W).$$

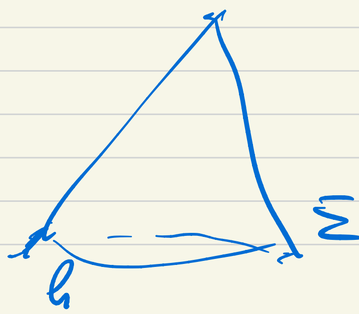


Lemma. Let $C(\Sigma)$ be a metric cone over a compact metric space. Assume that there is a point w^* (different from z^*) s.t. $C(\Sigma)$ is a metric cone w.r.t. w^*.

Then $C(\Sigma) \equiv \mathbb{R} \times C(W)$.

Definition (Hyperbolic suspension)

$$g = dr^2 + \sinh^2(r)\, h$$



Ex Let $\mathcal{Z} = \mathrm{Susp}_K(\Sigma)$ with $K \in \{-1, +1\}$. Show that $\sec_\Sigma \equiv +1$ iff $\sec_{\mathcal{Z}} \equiv K$.

Theorem Let $C(\Sigma)$ be a metric cone over a metric space (Σ, d_Σ).

Assume $C(\Sigma) \equiv \mathbb{R} \times C(W)$.

Then Σ is the spherical suspension over W.